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FINSLER STRUCTURES OF 4-th ROOT TYPE IN CANCER CELL EVOLUTION MODEL

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Abstract

The present work introduces a Finslerian model related to the classical Garner dynamical system, which models the cancer cell population growth.

The Finsler structure is determined by the energy of the deformation field - the difference of the fields, which describe the reduced and the proper biological models.

It is shown that a certain locally-Minkowski anisotropic $4-$th root structure, obtained by means of statistical fitting, is able to provide an evaluation of the overall cancer cell population growth, which occurs due to significant changes within the cancerous process.

The geometric background, the comparison relative to the Euclidean and to the Randers fitted structures, and the applicative advantages of the constructed geometric structure are discussed.

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1 Introduction

The anisotropic geometric structures are of great interest in modelling real-life phenomena (e.g., [2, 5, 12]). By applying the statistical techniques from [2], certain Finsler type structures were determined by the least square method fitting [7]. Three locally-Minkowski Finslerian structures were built, emerging from the Garner dynamical system of cancer cell population. The anisotropic structures - of Randers, Euclidean and 4-root type - were built on the system data, and were shown to provide information on the evolution of the cancer cell population ([7]).

The relevance of the grid density for the resulting structures was discussed in [6], and the corresponding locally-Minkowski norms of Randers and Euclidean

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type were considered, compared and their relevance towards the cancerous process, was presented in [8].

Emerging from the fact that the metric tensor fields related to these three structures are elements of the Hilbert space of bounded and continuous $(0,2)$-type d-tensors over the same differentiable manifold [12, 19], it was shown that the canonic Euclidean metric δ enhances the comparison between the statistically determined Finsler metrics and allows an evaluation of their norms, deviation angles and the conformal projections.

We shall further present the fitted 4-th root Finsler structure and compare it with the fitted Euclidean and Randers ones.

2 The Garner cancer cell evolution model

A cancerous tissue contains three types of cells: proliferating, quiescent and dead ones [16, 18, 20], whose abundance indicate the cancerous disease course. Solyanik suggested the first model for the evolution of the cancer cell population [21], which was further improved by Garner et al.[17]. The Garner dynamical system describes the evolution of the amounts of quiescent and proliferating cells:

$$\begin{cases} \dot{x} = x - x(x+y) + \dfrac{hxy}{1+kx^2} \\ \dot{y} = -ry + ax(x+y) - \dfrac{hxy}{1+kx^2}, \end{cases} \tag{1}$$

where x and y represent the scaled amounts of proliferating and quiescent cells, respectively. The other parameters of the system are:

- a measures the relative nutrient uptake by resting vs. proliferating cancerous cells;

- $r = d/b$ is the ratio between the death rate of quiescent cells and the birth rate of proliferating cells;

- h represents a growth factor that preferentially shifts cells from quiescent to proliferating state, it is inversely proportional to a;

- k represents a mild moderating factor.

We shall further consider the Garner dynamical system for the case of the fixed parameters ([17])

$$a = 1.998958904, \ r = 0.03, \ h = 1.236, \ k = 0.236.$$

This context was thoroughly described in [7, 8, 6].

3 Finsler structures and comparison of their metrics

A Finsler space is a couple (M, F), where M is a differential manifold endowed with a fundamental function $F : TM \to \mathbb{R}$, which satisfies certain requirements [9, 14, 12].

A dynamical system described by a system of second order differential equations is represented in the Finslerian framework as a semispray [12].

The components of the associated metric tensor $g = g_{ij}(x, y)dx^i \otimes dx^j$ are ([9, 14]):

$$g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j},$$

and they play a major role in constructing the specific Finslerian geometric objects, one of which is the $(0, 3)$-type, totally symmetric *Cartan tensor field* [9, 12, 14],

$$C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k} = \frac{1}{4}\frac{\partial^3 F^2}{\partial y^i \partial y^j \partial y^k}. \tag{2}$$

This tensor field characterizes the non-Riemannian nature of the structure.

A subsequently determined tensor field of type $(0, 1)$, which reflects the properties of the structure, is the *mean Cartan tensor field*,

$$I_i = g^{jk} C_{ijk} = \frac{\partial}{\partial y^i} \ln \sqrt{\det(g_{jk})}.$$

Deicke's Theorem (cf. [15, 11, 9]) proves that the mean Cartan tensor vanishes iff the Finsler structure is reducible to a Riemannian one.

Both the Finsler metric g_{ij} and the mean Cartan tensor field I_i, belong to Hilbert spaces of bounded and continuous d-tensor fields of the corresponding type, $(0, 2)$ and $(0, 1)$, respectively [12, 19]. Our goal is to compare Finsler metrics locally produced by the Solyanik differential system. To this aim, we chose the Hilbert structure provided by the canonical scalar product (whose matrix of components is $\delta = \mathrm{diag}(1, \ldots, 1)$),

$$\langle g, h\rangle_\delta = g_{ij} h_{kl} \delta^{ik} \delta^{jl} = Trace(g \cdot h^t). \tag{3}$$

Further, the induced by δ norm of a metric is

$$||g|| = \sqrt{\langle g, g\rangle}_\delta = \sqrt{\sum_{i,j} g_{ij}^2}. \tag{4}$$

The angle between two given metrics g and h, and the projection of g onto h is given by

$$\sphericalangle(g, h) = \arccos \frac{\langle g, h\rangle}{||g|| \cdot ||h||}, \qquad pr_g h = \frac{\langle h, g\rangle}{\langle g, g\rangle} g. \tag{5}$$

4 Comparison of the fitted Finslerian estimates

According to the fitting results and conclusions from [7, 8], the Garner dynamical system provides by fitting the following Finsler structures:

$$F_R(\dot{x},\dot{y}) \approx \sqrt{\dot{x}^2+\dot{y}^2} + 0.63\cdot\dot{x} - 0.27\cdot\dot{y}, \tag{6}$$
$$F_E(\dot{x},\dot{y}) \approx \sqrt{0.94\dot{x}^2 + 1.16\dot{x}\dot{y} + 0.50\dot{y}^2}, \tag{7}$$
$$F_Q(\dot{x},\dot{y}) \approx \sqrt[4]{-0.29\dot{x}^4 + 2.66\dot{x}^3\dot{y} + 2.44\dot{x}^2\dot{y}^2 + 1.07\dot{x}\dot{y}^3 + 0.25\dot{y}^4}, \tag{8}$$

of Randers, Euclidean and 4-th root type, respectively.

The structures of Randers and Euclidean type were mutually compared and then studied with respect to the canonical Euclidean structure, by considering the deviation angles, projections and relevant first order tensors[8].

Also, in [6], an improvement of the fitted 4-the root structure (8) firstly constructed in [7], was provided. In the following, we shall present the properties of this refined structure.

A straightforward calculation yields the components of the metric tensor field

$$\begin{cases} g_{Q11} = \frac{1}{1000F_Q^6}\left(84.1\dot{x}^6 - 1157.15\dot{x}^5\dot{y} + 1591.95\dot{x}^4\dot{y}^2 + 2624.6\dot{x}^3\dot{y}^3 \right.\\ \qquad\qquad \left. + 1917.15\dot{x}^2\dot{y}^4 + 997.5\dot{x}\dot{y}^5 + 161.89\dot{y}^6\right), \\ g_{Q12} = \frac{1}{1000F_Q^6}\left(-192.85\dot{x}^6 + 2653.35\dot{x}^5\dot{y} + 5100.525\dot{x}^4\dot{y}^2 + 3833.35\dot{x}^3\dot{y}^3 \right.\\ \qquad\qquad \left. + 1459.35\dot{x}^2\dot{y}^4 + 429.34\dot{x}\dot{y}^5 + 66.87\dot{y}^6\right), \\ g_{Q22} = \frac{1}{1000F_Q^6}\left(-1238.25\dot{x}^6 - 465.45\dot{x}^5\dot{y} + 1917.15\dot{x}^4\dot{y}^2 + 2635.4\dot{x}^3\dot{y}^3 \right.\\ \qquad\qquad \left. +1344.34\dot{x}^2\dot{y}^4 + 401.25\dot{x}\dot{y}^5 + 62.5\dot{y}^6\right). \end{cases} \tag{9}$$

All the following statements, regarding the properties of the metric g_Q, are consequences of [7, Prop 2.1 and 2.2] and [8, Prop. 4.1].

Proposition 4.1. *With respect to the standard Hilbert structure on the space of (0,2)-tensors, the fitted* 4*-root metric of the Finsler structure* (8) *has the norm*

$$||g_Q|| \approx \frac{1}{F_Q^6}\sqrt{p}, \tag{10}$$

where

$$\begin{aligned} p = \;& -3.85\dot{x}^{12} - 12.40\dot{x}^{10}\dot{y}^2 + 56.44\dot{x}^{11}\dot{y} - 22.12\dot{x}^9\dot{y}^3 \\ & -10.23\dot{x}^8F_Q^4 - 83.6\dot{x}^6\dot{y}^2F_Q^4 - 0.22\dot{x}\dot{y}^3F_Q^8 - 28.1\dot{x}^5\dot{y}^3F_Q^4 \\ & -51.03\dot{x}^7\dot{y}F_Q^4 + 2.5F_Q^{12} + 0.87\dot{x}^3\dot{y}F_Q^8 + 30.37\dot{x}^4F_Q^8 - 7.53\dot{x}^2\dot{y}^2F_Q^8. \end{aligned}$$

Figure 1 illustrates the norm of the 4-root metric g_Q after fixing the flagpole from the region of feasible directions determined by the Garner dynamical system.

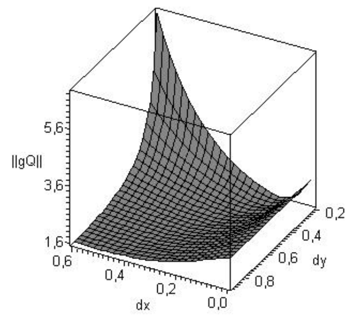

Figure 1: The Hilbert norm of the metric tensor g_Q

Proposition 4.2. *With respect to the standard Hilbert structure, the fitted* 4*-root metric of the Finsler structure* (8) *deviates from the canonic Euclidean, Finsler-Euclidean, and Finsler-Randers structures, by the following angles*

$$\begin{aligned}\sphericalangle(g_Q,\delta) = \ & arccos\Big(\tfrac{1}{F_Q^6||g_Q||}(-1.48\dot{x}^6 + 5.23\dot{x}^5\dot{y} + 4.92\dot{x}^4\dot{y}^2 + 1.44\dot{x}^3\dot{y}^3 \\ & \qquad -2.28F_Q^4\dot{x}^2 + 1.24F_Q^4\dot{x}\dot{y} + 0.63F_Q^4\dot{y}^2)\Big),\end{aligned}$$

$$\begin{aligned}\sphericalangle(g_Q,g_E) = \ & arccos\Big(\tfrac{1}{F_Q^6||g_Q||}(-1.09\dot{x}^6 + 6.53\dot{x}^5\dot{y} + 6.72\dot{x}^4\dot{y}^2 + 2.21\dot{x}^3\dot{y}^3 \\ & \qquad -1.79F_Q^4\dot{x}^2 + 1.54F_Q^4\dot{x}\dot{y} + 0.78F_Q^4\dot{y}^2)\Big),\end{aligned}$$

$$\sphericalangle(g_Q,g_R) = arccos\Big(\frac{r}{\alpha^{3/2}\sqrt{p}\sqrt{s}}\Big),$$

where p *is given in the previous Proposition and*

$$s = -0.22\dot{x}^3 + 0.60\dot{x}^2\dot{y} + 0.97\alpha\dot{x}^2 - 1.02\alpha\dot{x}\dot{y} + 5.28\alpha^2\dot{x} - 2.34\alpha^2\dot{y} + 4.32\alpha^3,$$

$$\begin{aligned}r = \ & -2.75\alpha^2\dot{x}^7 - 4.23\alpha^3\dot{x}^6 + (3.74F_Q^4 + 2.12\alpha^4)\dot{x}^5 \\ & +(3.20\alpha^5 + 3,76\cdot 10^{-10}\alpha F_Q^4)\dot{x}^4 - (0.06\alpha^6 + 4.88\alpha^2F_Q^4)\dot{x}^3 \\ & -(0.36\alpha^7 + 2.3\alpha^3F_Q^4)\dot{x}^2 + 1.15\alpha^3F_Q^4\dot{x}\dot{y} \\ & +(0.52\alpha^4F_Q^4 - 0.08\alpha^8 + 0.44F_Q^8)\dot{x} + (-0.21\alpha^4F_Q^4 + 0.95F_Q^8)\dot{y} \\ & +2.02\cdot 10^{-10}\alpha F_Q^8 - 0.05\alpha^9 + 1.3\alpha^5F_Q^4, \quad \alpha = \sqrt{\dot{x}^2+\dot{y}^2}.\end{aligned}$$

Proposition 4.3. *With respect to the standard Hilbert structure, the fitted* 4*-root metric of the Finsler structure* (8) *has the following projections onto the canonic Euclidean, Finsler-Euclidean and Finsler-Randers metrics, respectively:*

- $pr_\delta g_Q = \frac{u}{F_Q^6}\delta$, *where*

$$\begin{aligned} u = \ & -1.04\dot{x}^6 + 3.72\dot{x}^5\dot{y} + 3.48\dot{x}^4\dot{y}^2 + 1.02\dot{x}^3\dot{y}^3 \\ & -1.61\dot{x}^2F_Q^4 + 0.88\dot{x}\dot{y}F_Q^4 + 0.45\dot{y}^2F_Q^4, \end{aligned}$$

- $pr_{g_E} g_Q = \frac{v}{F_Q^6} g_E$, *where*

$$\begin{aligned} v = \ & -0.81\dot{x}^6 + 4.85\dot{x}^5\dot{y} + 5.0\dot{x}^4\dot{y}^2 + 1.64\dot{x}^3\dot{y}^3 \\ & -1.33\dot{x}^2F_Q^4 + 1.15\dot{x}\dot{y}F_Q^4 + 0.58\dot{y}^2F_Q^4, \end{aligned}$$

- $pr_{g_R} g_Q = \frac{r}{F_Q^6 \cdot s} g_R$, *where* s *is given in the previous Proposition.*

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