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THE ITERATIVE COMBINATIONS OF BERNSTEIN-DURRMEYER TYPE OPERATORS U_n^ρ

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Abstract

In this paper we study a Voronovskaja-type theorem for iterated combinations of Bernstein-Durrmeyer-type operators with parameter U_n^ρ introduced by Gonsha and Paltanea in [2].

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1 Introduction

Denote by $L_B[0,1]$ the space of bounded Lebesgue integrable functions on $[0,1]$ and by Π_n the space of polynomials of degree at most $n \in \mathbb{N}$. The Bernstein-Durrmeyer-type operators with parameter $U_n^\rho : L_B[0,1] \to \Pi_n$ for $f \in L_B[0,1]$, $\rho > 0$ and $n \geq 1$, given by

$$\begin{aligned} U_n^\rho(f,x) &= \sum_{k=0}^{n} F_{n,k}^\rho(f) \cdot p_{n,k}(x) \\ &= \sum_{k=1}^{n-1} \left(\int_0^1 f(t)\, \mu_{n,k}^\rho(t)\, dt \right) \cdot p_{n,k}(x) + f(0)(1-x)^n + f(1)x^n, \quad (1) \end{aligned}$$

where

$$\begin{aligned} p_{n,k}(x) &= \binom{n}{k} x^k (1-x)^{n-k}\ , \ x \in [0,1],\ 0 \leq k \leq n \\ \mu_{n,k}^\rho(x) &= \frac{x^{k\rho-1}(1-x)^{(n-k)\rho-1}}{B(k\rho,(n-k)\rho)},\ x \in [0,1],\ 1 \leq k \leq n-1 \end{aligned}$$

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and

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt,\ \ x,y>0,$$

were introduced in Păltănea [5] and investigated in Gonska and Păltănea [2] and in other papers.

The purpose of this paper is to accelerate the rate of convergence of operators U_n^ρ. For this we apply the iterative technique given by Micchelli combination who has used it to improve the order of approximation by Bernstein polynomials.

Gupta and Vasishtha [3] and Finta, Z., Govil, N.K., Gupta, V. [1] have recently estimated some direct results for some summation-integral type operators. In [3], the authors point out that iterative combinations can be applied only for those operators which reproduce linear functions. We disagree with their claim and state that iterative combinations can be applied even for those operators which do not reproduce the linear functions. Here we show that iterative combinations can easily be used for equations not reproducing linear functions. We use iterative combinations of ordinary approximations, for the operators U_n^ρ and prove that the order of approximation by these operators turns out to be $O(n^{-k})$ for sufficiently smooth functions.

For $f\in L_B[0,1]$ the operators U_n^ρ can be written as

$$U_n^\rho(f,x)=\int_0^1 K_n^\rho(t,x)\,f(t)\,dt,$$

where

$$K_n^\rho(t,x)=\sum_{k=1}^{n-1}p_{n,k}(x)\,\mu_{n,k}^\rho(t)+(1-x)^n\,\delta(t)+x^n\delta(1-t),$$

$\delta(t)$ being the Dirac-function, is the kernel of U_n^ρ.

Section 2 of this paper contains some definitions and auxiliary results. In Section 3 we establish an asymptotic Voronovskaja type formula and the degree of approximation to a class of functions.

2 Auxiliary results

Next we use the following notation $e_j(t)=t^j$, $t\in[0,1]$, for $j\geq 0$.

For $r\in\mathbb{N}\cup\{0\}$, the r-th order moment for operators U_n^ρ is defined as

$$M_{n,r}^\rho(x)=U_n^\rho\left((e_1-xe_0)^r,x\right).$$

In [2], Gonska and Păltănea have given the following result:

Lemma 1. *For $x\in[0,1]$ and $n\in\mathbb{N}$ we have*

$$M_{n,0}^\rho(x)=1,\ \ M_{n,1}^\rho(x)=0$$

and, for $r \geq 1$,

$$\begin{aligned} M_{n,r+1}^{\rho}(x) &= \frac{r(\rho+1)x(1-x)}{n\rho+r}M_{n,r-1}^{\rho}(x) \\ &+\frac{r(1-2x)}{n\rho+r}M_{n,r}^{\rho}(x)+\frac{\rho x(1-x)}{n\rho+r}\left(M_{n,r}^{\rho}(x)\right)'. \end{aligned} \tag{2}$$

Whence

$$M_{n,2}^{\rho}(x)=\frac{(\rho+1)x(1-x)}{n\rho+r}. \tag{3}$$

Corollary 1. *i) The function $M_{n,r}^{\rho}(x)$ is a polynomial in x of degree r, $r \geq 2$.*

ii) For every $x \in [0,1]$, $M_{n,r}^{\rho}(x) = O\left(n^{-\left[\frac{r+1}{2}\right]}\right)$ where $[\alpha]$ is the integer part of α.

Definition 1. *For $k \in \mathbb{N}$. We define the iterative combination $U_{n,k}^{\rho} : L_B[0,1] \rightarrow \Pi_n$ of the operators U_n^ρ by*

$$\begin{aligned} U_{n,k}^{\rho}(f,x) &= \left(I-(I-U_n^{\rho})^k\right)(f,x) \\ &= \sum_{p=1}^{k}(-1)^{p-1}\binom{k}{p}U_n^{\rho,p}(f,x), \end{aligned} \tag{4}$$

where $U_n^{\rho,0} := U_n^\rho$ and $U_n^{\rho,p} = U_n^\rho\left(U_n^{\rho,p-1}\right)$ for $p \in \mathbb{N}$.

For the iterative operators $U_n^{\rho,p}$, the r-th order moment is given by

$$M_{n,r}^{\rho,p}(x) = U_n^{\rho,p}\left((e_1-xe_0)^r, x\right).$$

Lemma 2. *For $p \in \mathbb{N}$ and $x \in [0,1]$ the following recurrence relation holds*

$$M_{n,r}^{\rho,p+1}(x)=\sum_{j=0}^{r}\binom{r}{j}\sum_{i=0}^{r-j}\frac{1}{i!}M_{n,i+j}^{\rho}(x)D^i\left(M_{n,r-j}^{\rho,p}(x)\right).$$

where $D^i$is the i-order derivative.

Proof. We can write

$$\begin{aligned} M_{n,r}^{\rho,p+1}(x) &= U_n^{\rho,p+1}\left((e_1-xe_0)^r,x\right)=U_n^{\rho}\left(U_n^{\rho,p}\left((u-x)^r,t\right),x\right) \\ &= \sum_{j=0}^{r}\binom{r}{j}U_n^{\rho}\left((t-x)^j\cdot U_n^{\rho,p}\left((u-t)^{r-j},t\right),x\right) \end{aligned} \tag{5}$$

Since $U_n^{\rho,p}\left((u-t)^{r-j},t\right)$ is a polynomial in x of degree $r-j$, by Taylor's formula we can write it as:

$$U_n^{\rho,p}\left((u-t)^{r-j},t\right)=\sum_{i=0}^{r-j}\frac{(t-x)^i}{i!}D^i\left(M_{n,r-j}^{\rho,p}(x)\right) \tag{6}$$

From (5) and (6) we obtain Lemma. □

Lemma 3. *For $p \in \mathbb{N}$, $r \in \mathbb{N} \cup \{0\}$ and $x \in [0,1]$ we have*

$$M_{n,r}^{\rho,p}(x) = O\left(n^{-\left[\frac{r+1}{2}\right]}\right). \tag{7}$$

Proof. For the proof we use the induction method with regard to p.

For $p = 1$ the result follows from Lemma 1. Suppose (7) is true for a certain p. Then $M_{n,r-j}^{\rho,p}(x) = O\left(n^{-\left[\frac{r-j+1}{2}\right]}\right)$, $0 \le j \le p$. Also, since $M_{n,r-j}^{\rho,p}(x)$ is a polynomial in x of degree $r-j$, we obtain $D^i\left(M_{n,r-j}^{\rho,p}(x)\right) = O\left(n^{-\left[\frac{r-j+1}{2}\right]}\right)$ for any $0 \le i \le r-j$. From Lemma 2 we obtain

$$\begin{aligned} M_{n,r}^{\rho,p+1}(x) &= \sum_{j=0}^{r}\binom{r}{j}\sum_{i=0}^{r-j}\frac{1}{i!}O\left(n^{-\left[\frac{r-j+1}{2}\right]}\right)\cdot O\left(n^{-\left[\frac{i+j+1}{2}\right]}\right) \\ &= \sum_{j=0}^{r}\sum_{i=0}^{r-j}O\left(n^{-\left[\frac{r+i+1}{2}\right]}\right) = O\left(n^{-\left[\frac{r+1}{2}\right]}\right). \end{aligned}$$

□

Lemma 4. *For $k, l \in \mathbb{N}$ we have*

$$U_{n,k}^{\rho}\left((e_1 - xe_0)^l, x\right) = O\left(n^{-k}\right). \tag{8}$$

Proof. Using the principle of mathematical induction with regard to k. For $k = 1$ the result follows from Lemma 1. Suppose (8) is true for a certain k, then by (4) we get

$$\begin{aligned} U_{n,k+1}^{\rho}\left((e_1 - xe_0)^l, x\right) &= \sum_{p=1}^{k+1}(-1)^{p-1}\binom{k+1}{p}U_n^{\rho,p}\left((e_1 - xe_0)^l, x\right) \\ &= I_1 + I_2 \end{aligned} \tag{9}$$

where

$$I_1 = U_{n,k}^{\rho}\left((e_1 - xe_0)^l, x\right)$$

and

$$I_2 = \sum_{p=1}^{k+1}(-1)^{p-1}\binom{k}{p-1}U_n^{\rho,p}\left((e_1 - xe_0)^l, x\right).$$

From Lemma 2 we obtain

$$\begin{aligned} I_2 &= \sum_{p=0}^{k}(-1)^p\binom{k}{p}U_n^{\rho,p+1}\left((e_1-xe_0)^l,x\right)=\sum_{p=0}^{k}(-1)^p\binom{k}{p}M_{n,l}^{\rho,p+1}(x) \\ &= M_{n,l}^{\rho}(x)-\sum_{j=1}^{l}\binom{l}{j}\sum_{i=0}^{l-j}\frac{1}{i!}D^i\left(U_{n,k}^{\rho}\left((e_1-xe_0)^{l-j},x\right)\right)\cdot M_{n,i+j}^{\rho}(x) \\ &\quad -\sum_{i=0}^{l}\frac{1}{i!}D^i\left(U_{n,k}^{\rho}\left((e_1-xe_0)^{l},x\right)\right)\cdot M_{n,i}^{\rho}(x) \\ &= -\sum_{j=1}^{l-1}\binom{l}{j}\sum_{i=0}^{l-j}\frac{1}{i!}D^i\left(U_{n,k}^{\rho}\left((e_1-xe_0)^{l-j},x\right)\right)\cdot M_{n,i+j}^{\rho}(x) \\ &\quad -\sum_{i=1}^{l}\frac{1}{i!}D^i\left(U_{n,k}^{\rho}\left((e_1-xe_0)^{l},x\right)\right)\cdot M_{n,i}^{\rho}(x). \end{aligned} \tag{10}$$

From (9), (10) and Lemma 1 we obtain

$$U_{n,k+1}^{\rho}\left((e_1-xe_0)^l,x\right)=O\left(n^{-(k+1)}\right).$$

□

3 Main results

Theorem 1. *(Voronovskaja type asymptotic formula). Let $f\in L_B[0,1]$ admitting a derivative of order $2k$ at a point $x\in[0,1]$. Then*

$$\lim_{n\to\infty}n^k\left[U_{n,k}^{\rho}(f,x)-f(x)\right]=\sum_{r=1}^{2k}\frac{f^{(r)}(x)}{r!}Q(r,k,x) \tag{11}$$

and

$$\lim_{n\to\infty}n^k\left[U_{n,k+1}^{\rho}(f,x)-f(x)\right]=0, \tag{12}$$

where $Q(r,k,x)$ are certain polynomials in x of degree r.

Further, the limits in (11) and (12) hold uniformly in $[0,1]$ if $f^{(2k)}$ is continuous in $[0,1]$.

Proof. Since $f^{(2k)}$ exists at a point $x\in[0,1]$, using Taylor's expresion we obtain

$$f(t)=\sum_{r=0}^{2k}\frac{f^{(r)}(x)}{r!}(t-x)^r+o(t,x)\cdot(t-x)^{2k},$$

where $o(t,x) \underset{t\to x}{\to} 0$ and is a bounded and integrable function in $[0,1]$. Applying $U_{n,k}^{\rho}$ we obtain

$$\begin{aligned} n^k\left[U_{n,k}^{\rho}(f,x)-f(x)\right] &= n^k\sum_{r=1}^{2k}\frac{f^{(r)}(x)}{r!}U_{n,k}^{\rho}\left((t-x)^r,x\right) \\ &\quad +n^kU_{n,k}^{\rho}\left(o(t,x)\cdot(t-x)^{2k},x\right) \\ &=: I_3+I_4. \end{aligned}$$

From Lemma 4, we have

$$I_3=\sum_{r=1}^{2k}\frac{f^{(r)}(x)}{r!}Q(r,k,x)+O\left(n^{-1}\right),$$

where $Q(r,k,x)$ is the coefficient of n^{-k} in $U_{n,k}^{\rho}\left((t-x)^r,x\right)$.

For a given $\varepsilon>0$ we can find a $\delta>0$ such that $|o(t,x)|<\varepsilon$ anytime $0<|t-x|<\delta$ and for $|t-x|\geq\delta$ we have $|o(t,x)|\leq K$, where K is a strictly positive constant. If $\phi_\delta(t)$ denotes the characteristic function of the interval $(x-\delta,x+\delta)$, then

$$\begin{aligned} |I_4| &\leq n^k\sum_{p=1}^{k}\binom{k}{p}U_n^{\rho,p}\left(|o(t,x)|\cdot(t-x)^{2k}\phi_\delta(t),x\right) \\ &\quad +n^k\sum_{p=1}^{k}\binom{k}{p}U_n^{\rho,p}\left(|o(t,x)|\cdot(t-x)^{2k}(1-\phi_\delta(t)),x\right) \\ &=: I_5+I_6 \end{aligned}$$

From Lemma 3, we have $I_5=\varepsilon\cdot O(1)$ and

$$I_6\leq n^k\sum_{p=1}^{k}\binom{k}{p}U_n^{\rho,p}\left(K\cdot(t-x)^{2s}\delta^{2k-2s},x\right)=O\left(n^{k-s}\right),$$

for any integer $s>k$. Since ε is arbitrary we obtain $|I_4|=O\left(\frac{1}{n}\right)$. From the estimates of I_3 and I_4 we obtain (11).

Similarly assertion (12) folows from the fact that $U_{n,k+1}^{\rho}\left((t-x)^r,x\right)=O\left(n^{-k-1}\right)$ for all $r\in\mathbb{N}$.

The uniform convergence results due to uniform continuity of $f^{(2k)}$ on $[0,1]$ which insolves δ independent of x and the uniformity of the term $O\left(\frac{1}{n}\right)$ in the estimate of I_3. □

Next we give an estimate of the degree of approximation of a function with specified smoothness.

Theorem 2. *Let $1 \leq p \leq 2k$ be an integer and $f^{(p)} \in C[0,1]$. We consider $\|\cdot\|$ is sup-norm on $[0,1]$ and $\omega(g,\delta)$ is the modulus of continuity of g on $[0,1]$. Then for a sufficiently large n we have*

$$\left\|U_{n,k}^\rho(f,x) - f(x)\right\| \leq \max\left\{C_1 \cdot n^{-\frac{p}{2}}\omega\left(f^{(p)}, n^{-\frac{1}{2}}\right), C_2 \cdot n^{-k}\right\}, \tag{13}$$

where $C_1 = C_1(k,p)$ and $C_2 = C_2(f,k,p)$.

Proof. From Taylor's expansion we can write

$$f(t) - f(x) = \sum_{r=1}^{p} \frac{f^{(r)}(x)}{r!}(t-x)^r + (t-x)^p \cdot \frac{f^{(p)}(\xi) - f^{(p)}(x)}{p!},$$

where ξ is between t and x.
Using $U_{n,k}^\rho$ we obtain

$$\begin{aligned} U_{n,k}^\rho(f,x) - f(x) &= \sum_{r=1}^{p} \frac{f^{(r)}(x)}{r!} U_{n,k}^\rho\left((t-x)^r, x\right) \\ &\quad + U_{n,k}^\rho\left((t-x)^p \cdot \frac{f^{(p)}(\xi) - f^{(p)}(x)}{p!}, x\right) \\ &=: I_7 + I_8. \end{aligned}$$

Applying Lemma 4 we have

$$I_7 = \sum_{r=1}^{p} \frac{f^{(r)}(x)}{r!} U_{n,k}^\rho\left((t-x)^r, x\right) = O\left(n^{-k}\right),$$

uniformly for every $x \in [0,1]$.
Since $f^{(p)}$ is continuous in $[0,1]$, we obtain

$$\left|f^{(p)}(\xi) - f^{(p)}(x)\right| \leq \left(1 + |t-x|\,\delta^{-1}\right) \cdot \omega\left(f^{(p)}, \delta\right), \text{ for any } \delta > 0.$$

Using Schwarz's inequality and Lemma 3, we have

$$\begin{aligned} |I_8| &\leq U_{n,k}^\rho\left(|t-x|^p \cdot \frac{\left(1 + |t-x|\,\delta^{-1}\right) \cdot \omega\left(f^{(p)}, \delta\right)}{p!}, x\right) \\ &\leq \frac{\omega\left(f^{(p)}, \delta\right)}{p!} \sum_{r=1}^{k} \binom{k}{r} U_n^{\rho,r}\left(|t-x|^p \cdot \left(1 + (t-x)\,\delta^{-1}\right), x\right) \\ &\leq \frac{\omega\left(f^{(p)}, \delta\right)}{p!} \sum_{r=1}^{k} \binom{k}{r} \left[U_n^{\rho,r}\left(|t-x|^p, x\right) + U_n^{\rho,r}\left(|t-x|^{p+1}\delta^{-1}, x\right)\right]. \end{aligned}$$

Choosing $\delta = n^{-\frac{1}{2}}$, we obtain

$$|I_8| \leq \omega\left(f^{(p)}, n^{-\frac{1}{2}}\right) \cdot O\left(n^{-\frac{p}{2}}\right), \text{ uniformly in } [0,1].$$

From the estimates of I_7 and I_8, the theorem follows. □

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