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ON THE GROUP OF TRANSFORMATIONS OF SYMMETRIC CONFORMAL METRICAL N-LINEAR CONNECTIONS ON A HAMILTON SPACE OF ORDER K

Monica PURCARU[1]

Abstract

In present paper we obtain in a Hamilton space of order k the transformation laws of the torsion and curvature tensor fields, with respect to the transformations of the group $\mathcal{T}_N$ of the transformations of $N-$linear connections having the same nonlinear connection N.

We also determine in a Hamilton space of order k the set of all metrical semisymmetric $N-$linear connections, $\overset{ms}{\mathcal{T}}_N$, in the case when the nonlinear connection is fixed and we prove that $\overset{ms}{\mathcal{T}}_N$, together with the composition of mappings's a group. We obtain some important invariants of group $\overset{ms}{\mathcal{T}}_N$ and give their properties. We also study the transformation laws of the torsion $d-$tensor fields with respect to the transformations of the group $\overset{ms}{\mathcal{T}}_N$.

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1 Introduction

The notion of Hamilton space was introduced by R. Miron in [2], [3]. The Hamilton spaces appear as dual via Legendre transformation, of the Lagrange spaces.

Differential geometry of the dual bundle of $k-$osculator bundle was introduced and studied by R. Miron [5].

In the present section we keep the general setting from R. Miron [5], and subsequently we recall only some needed notions. For more details see [5].

Let M be a real $n-$dimensional C^∞ $-$manifold and let $\left(T^{*k}M, \pi^{*k}, M\right)$, $(k \geq 2, k \in \mathbb{N})$ be the dual bundle of $k-$osculator bundle (or $k-$cotangent bundle), where the total space is:

$$T^{*k}M = T^{k-1}M \times T^*M. \tag{1.1}$$

[1]"Transilvania" University of Braşov, Faculty of Mathematics and Informatics Iuliu Maniu 50, Braşov 500091, Romania, e-mail: mpurcaru@unitbv.ro

Let $\left(x^i, y^{(1)i}, ..., y^{(k-1)i}, p_i\right), (i = 1, 2, ..., n)$, be the local coordinates of a point $u = \left(x, y^{(1)}, ..., y^{(k-1)}, p\right) \in T^{*k}M$ in a local chart on $T^{*k}M$. The change of coordinates on the manifold $T^{*k}M$ is:

$$\begin{cases} \tilde{x}^i = \tilde{x}^i\left(x^1, ..., x^n\right), \det\left(\frac{\partial \tilde{x}^i}{\partial x^j}\right) \neq 0, \\ \tilde{y}^{(1)i} = \frac{\partial \tilde{x}^i}{\partial x^j} y^{(1)j}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ (k-1)\,\tilde{y}^{(k-1)i} = \frac{\partial \tilde{y}^{(k-2)i}}{\partial x^j} y^{(1)j} + ... + (k-1)\frac{\partial \tilde{y}^{(k-2)i}}{\partial y^{(k-2)j}} y^{(k-1)j}, \\ \tilde{p}_i = \frac{\partial x^j}{\partial \tilde{x}^i} p_j, \end{cases} . \tag{1.2}$$

We denote by $\widetilde{T^{*k}M} = T^{*k}M \setminus \{0\}$, where $0 : M \to T^{*k}M$ is the null section of the projection π^{*k}.

Let us consider the tangent bundle of the differentiable manifold $T^{*k}M$: $\left(TT^{*k}M, d\pi^{*k}, T^{*k}M\right)$, where $d\pi^{*k}$ is the canonical projection and the vertical distribution $V : u \in T^{*k}M \to V(u) \in T_uT^{*k}M$, locally generated by the vector fields: $\left\{\frac{\partial}{\partial y^{(1)i}}, ..., \frac{\partial}{\partial y^{(k-1)i}}, \frac{\partial}{\partial p_i}\right\}$ at every point $u \in T^{*k}M$.

The following $\mathcal{F}\left(T^{*k}M\right) -$ linear mapping: $J : \chi\left(T^{*k}M\right) \to \chi\left(T^{*k}M\right)$, defined by:

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^{(1)i}}, J\left(\frac{\partial}{\partial y^{(1)i}}\right) = \frac{\partial}{\partial y^{(2)i}}, ..., J\left(\frac{\partial}{\partial y^{(k-2)i}}\right) =$$

$$= \frac{\partial}{\partial y^{(k-1)i}}, J\left(\frac{\partial}{\partial y^{(k-1)i}}\right) = 0, J\left(\frac{\partial}{\partial p_i}\right) = 0, \quad \forall u \in \widetilde{T^{*k}M} \tag{1.3}$$

is a tangent structure on $T^{*k}M$.

We denote with N a nonlinear connection on the manifold $T^{*k}M$, with the coefficients:

$$\left(\underset{(1)}{N}{}^j{}_i\left(x, y^{(1)}, ..., y^{(k-1)}, p\right), ..., \underset{(k-1)}{N}{}^j{}_i\left(x, y^{(1)}, ..., y^{(k-1)}, p\right), N_{ij}(x, y^{(1)}, ..., y^{(k-1)}, p)\right),$$

$$(i, j = 1, 2, ..., n).$$

The tangent space of $T^{*k}M$ in the point $u \in T^{*k}M$ is given by the direct sum of vector spaces:

$$T_u\left(T^{*k}M\right) = N_{0,u} \oplus N_{1,u} \oplus ... \oplus N_{k-2,u} \oplus V_{k-1,u} \oplus W_{k,u}, \ \forall u \in T^{*k}M \tag{1.4}$$

A local adapted basis to the direct decomposition (1.4) is given by:

$$\left\{\frac{\delta}{\delta x^i}, \frac{\delta}{\delta y^{(1)i}}, ..., \frac{\delta}{\delta y^{(k-1)i}}, \frac{\delta}{\delta p_i}\right\}, (i = 1, 2, ..., n), \tag{1.5}$$

where:

$$\begin{cases} \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \underset{(1)}{N}{}^j{}_i \frac{\partial}{\partial y^{(1)j}} - ... - \underset{(k-1)}{N}{}^j{}_i \frac{\partial}{\partial y^{(k-1)j}} + N_{ij} \frac{\partial}{\partial p_j}, \\ \frac{\delta}{\delta y^{(1)i}} = \frac{\partial}{\partial y^{(1)}{}_i} - \underset{(1)}{N}{}^j{}_i \frac{\partial}{\partial y^{(2)j}} - ... - \underset{(k-2)}{N}{}^j{}_i \frac{\partial}{\partial y^{(k-1)j}}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{\delta}{\delta y^{(k-1)i}} = \frac{\partial}{\partial y^{(k-1)i}}, \\ \frac{\delta}{\delta p_i} = \frac{\partial}{\partial p_i}. \end{cases} \tag{1.6}$$

Under a change of local coordinates on $T^{*k}M$, the vector fields of the adapted basis transform by the rule:

$$\frac{\delta}{\delta x^i} = \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\delta}{\delta \tilde{x}^j}, \frac{\delta}{\delta y^{(1)i}} = \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\delta}{\delta \tilde{y}^{(1)j}}, ..., \frac{\delta}{\delta y^{(k-1)i}} = \frac{\partial \tilde{x}^j}{\partial x^i} \frac{\delta}{\delta \tilde{y}^{(k-1)j}}, \frac{\delta}{\delta p_i} = \frac{\delta x^i}{\delta \tilde{x}^j} \frac{\delta}{\delta \tilde{p}_j}. \tag{1.7}$$

The dual basis of the adapted basis (1.5) is given by:

$$\left\{ \delta x^i, \delta y^{(1)i}, ..., \delta y^{(k-1)i}, \delta p_i \right\}, \tag{1.8}$$

where:

$$\begin{cases} dx^i = \delta x^i, \\ dy^{(1)i} = \delta y^{(1)i} - \underset{(1)}{N}{}^i{}_j \delta x^j, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ dy^{(k-1)i} = \delta y^{(k-1)i} - \underset{(1)}{N}{}^i{}_j \delta y^{(k-2)j} - ... - \underset{(k-2)}{N}{}^i{}_j \delta y^{(1)j} - \underset{(k-1)}{N}{}^i{}_j \delta x^j, \\ dp_i = \delta p_i + N_{ji} \delta x^j. \end{cases} \tag{1.9}$$

With respect to (1.2) the covector fields (1.8) are transformed by the rules:

$$\begin{array}{l} \delta \tilde{x}^i = \frac{\partial \tilde{x}^i}{\partial x^j} \delta x^j, \delta \tilde{y}^{(1)i} = \frac{\partial \tilde{x}^i}{\partial x^j} \delta y^{(1)j}, ..., \delta \tilde{y}^{(k-1)i} = \frac{\partial \tilde{x}^i}{\partial x^j} \delta y^{(k-1)j}, \\ \delta \tilde{p}_i = \frac{\partial x^j}{\partial \tilde{x}^i} \delta p_j. \end{array} \tag{1.10}$$

Let D be an $N-$linear connection on $T^{*k}M$, with the local coefficients in the adapted basis (1.5) :

$$D\Gamma(N) = \left(H^i{}_{jh}, \underset{(\alpha)}{C}{}^i{}_{jh}, C_i^{jh} \right), (\alpha = 1, ..., k-1). \tag{1.11}$$

An $N-$linear connection D is uniquely represented in the adapted basis in the following form:

$$\begin{cases} D_{\frac{\delta}{\delta x^j}}\frac{\delta}{\delta x^i}=H^s{}_{ij}\frac{\delta}{\delta x^s}, D_{\frac{\delta}{\delta x^j}}\frac{\delta}{\delta y^{(\alpha)i}}=H^s{}_{ij}\frac{\delta}{\delta y^{(\alpha)s}}, (\alpha=1,...,k-1), \\ D_{\frac{\delta}{\delta x^j}}\frac{\delta}{\delta p_i}=-H^i_{sj}\frac{\delta}{\delta p_s}, \\ D_{\frac{\delta}{\delta y^{(\alpha)j}}}\frac{\delta}{\delta x^i}=\underset{(\alpha)}{C}{}^s{}_{ij}\frac{\delta}{\delta x^s}, D_{\frac{\delta}{\delta y^{(\alpha)j}}}\frac{\delta}{\delta y^{(\beta)i}}=\underset{(\alpha)}{C}{}^s{}_{ij}\frac{\delta}{\delta y^{(\beta)s}}, \\ D_{\frac{\delta}{\delta y^{(\alpha)j}}}\frac{\delta}{\delta p_i}=-\underset{(\alpha)}{C}{}^i{}_{sj}\frac{\delta}{\delta p_s}, (\alpha,\beta=1,...,k-1), \\ D_{\frac{\delta}{\delta p_j}}\frac{\delta}{\delta x^i}=C_i{}^{js}\frac{\delta}{\delta x^s}, D_{\frac{\delta}{\delta p_j}}\frac{\delta}{\delta y^{(\alpha)i}}=C_i{}^{js}\frac{\delta}{\delta y^{(\alpha)s}}, (\alpha=1,...,k-1), \\ D_{\frac{\delta}{\delta p_j}}\frac{\delta}{\delta p_i}=-C_s{}^{ij}\frac{\delta}{\delta p_s}. \end{cases} \tag{1.12}$$

2 The transformations of the $d-$tensors of torsion and curvature

In the following, we shall study the Abelian group $\mathcal{T}_N$. Its elements are the transformations $t : D\Gamma(N) \to D\bar{\Gamma}(N)$ given by (see(2.11)-p.131 [6]):

$$\begin{cases} \underset{(\alpha)}{\bar{N}}{}^i{}_j=\underset{(\alpha)}{N}{}^i{}_j, \quad (\alpha=1,...,k-1), \\ \bar{N}_{ij}=N_{ij}, \\ \bar{H}^i{}_{jh}=H^i{}_{jh}-B^i{}_{jh}, \\ \underset{(\alpha)}{\bar{C}}{}^i{}_{jh}=\underset{(\alpha)}{C}{}^i{}_{jh}-\underset{(\alpha)}{D}{}^i{}_{jh}, \quad (\alpha=1,...,k-1), \\ \bar{C}_i^{jh}=C_i^{jh}-D_i^{jh}, (i,j,h=1,2,...,n). \end{cases} \tag{2.1}$$

Firstly, we shall study the transformations of the $d-$tensors of torsion of $D\Gamma(N)$.

Proposition 2.1. *The transformations of the Abelian group $\mathcal{T}_N$, given by* (2.1) *lead to the transformations of the $d-$tensors of torsion in the following way:*

$$\begin{cases} \underset{(01)}{\bar{r}}{}^i{}_{jh}=\underset{(01)}{r}{}^i{}_{jh}, \ \underset{(\alpha\beta)}{\bar{B}}{}^i{}_{jh}=\underset{(\alpha\beta)}{B}{}^i{}_{jh}, \underset{(\alpha)}{\bar{B}}{}_{ijh}=\underset{(\alpha)}{B}{}_{ijh}, \underset{(0)}{\bar{B}}{}^h{}_{ij}=\underset{(0)}{B}{}^h{}_{ij}, \\ \underset{(\alpha\beta)}{\bar{B}}{}_{ijh}=\underset{(\alpha\beta)}{B}{}_{ijh}, \underset{(\alpha)}{\bar{B}}{}_i^{jh}=\underset{(\alpha)}{B}{}_i^{jh}, \ \underset{(01)}{\bar{R}}{}^i{}_{jh}=\underset{(01)}{R}{}^i{}_{jh}, \underset{(02)}{\bar{R}}{}^i{}_{jh}=\underset{(02)}{R}{}^i{}_{jh}, ..., \\ \underset{(0,k-1)}{\bar{R}}{}^i{}_{jh}=\underset{(0,k-1)}{R}{}^i{}_{jh}, \underset{(0)}{\bar{R}}{}_{ijh}=\underset{(0)}{R}{}_{ijh}, \quad (\alpha\leq\beta, \alpha,\beta=1,...,k-1), \end{cases} \tag{2.2}$$

$$\bar{T}^i{}_{jh}=T^i{}_{jh}+\left(B^i{}_{hj}-B^i{}_{jh}\right), \tag{2.3}$$

$$\underset{(\alpha)}{\bar{S}}{}^i{}_{jh}=\underset{(\alpha)}{S}{}^i{}_{jh}+(\underset{(\alpha)}{D}{}^i{}_{hj}-\underset{(\alpha)}{D}{}^i{}_{jh}), \bar{S}_i^{jh}=S_i^{jh}+\left(D_i^{hj}-D_i^{jh}\right), \tag{2.4}$$

$$\begin{aligned} &\overline{\underset{(\alpha\beta)}{\overset{(\gamma)}{C}}}{}^i{}_{jh}=\underset{(\alpha\beta)}{\overset{(\gamma)}{C}}{}^i{}_{jh}, (\alpha\leq\beta, \alpha,\beta,\gamma=1,...,,k-1), \\ &\underset{(\alpha 1)}{\bar{C}_i^{jh}}=\underset{(\alpha 1)}{C_i^{jh}}, ..., \underset{(\alpha,k-1)}{\bar{C}_i^{jh}}=\underset{(\alpha,k-1)}{C_i^{jh}}, (\alpha=1,...,k-1). \end{aligned} \tag{2.5}$$

Proof. Using $(5.3), (5.3'), (5.3") - p.131, [5], (5.12) - p.135, [5], (6.3) - p.161, [5]$ and (2.1) we have the results. □

Now, we shall study the transformations of the $d-$tensors of curvature of $D\Gamma(N)$. We get:

Proposition 2.2. *The transformations of the Abelian group $\mathcal{T}_N$, given by (2.1) lead to the transformations of the d−tensors of curvature in the following way:*

$$\bar{R}_m{}^i{}_{jh} = R_m{}^i{}_{jh} - \underset{(1)}{D}{}^i{}_{ms}\underset{(01)}{R}{}^s{}_{jh} - ... - \underset{(k-1)}{D}{}^i{}_{ms}\underset{(0,k-1)}{R}{}^s{}_{jh} - D_m{}^{is}\underset{(0)}{R}{}_{sjh} - \\ -B^i{}_{ms}T^s{}_{jh} + \mathcal{A}_{jh}\left\{B^s{}_{mj}\cdot B^i{}_{sh} - B^i{}_{mj|h}\right\}, \tag{2.6}$$

$$\underset{(\alpha)}{\bar{P}}{}_m{}^i{}_{jh} = \underset{(\alpha)}{P}{}_m{}^i{}_{jh} - B^i_{mj}\overset{(\alpha)}{|}_h + \underset{(\alpha)}{D}{}^i{}_{mh|j} - B^i{}_{ms}\underset{(\alpha)}{C}{}^s{}_{jh} + \underset{(\alpha)}{D}{}^i{}_{ms}H^s_{hj} + B^s{}_{mj}\underset{(\alpha)}{D}{}^i{}_{sh} - \\ -\underset{(\alpha)}{D}{}^s{}_{mh}B^i{}_{sj} - \underset{(1)}{D}{}^i{}_{ms}\underset{(\alpha 1)}{B}{}^s{}_{jh} - ... - \underset{(k-1)}{D}{}^i{}_{ms}\underset{(\alpha,k-1)}{B}{}^s{}_{jh} - D^{is}_m\underset{(\alpha)}{B}{}_{sjh}, \\ (\alpha = 1, ..., k-1), \tag{2.7}$$

$$\bar{P}_m{}^i{}_j{}^h = P_m{}^i{}_j{}^h - B^i_{mj}|^h + D^{ih}_{m|j} - B^i_{ms}C^{sh}_j - D^{is}_m H^h_{sj} + B^s_{mj}D^{ih}_s - \\ -D^{sh}_m B^i_{sj} - \underset{(1)}{D}{}^i{}_{ms}\underset{(1)}{B}{}_j{}^{sh} - ... - \underset{(k-1)}{D}{}^i{}_{ms}\underset{(k-1)}{B}{}_j{}^{sh} - D^{is}_m\underset{(0)}{B}{}^h{}_{sj}, \tag{2.8}$$

$$\underset{(\alpha\beta)}{\bar{S}}{}_m{}^i{}_{jh} = \underset{(\alpha\beta)}{S}{}_m{}^i{}_{jh} - \underset{(\alpha)}{D}{}^i{}_{mj}\overset{(\beta)}{|}_h + \underset{(\beta)}{D}{}^i{}_{mh}\overset{(\alpha)}{|}_j - \underset{(\alpha)}{D}{}^i{}_{ms}\underset{(\beta)}{C}{}^s{}_{jh} + \underset{(\beta)}{D}{}^i{}_{ms}\underset{(\alpha)}{C}{}^s{}_{jh} + \\ +\underset{(\alpha)}{D}{}^s{}_{mj}\underset{(\beta)}{D}{}^i{}_{sh} - \underset{(\beta)}{D}{}^s{}_{mh}\underset{(\alpha)}{D}{}^i{}_{sj} - \underset{(1)}{D}{}^i{}_{ms}\underset{(\alpha\beta)}{\overset{(1)}{C}}{}^s{}_{jh} - ... - \underset{(k-1)}{D}{}^i{}_{ms}\underset{(\alpha\beta)}{\overset{(k-1)}{C}}{}^s{}_{jh} - \\ -D^{is}{}_m\underset{(\alpha\beta)}{B}{}_{sjh}, \quad (\alpha \leq \beta, \alpha, \beta = 1, ..., k-1). \tag{2.9}$$

$$\underset{(\alpha)}{\bar{S}}{}_m{}^i{}_j{}^h = \underset{(\alpha)}{S}{}_m{}^i{}_j{}^h - \underset{(\alpha)}{D}{}^i{}_{mj}|^h + D_m{}^{ih}\overset{(\alpha)}{|}_j - C^{sh}_j\underset{(\alpha)}{D}{}^i{}_{ms} - \underset{(\alpha)}{C}{}^h{}_{sj}D^{is}_m + \underset{(\alpha)}{D}{}^s{}_{mj}D^{ih}_s - \\ -D^{sh}_m\underset{(\alpha)}{D}{}^i{}_{sj} - \underset{(1)}{D}{}^i{}_{ms}\underset{(\alpha 1)}{C}{}_j{}^{sh} - ... - \underset{(k-1)}{D}{}^i{}_{ms}\underset{(\alpha,k-1)}{C}{}_j{}^{sh} - D^{is}_m\underset{(\alpha)}{C}{}^h{}_{sj}, \\ (\alpha = 1, ..., k-1) \tag{2.10}$$

$$\bar{S}_m{}^{ijh} = S_m{}^{ijh} + D_m{}^{is}S_s{}^{jh} + \mathcal{A}_{jh}\left\{-D_m{}^{ij}|^h + D_m{}^{sj}D_s{}^{ih}\right\}, \tag{2.11}$$

where $\mathcal{A}_{ij}$ denotes the alternate summation and $|_m$, $\overset{(\alpha)}{|}_m$ and $|^m$ denote the h-covariant derivative, the v_α-covariant derivative and the w_k-covariant derivative with respect to $D\Gamma(N)$ respectively, $\alpha = 1, ..., k-1$.

Proof. Using (6.4)-p.161, [5], (2.1) and (5.2)'-p.156, [5] we have the results. □

We shall consider the tensor fields:

$$K_m{}^i{}_{jh} = R_m{}^i{}_{jh} - \underset{(1)}{C}{}^i{}_{ms}\underset{(01)}{R}{}^s{}_{jh} - ... - \underset{(k-1)}{C}{}^i{}_{ms}\underset{(0,k-1)}{R}{}^s{}_{jh} - C_m{}^{is}\underset{(0)}{R}{}_{sjh}, \tag{2.12}$$

$$\underset{(\alpha)}{\mathcal{P}}{}_m{}^i{}_{jh} = \mathcal{A}_{jh}\{\underset{(\alpha)}{P}{}_m{}^i{}_{jh} - \underset{(1)}{C}{}^i_{ms}\underset{(\alpha 1)}{B}{}^s{}_{jh} - \ldots - \underset{(k-1)}{C}{}^i{}_{ms}\underset{(\alpha,k-1)}{B}{}^s{}_{jh} - C^{is}_m\underset{(\alpha)}{B}{}_{sjh}\},$$
$$(\alpha = 1,\ldots,k-1), \tag{2.13}$$

$$\underset{(\alpha)}{\mathcal{P}'}{}_m{}^i{}_{jh} = \underset{(\alpha)}{P}{}_m{}^i{}_{jh} - \underset{(1)}{C}{}^i_{ms}\underset{(\alpha 1)}{B}{}^s{}_{jh} - \ldots - \underset{(k-1)}{C}{}^i{}_{ms}\underset{(\alpha,k-1)}{B}{}^s{}_{jh} - C^{is}_m\underset{(\alpha)}{B}{}_{sjh}, \quad (\alpha = 1,\ldots,k-1), \tag{2.14}$$

$$\mathcal{P}_m{}^i{}_j{}^h = \mathcal{A}_{jh}\left\{P_m{}^i{}_j{}^h - \underset{(1)}{C}{}^i_{ms}\underset{(1)}{B}{}^{sh}{}_j - \ldots - \underset{(k-1)}{C}{}^i{}_{ms}\underset{(k-1)}{B}{}^{sh}{}_j - C^{is}_m\underset{(0)}{B}{}^h{}_{sj}\right\}, \tag{2.15}$$

$$\mathcal{P}'{}_m{}^i{}_j{}^h = P_m{}^i{}_j{}^h - \underset{(1)}{C}{}^i_{ms}\underset{(1)}{B}{}^{sh}{}_j - \ldots - \underset{(k-1)}{C}{}^i{}_{ms}\underset{(k-1)}{B}{}^{sh}{}_j - C^{is}_m\underset{(0)}{B}{}^h{}_{sj}, \tag{2.16}$$

$$\underset{(\alpha\beta)}{\mathcal{S}}{}_m{}^i{}_{jh} = \mathcal{A}_{jh}\left\{\underset{(\alpha\beta)}{S}{}_m{}^i{}_{jh} - \underset{(1)}{C}{}^i_{ms}\overset{(1)}{\underset{(\alpha\beta)}{C}}{}^s{}_{jh} - \underset{(k-1)}{C}{}^i{}_{ms}\overset{(k-1)}{\underset{(\alpha\beta)}{C}}{}^s{}_{jh} - C^{is}_m\underset{(\alpha\beta)}{B}{}_{sjh}\right\},$$
$$(\alpha \le \beta,\ \alpha,\beta = 1,\ldots,k-1), \tag{2.17}$$

$$\underset{(\alpha\beta)}{\mathcal{S}'}{}_m{}^i{}_{jh} = \underset{(\alpha\beta)}{S}{}_m{}^i{}_{jh} - \underset{(1)}{C}{}^i_{ms}\overset{(1)}{\underset{(\alpha\beta)}{C}}{}^s{}_{jh} - \ldots - \underset{(k-1)}{C}{}^i{}_{ms}\overset{(k-1)}{\underset{(\alpha\beta)}{C}}{}^s{}_{jh} - C^{is}_m\underset{(\alpha\beta)}{B}{}_{sjh}, \quad (\alpha \le \beta,\ \alpha,\beta = 1,\ldots,k-1), \tag{2.18}$$

$$\underset{(\alpha)}{\mathcal{S}}{}_m{}^i{}_j{}^h = \mathcal{A}_{jh}\left\{\underset{(\alpha)}{S}{}_m{}^i{}_j{}^h - \underset{(1)}{C}{}^i_{ms}\underset{(\alpha 1)}{C}{}^{sh}{}_j - \ldots - \underset{(k-1)}{C}{}^i{}_{ms}\underset{(\alpha,k-1)}{C}{}^{sh}{}_j - C^{is}_m\underset{(\alpha)}{C}{}^h{}_{sj}\right\},$$
$$(\alpha = 1,\ldots,k-1), \tag{2.19}$$

$$\underset{(\alpha)}{\mathcal{S}'}{}_m{}^i{}_j{}^h = \underset{(\alpha)}{\mathcal{S}}{}_m{}^i{}_j{}^h - \underset{(1)}{C}{}^i_{ms}\underset{(\alpha 1)}{C}{}^{sh}{}_j - \ldots - \underset{(k-1)}{C}{}^i{}_{ms}\underset{(\alpha,k-1)}{C}{}^{sh}{}_j - C^{is}_m\underset{(\alpha)}{C}{}^h{}_{sj}, (\alpha = 1,\ldots,k-1). \tag{2.20}$$

Proposition 2.3. *By a transformation of the Abelian group $\mathcal{T}_N$, given by (2.1) ,the tensor fields $K_m{}^i{}_{jh}$, $\underset{(\alpha)}{\mathcal{P}}{}_m{}^i{}_{jh}$, $\underset{(\alpha)}{\mathcal{P}'}{}_m{}^i{}_{jh}$, $\mathcal{P}_m{}^i{}_j{}^h$, $\mathcal{P}'{}_m{}^i{}_j{}^h$, $\underset{(\alpha\beta)}{\mathcal{S}}{}_m{}^i{}_{jh}$, $\underset{(\alpha\beta)}{\mathcal{S}'}{}_m{}^i{}_{jh}$, $\underset{(\alpha)}{\mathcal{S}}{}_m{}^i{}_j{}^h$, $\underset{(\alpha)}{\mathcal{S}'}{}_m{}^i{}_j{}^h$ are transformed according to the following laws:*

$$\bar{\mathcal{K}}_m{}^i{}_{jh} = \mathcal{K}_m{}^i{}_{jh} - B^i{}_{ms}T^s{}_{jh} + \mathcal{A}_{jh}\left\{B^s{}_{mj}B^i{}_{sh} - B^i{}_{mj|h}\right\}, \tag{2.21}$$

$$\underset{(\alpha)}{\bar{\mathcal{P}}}{}_m{}^i{}_{jh} = \underset{(\alpha)}{\mathcal{P}}{}_m{}^i{}_{jh} + \underset{(\alpha)}{D}{}^i{}_{ms}T^s_{jh} - B^i{}_{ms}\underset{(\alpha)}{S}{}^s{}_{jh} + \mathcal{A}_{jh}\left\{-B^i_{mj}\overset{(\alpha)}{|}{}_h + \underset{(\alpha)}{D}{}^i{}_{mh|j} + \right.$$
$$\left. + B^s{}_{mj}\underset{(\alpha)}{D}{}^i{}_{sh} - \underset{(\alpha)}{D}{}^s{}_{mh}B^i_{sj}\right\},\ \ (\alpha = 1,\ldots,k-1), \tag{2.22}$$

$$\underset{(\alpha)}{\bar{\mathcal{P}}'}{}_m{}^i{}_{jh} = \underset{(\alpha)}{\mathcal{P}'}{}_m{}^i{}_{jh} - B^i_{mj}\overset{(\alpha)}{|}{}_h + \underset{(\alpha)}{D}{}^i{}_{mh|j} - B^i{}_{ms}\underset{(\alpha)}{C}{}^s{}_{jh} - \underset{(\alpha)}{D}{}^i{}_{ms}H^s_{jh} +$$
$$+ B^s{}_{mj}\underset{(\alpha)}{D}{}^i{}_{sh} - \underset{(\alpha)}{D}{}^s{}_{mh}B^i_{sj},\ \ (\alpha = 1,\ldots,k-1), \tag{2.23}$$

$$\bar{\mathcal{P}}_m{}^i{}_j{}^h = \mathcal{P}_m{}^i{}_j{}^h + \mathcal{A}_{jh}\Big\{-B^i_{mj}|^h + D^{ih}{}_{m|j} - B^i{}_{ms}C^{sh}_j - D^{is}_m H^h_{sj} + \\ + B^s_{mj}D^{ih}_s - D^{sh}_m B^i_{sj}\Big\}, \quad (\alpha = 1, ..., k-1), \tag{2.24}$$

$$\bar{\mathcal{P}}'_m{}^i{}_j{}^h = \mathcal{P}'_m{}^i{}_j{}^h - B^i_{mj}\Big|^h + D_m{}^{ih}{}_{|j} - B^i{}_{ms}C^{sh}_j - D^{is}_m H^h_{sj} + B^s_{mj}D^{ih}_s - D^{sh}_m B^i_{sj}, \tag{2.25}$$

$$\underset{(\alpha\beta)}{\bar{\mathcal{S}}_m}{}^i{}_{jh} = \underset{(\alpha\beta)}{\mathcal{S}_m}{}^i{}_{jh} - \underset{(\alpha)}{D}{}^i{}_{ms}\underset{(\beta)}{S}{}^s{}_{jh} - \underset{(\beta)}{D}{}^i{}_{ms}\underset{(\alpha)}{S}{}^s{}_{jh} + \mathcal{A}_{jh}\Big\{-\underset{(\alpha)}{D}{}^i{}_{mj}\overset{(\beta)}{|}{}_h + \underset{(\beta)}{D}{}^i{}_{mh}\overset{(\alpha)}{|}{}_j + \\ + \underset{(\alpha)}{D}{}^s{}_{mj}\underset{(\beta)}{D}{}^i{}_{sh} - \underset{(\beta)}{D}{}^s{}_{mh}\underset{(\alpha)}{D}{}^i{}_{sj}\Big\}, \quad (\alpha \le \beta, \alpha, \beta = 1, ..., k-1), \tag{2.26}$$

$$\underset{(\alpha\beta)}{\bar{\mathcal{S}}'_m}{}^i{}_{jh} = \underset{(\alpha\beta)}{\mathcal{S}'_m}{}^i{}_{jh} - \underset{(\alpha)}{D}{}^i{}_{mj}\overset{(\beta)}{|}{}_h + \underset{(\beta)}{D}{}^i{}_{mh}\overset{(\alpha)}{|}{}_j - \underset{(\alpha)}{D}{}^i{}_{ms}\underset{(\beta)}{C}{}^s{}_{jh} + \underset{(\beta)}{D}{}^i{}_{ms}\underset{(\alpha)}{C}{}^s{}_{hj} + \\ + \underset{(\alpha)}{D}{}^s{}_{mj}\underset{(\beta)}{D}{}^i{}_{sh} - \underset{(\beta)}{D}{}^s{}_{mh}\underset{(\alpha)}{D}{}^i{}_{sj}, \quad (\alpha \le \beta, \alpha, \beta = 1, ..., k-1), \tag{2.27}$$

$$\underset{(\alpha)}{\bar{\mathcal{S}}_m}{}^i{}_j{}^h = \underset{(\alpha)}{\mathcal{S}_m}{}^i{}_j{}^h + \mathcal{A}_{jh}\Big\{-\underset{(\alpha)}{D}{}^i{}_{mj}|^h + D^{ih}_m\overset{(\alpha)}{|}{}_j - C^{sh}_j\underset{(\alpha)}{D}{}^i{}_{ms} - \underset{(\alpha)}{C}{}^h{}_{sj}D^{is}_m + \\ + \underset{(\alpha)}{D}{}^s{}_{mj}D^{ih}_s - D^{sh}_m\underset{(\alpha)}{D}{}^i{}_{sj}\Big\}, \quad (\alpha = 1, ..., k-1), \tag{2.28}$$

$$\underset{(\alpha)}{\bar{\mathcal{S}}'_m}{}^i{}_j{}^h = \underset{(\alpha)}{\mathcal{S}'_m}{}^i{}_j{}^h - \underset{(\alpha)}{D}{}^i{}_{mj}|^h + D^{ih}_m\overset{(\alpha)}{|}{}_j - C^{sh}_j\underset{(\alpha)}{D}{}^i{}_{ms} - \underset{(\alpha)}{C}{}^h{}_{sj}D^{is}_m + \\ + \underset{(\alpha)}{D}{}^s{}_{mj}D^{ih}_s - D^{sh}_m\underset{(\alpha)}{D}{}^i{}_{sj}, \quad (\alpha = 1, ..., k-1), \tag{2.29}$$

Proof. From (2.7) we get:

$$\mathcal{A}_{jh}\Big\{\underset{(\alpha)}{\bar{P}_m}{}^i{}_{jh}\Big\} = \mathcal{A}_{jh}\Big\{\underset{(\alpha)}{P_m}{}^i{}_{jh}\Big\} + \mathcal{A}_{jh}\Big\{\underset{(\alpha)}{D}{}^i{}_{ms}H^s_{hj} - B^i_{ms}\underset{(\alpha)}{C}{}^s{}_{jh}\Big\} + \\ + \mathcal{A}_{jh}\Big\{-B^i{}_{mj}\overset{(\alpha)}{|}{}_h + \underset{(\alpha)}{D}{}^i{}_{mh|j} + B^s_{mj}\underset{(\alpha)}{D}{}^i{}_{sh} - \underset{(\alpha)}{D}{}^s{}_{mh}B^i_{sj} - \underset{(1)}{D}{}^i{}_{ms}\underset{(\alpha1)}{B}{}^s{}_{jh} - ... - \\ - \underset{(k-1)}{D}{}^i{}_{ms}\underset{(\alpha,k-1)}{B}{}^s{}_{jh} - D^{is}_m\underset{(\alpha)}{B}{}_{sjh}\Big\}$$

Using (2.30),(6.3) -p.161,[5] and (2.1) we have:

$$\mathcal{A}_{jh}\left\{\underset{(\alpha)}{\bar{P}_m}{}^i{}_{jh}\right\}=\mathcal{A}_{jh}\left\{\underset{(\alpha)}{P_m}{}^i{}_{jh}\right\}-\underset{(\alpha)}{D}{}^i{}_{ms}T^s_{jh}-B^i_{ms}\underset{(\alpha)}{S}{}^s{}_{jh}+$$
$$+\mathcal{A}_{jh}\left\{-B^i{}_{mj}\overset{(\alpha)}{|}_h+\underset{(\alpha)}{D}{}^i{}_{mh|j}+B^s_{mj}\underset{(\alpha)}{D}{}^i{}_{sh}-\underset{(\alpha)}{D}{}^s{}_{mh}B^i_{sj}\right\}+$$
$$+\mathcal{A}_{jh}\left\{(\underset{(1)}{\bar{C}}{}^i{}_{ms}-\underset{(1)}{C}{}^i{}_{ms})\underset{(\alpha 1)}{B}{}^s{}_{jh}+...+(\underset{(k-1)}{\bar{C}}{}^i{}_{ms}-\underset{(k-1)}{C}{}^i{}_{ms})\underset{(\alpha,k-1)}{B}{}^s{}_{jh}+\right.$$
$$\left.+(\bar{C}^{is}_m-C^{is}_m)\underset{(\alpha)}{B}{}_{sjh}\right\}.$$

If we separate the terms and using (2.13) we get (2.22).
Analogous we obtain the other formulas. □

3 Metrical semisymmetric $N-$linear connections of the space $H^{(k)n}$

Let $H^{(k)n}=(M,H)$ be a Hamilton space of order k, and let N be the canonical nonlinear connection of space $H^{(k)n}=(M,H)$ $([5])-p.192)$.

We consider the adapted basis $\left\{\frac{\delta}{\delta x^i},\frac{\delta}{\delta y^{(1)i}},...,\frac{\delta}{\delta y^{(k-1)i}},\frac{\delta}{\delta p_i}\right\}$ and its dual basis $\left\{\delta x^i,\delta y^{(1)i},...,\delta y^{(k-1)i},\delta p_i\right\}$ determined by N and by the distribution W_k. Let

$$g^{ij}(x,y^{(1)},...,y^{(k-1)},p)=\frac{1}{2}\frac{\partial^2 H}{\partial p_i\partial p_j} \tag{3.1}$$

be the fundamental tensor of the space $H^{(k)n}$.

The d-tensor field g^{ij} being nonsingular on $\widetilde{T^{*k}M}$ there is a d-tensor field g_{ij} covariant of order 2, symmetric, uniquely determined, at every point $u\in\widetilde{T^{*k}M}$, by

$$g_{ij}g^{jk}=\delta^k_i. \tag{3.2}$$

Definition 3.1. ([5]) *An $N-$linear connection D is called compatible to the fundamental tensor g^{ij} of the Hamiltonian space of order k, $H^{(k)n}=(M,H)$, or it is metrical if g^{ij} is covariant constant (or absolutelyy parallel) with respect to D, i.e.*

$$g^{ij}{}_{|h}=0,\quad g^{ij}\overset{(\alpha)}{|}_h=0,\quad g^{ij}\,|^h=0,\ (\alpha=1,...,k-1). \tag{3.3}$$

The tensorial equations (3.3) imply:

$$g^{ij}{}_{|h}=0,\quad g^{ij}\overset{(\alpha)}{|}_h=0,\quad g^{ij}\,|^h=0,\ (\alpha=1,...,k-1). \tag{3.4}$$

Theorem 3.1. ([5]) *1. In a Hamilton space of order k, $H^{(k)n} = (M, H)$, there is a unique N-linear connection D, with the coefficients* $D\Gamma(N) = \left(H^i{}_{jh}, \underset{(1)}{C}{}^i{}_{jh}, ..., \underset{(k-1)}{C}{}^i{}_{jh}, C_i{}^{jh} \right)$ *verifying the axioms:*

1°. N is the canonical nonlinear connection of $H^{(k)n}$.

2°. The fundamental tensor g^{ij} is $h-, v_\alpha-$, and w_k- covariant constant: $g^{ij}{}_{|h} = 0, \quad g^{ij} \overset{(\alpha)}{|}_h = 0, \quad g^{ij} |^h = 0.$

3°. $D\Gamma(N)$ is $h-, v_\alpha-$, and w_k- torsion free $T^i{}_{jh} = \underset{(\alpha)}{S}{}^i{}_{jh} = S_i{}^{jh} = 0.$ $(\alpha = 1, ..., k-1)$

2. The connection $D\Gamma(N)$ has the coefficients given by the generalized Christoffel symbols:

$$\begin{aligned} &H^i_{jh} = \tfrac{1}{2} g^{is} \left(\tfrac{\delta g_{sh}}{\delta x^j} + \tfrac{\delta g_{js}}{\delta x^h} - \tfrac{\delta g_{jh}}{\delta x^s} \right), \\ &\underset{(\alpha)}{C}{}^i{}_{jh} = \tfrac{1}{2} g^{is} \left(\tfrac{\delta g_{sh}}{\delta y^{(\alpha)j}} + \tfrac{\delta g_{js}}{\delta y^{(\alpha)h}} - \tfrac{\delta g_{jh}}{\delta y^{(\alpha)s}} \right), \quad (\alpha = 1, ..., k-1), \\ &C_i{}^{jh} = -\tfrac{1}{2} g_{is} \left(\tfrac{\partial g^{sh}}{\partial p_j} + \tfrac{\partial g^{js}}{\partial p_h} - \tfrac{\partial g^{jh}}{\partial p_s} \right), \end{aligned} \tag{3.5}$$

3. This connection depends only on the fundamental function H of the space $H^{(k)n}$ and on the canonical nonlinear connection N.

Definition 3.2. *([5]) The connection $D\Gamma(N)$ with the coefficients (3.5) will be denoted by $C\Gamma(N)$ and called canonical for the space $H^{(k)n}$.*

The canonical metrical N-linear connection: $C\Gamma(N)$ has zero torsions $T^i_{jh}, \underset{(\alpha)}{S}{}^i{}_{jh}, S_i^{jh}$, $(\alpha = 1, ..., k-1)$.

The Obata's operators, are given by:

$$\Omega^{ij}_{hk} = \frac{1}{2} \left(\delta^i_h \delta^j_k - g_{hk} g^{ij} \right), \; \Omega^{*ij}_{\;\;hk} = \frac{1}{2} \left(\delta^i_h \delta^j_k + g_{hk} g^{ij} \right). \tag{3.6}$$

There is inferred:

Proposition 3.1. *The Obata's operators have the following properties:*

$$\Omega^{ir}_{sj} + \Omega^{*ir}_{\;\;sj} = \delta^i_s \delta^r_j, \tag{3.7}$$

$$\Omega^{ir}_{sj} \Omega^{sn}_{mr} = \Omega^{in}_{mj}, \; \Omega^{*ir}_{\;\;sj} \Omega^{*sn}_{\;\;mr} = \Omega^{*in}_{\;\;mj}, \; \Omega^{ir}_{sj} \Omega^{*sn}_{\;\;mr} = \Omega^{*ir}_{\;\;sj} \Omega^{sn}_{mr} = 0, \tag{3.8}$$

$$\Omega^{ir}_{rj} = \Omega^{ir}_{si} = 0, \; \Omega^{ir}_{ij} = \frac{1}{2} (n-1) \delta^r_j, \; \Omega^{*ir}_{\;\;ij} = \frac{1}{2} (n+1) \delta^r_j. \tag{3.9}$$

Theorem 3.2. ([4], [5]) *There is a unique metrical connection $\bar{D}\Gamma(N)$* $= \left(\bar{H}^i{}_{jh}, \underset{(\alpha)}{\bar{C}}{}^i{}_{jh}, \bar{C}_i{}^{jh} \right)$, $(\alpha = 1, ..., k-1)$ *metrical with respect to the fundamental tensor g^{ij} of space $H^{(k)n}$ having as torsion d− tensor fields $T^i{}_{jh}, \underset{(\alpha)}{S}{}^i{}_{jh}, S_i{}^{jh}$ apriori given.*

The coefficients of $\bar{D}\Gamma(N)$ *have the following expressions:*

$$\begin{array}{l}\bar{H}^i{}_{jh}=\frac{1}{2}g^{is}\left(\frac{\delta g_{sh}}{\delta x^j}+\frac{\delta g_{js}}{\delta x^h}-\frac{\delta g_{jh}}{\delta x^s}\right)+\\+\frac{1}{2}g^{is}\left(g_{sm}T^m{}_{jh}-g_{jm}T^m{}_{sh}+g_{hm}T^m{}_{js}\right),\\ \underset{(\alpha)}{\bar{C}}{}^i{}_{jh}=\frac{1}{2}g^{is}\left(\frac{\delta g_{sh}}{\delta y^{(\alpha)j}}+\frac{\delta g_{js}}{\delta y^{(\alpha)h}}-\frac{\delta g_{jh}}{\delta y^{(\alpha)s}}\right)+\\+\frac{1}{2}g^{is}\left(g_{sm}\underset{(\alpha)}{S}{}^m{}_{jh}-g_{jm}\underset{(\alpha)}{S}{}^m{}_{sh}+g_{hm}\underset{(\alpha)}{S}{}^m{}_{js}\right),\\ \bar{C}_i{}^{jh}=-\frac{1}{2}g_{is}\left(\frac{\partial g^{sh}}{\partial p_j}+\frac{\partial g^{js}}{\partial p_h}-\frac{\partial g^{jh}}{\partial p_s}\right)-\\-\frac{1}{2}g_{is}\left(g^{sm}S_m{}^{jh}-g^{jm}S_m{}^{sh}+g^{hm}S_m{}^{js}\right).\end{array}\tag{3.10}$$

A class of metrical N-linear connections, which has interesting properties is that of metrical semisymmetric N-linear connection.

Definition 3.3. ([1]) An $N-$linear connection on $T^{*(k)}M$ is called semisymmetric if:

$$T^i{}_{jh}=\frac{1}{2}\left(-\delta^i_j\sigma_h+\delta^i_h\sigma_j\right),\ \underset{(\alpha)}{S}{}^i{}_{jh}=\frac{1}{2}\left(-\delta^i_j\underset{(\alpha)}{\tau}{}_h+\delta^i_h\underset{(\alpha)}{\tau}{}_j\right),\ S_i{}^{jh}$$
$$=-\frac{1}{2}\left(-\delta^j_i v^h+\delta^h_i v^j\right),\ (\alpha=1,...,k-1),\tag{3.11}$$

where $\sigma,\ \underset{(\alpha)}{\tau}\in\chi^*\left(T^{*k}M\right)$ and $v\in\chi\left(T^{*k}M\right)$.

Theorem 3.3. *The set of all metrical semisymmetric $N-$linear connections with local coefficients* $D\Gamma(N)=\left(H^i{}_{jh},\underset{(1)}{C}{}^i{}_{jh},...,\underset{(k-1)}{C}{}^i{}_{jh},C_i{}^{jh}\right)$ *is given by:*

$$\begin{cases}H^i{}_{jh}=\overset{0}{H}{}^i{}_{jh}+\frac{1}{2}\left(-g_{jh}g^{is}\sigma_s+\sigma_j\delta^i_h\right),\\ \underset{(\alpha)}{C}{}^i{}_{jh}=\underset{(\alpha)}{\overset{0}{C}}{}^i{}_{jh}+\frac{1}{2}\left(-g_{jh}g^{is}\underset{(\alpha)}{\tau}{}_s+\underset{(\alpha)}{\tau}{}_j\delta^i_h\right),(\alpha=1,...,k-1),\\ C_i{}^{jh}=\overset{0}{C}{}_i{}^{jh}+\frac{1}{2}\left(-g^{jh}g_{is}v^s+v^j\delta^h_i\right),\end{cases}\tag{3.12}$$

where $C\Gamma(N)=\left(\overset{0}{H}{}^i{}_{jh},\underset{(1)}{\overset{0}{C}}{}^i{}_{jh},...,\underset{(k-1)}{\overset{0}{C}}{}^i{}_{jh},\overset{0}{C}{}_i{}^{jh}\right)$ *are the local coefficients of the canonical metrical $N-$linear connection and* $\sigma,\ \underset{(\alpha)}{\tau}\in\chi^*\left(T^{*k}M\right),(\alpha=1,...,k-1)$ *and* $v\in\chi\left(T^{*k}M\right)$.

Proof. Using Theorem 3.2 and Definition 3.3 we obtain the results by direct calculation. □

4 The group of transformations of metrical semisymmetric $N-$linear connections

Let N be a given nonlinear connection on $T^{*k}M$.

Remark 4.1. *The relations* (3.12) *give the transformations of metrical semisymmetric $N-$linear connections of space $H^{(k)n}$, corresponding to the same nonlinear connection N. We shall denote a transformation of this form by:* $t(\sigma, \underset{(\alpha)}{\tau}, v)$: $D\Gamma(N) \longrightarrow \bar{D}\Gamma(N)$, $(\alpha = 1, ..., k-1)$. *It is given by:*

$$\begin{cases} \bar{H}^i{}_{jh} = H^i{}_{jh} + \frac{1}{2}\left(-g_{jh}g^{is}\sigma_s + \sigma_j\delta^i_h\right), \\ \underset{(\alpha)}{\bar{C}}{}^i{}_{jh} = \underset{(\alpha)}{C}{}^i{}_{jh} + \frac{1}{2}\left(-g_{jh}g^{is}\underset{(\alpha)}{\tau}{}_s + \underset{(\alpha)}{\tau}{}_j\delta^i_h\right), (\alpha = 1, ..., k-1), \\ \bar{C}_i^{jh} = C_i^{jh} + \frac{1}{2}\left(-g^{jh}g_{is}v^s + v^j\delta^h_i\right). \end{cases} \tag{4.1}$$

Thus we have:

Theorem 4.1. *The set* $\overset{ms}{\mathcal{T}}_N$ *of all transformations* $t(\sigma, \underset{(\alpha)}{\tau}, v)$: $D\Gamma(N) \longrightarrow \bar{D}\Gamma(N)$,
$(\alpha = 1, ..., k-1)$, *of metrical semisymmetric $N-$linear connections, given by* (4.1) *is an Abelian group, together with the mapping product.*

This group acts on the set of all metrical semisymmetric $N-$linear connections, corresponding to the same nonlinear connection N, transitively.

Theorem 4.2. *By means of a transformation* (4.1), *the tensor fields:*
$\mathcal{K}_m{}^i{}_{jh}, \underset{(\alpha)}{\mathcal{P}_m}{}^i{}_{jh}, (\alpha = 1, ..., k-1), \mathcal{P}_m{}^i{}_j{}^h, \underset{(\alpha\beta)}{\mathcal{S}_m}{}^i{}_{jh}, (\alpha \leq \beta, \alpha, \beta = 1, ..., k-1), \mathcal{S}_m{}^{ijh}$
given in (2.12), (2.13), (2.15), (2.17) *and* (2.11) *are changed by the laws:*

$$\bar{\mathcal{K}}_m{}^i{}_{jh} = \mathcal{K}_m{}^i{}_{jh} + \mathcal{A}_{jh}\left\{\Omega^{ir}_{jm}\sigma_{rh}\right\}, \tag{4.2}$$

$$\underset{(\alpha)}{\bar{\mathcal{P}}_m}{}^i{}_{jh} = \underset{(\alpha)}{\mathcal{P}_m}{}^i{}_{jh} + \mathcal{A}_{jh}\left\{\Omega^{ir}_{jm}\underset{(\alpha)}{\gamma}{}_{rh}\right\}, (\alpha = 1, ..., k-1) \tag{4.3}$$

$$\bar{\mathcal{P}}_m{}^i{}_j{}^h = \mathcal{P}_m{}^i{}_j{}^h + \mathcal{A}_{jh}\left\{\Omega^{ir}_{jm}\sigma_r\,|^h + \Omega^{ij}_{rm}v^r{}_{|h} + \Omega^{is}_{rm}\left(H^h{}_{sj}v^r + C_j{}^{rh}\sigma_s\right) + \right. \tag{4.4}$$
$$\left. + \frac{1}{2}\Omega^{ij}_{hm}\sigma_r v^r + \frac{1}{4}\delta^h_m g_{jr}g^{is}\sigma_s v^r + \frac{1}{4}\delta^i_h g_{rm}g^{js}\sigma_s v^r - \frac{1}{4}g_{js}g^{ih}\sigma_m v^s - \frac{1}{4}g_{jm}g^{rh}\sigma_r v^i\right\},$$

$$\underset{(\alpha\beta)}{\bar{\mathcal{S}}_m}{}^i{}_{jh} = \underset{(\alpha\beta)}{\mathcal{S}_m}{}^i{}_{jh} + \mathcal{A}_{jh}\left\{\Omega^{ir}_{jm}\underset{(\alpha\beta)}{\tau}{}_{rh}\right\}, (\alpha \leq \beta, \alpha, \beta = 1, ..., k-1), \tag{4.5}$$

$$\bar{S}_m{}^{ijh} = S_h{}^{ijh} + \mathcal{A}_{jh}\left\{\Omega^{ij}_{rm}v^{rh}\right\}, \tag{4.6}$$

where:

$$\sigma_{rh} = -\sigma_r\sigma_h + \sigma_{r|h} + \frac{1}{4}g_{rh}\cdot\sigma, (\sigma = g^{rs}\sigma_r\sigma_s), \tag{4.7}$$

$$\begin{gathered} \underset{(\alpha)}{\gamma}{}_{rh} = -\left(\sigma_r\underset{(\alpha)}{\tau}{}_h + \sigma_h\underset{(\alpha)}{\tau}{}_r\right) + \sigma_r\overset{(\alpha)}{|}{}_h + \underset{(\alpha)}{\tau}{}_{r|h} + \tfrac{1}{4}g_{rh}\underset{(\alpha)}{\gamma}, \\ \left(\underset{(\alpha)}{\gamma} = g^{rs}\left(\sigma_r\underset{(\alpha)}{\tau}{}_s + \sigma_s\underset{(\alpha)}{\tau}{}_r\right)\right), (\alpha = 1, ..., k-1), \end{gathered} \tag{4.8}$$

$$\underset{(\alpha\beta)}{\tau}{}_{rh} = -\left(\underset{(\alpha)}{\tau}{}_r \underset{(\beta)}{\tau}{}_h + \underset{(\alpha)}{\tau}{}_h \underset{(\beta)}{\tau}{}_r\right) + \underset{(\alpha)}{\tau}{}_r \overset{(\beta)}{|}{}_h + \underset{(\beta)}{\tau}{}_r \overset{(\alpha)}{|}{}_h + \tfrac{1}{4} g_{rh} \underset{(\alpha\beta)}{\tau},$$
$$\left(\underset{(\alpha\beta)}{\tau} = g^{rs}\left(\underset{(\alpha)}{\tau}{}_r \underset{(\beta)}{\tau}{}_s + \underset{(\alpha)}{\tau}{}_s \underset{(\beta)}{\tau}{}_r\right)\right), \ (\alpha \leq \beta, \ \alpha, \beta = 1, ..., k-1), \tag{4.9}$$

$$v^{rh} = v^r v^h + v^r \,|^h - \frac{1}{4} g^{rh} v, \ (v = g_{rs} v^r v^s). \tag{4.10}$$

Proof. Using (2.1) and (4.1) we get:

$$B^i{}_{jh} = \frac{1}{2}\left(-\sigma_j \delta^i_h + g_{jh} g^{is} \sigma_s\right) = -\Omega^{is}_{hj} \sigma_s, \ \underset{(\alpha)}{D}{}^i{}_{jh} = \frac{1}{2}\left(-\underset{(\alpha)}{\tau}{}_j \delta^i_h + g_{jh} g^{is} \underset{(\alpha)}{\tau}{}_s\right) =$$
$$-\Omega^{is}_{hj} \underset{(\alpha)}{\tau}{}_s, \ (\alpha = 1, ..., k-1), D_i^{jh} = \frac{1}{2}\left(-v^j \delta^h_i + g^{jh} g_{is} v^s\right) = -\Omega^{jh}_{si} v^s. \tag{4.11}$$

By applying Proposition 2.3, relations (3.11) and (4.11) we obtain the results. □

Using these results we can determine some invariants of group $\overset{ms}{\mathcal{T}}_N$. To this end we eliminate σ_{ij}, $\underset{(\alpha)}{\gamma}{}_{ij}$, $\underset{(\alpha\beta)}{\tau}{}_{ij}$ and v^{ij} from (4.2), (4.3), (4.5) and (4.6) and we obtain:

Theorem 4.3. *For $n > 2$ the following tensor fields: $H_m{}^i{}_{jh}$, $\underset{(\alpha)}{N}{}_m{}^i{}_{jh}$, $(\alpha = 1, ..., k-1)$, $\underset{(\alpha\beta)}{M}{}_m{}^i{}_{jh}$, $M_m{}^{ijh}$, are invariants of group $\overset{ms}{\mathcal{T}}_N$:*

$$H_m{}^i{}_{jh} = \mathcal{K}_m{}^i{}_{jh} + \frac{2}{n-2} \mathcal{A}_{jh} \left\{\Omega^{ir}_{jm}\left(\mathcal{K}_{rh} - \frac{g_{rh}\mathcal{K}}{2(n-1)}\right)\right\}, \tag{4.12}$$

$$\underset{(\alpha)}{N}{}_m{}^i{}_{jh} = \underset{(\alpha)}{\mathcal{P}}{}_m{}^i{}_{jh} + \frac{2}{n-2} \mathcal{A}_{jh} \left\{\Omega^{ir}_{jm}\left(\underset{(\alpha)}{\mathcal{P}}{}_{rh} - \frac{g_{rh}\underset{(\alpha)}{\mathcal{P}}}{2(n-1)}\right)\right\}, \ (\alpha = \overline{1, k-1}) \tag{4.13}$$

$$\underset{(\alpha\beta)}{M}{}_m{}^i{}_{jh} = \underset{(\alpha\beta)}{\mathcal{S}}{}_m{}^i{}_{jh} + \frac{2}{n-2} \mathcal{A}_{jh} \left\{\Omega^{ir}_{jm}\left(\underset{(\alpha\beta)}{\mathcal{S}}{}_{rh} - \frac{g_{rh}\underset{(\alpha\beta)}{\mathcal{S}}}{2(n-1)}\right)\right\}, (\alpha \leq \beta; \alpha, \beta = \overline{1, k-1}) \tag{4.14}$$

$$M_m{}^{ijh} = S_m{}^{ijh} + \frac{2}{n-2} \mathcal{A}_{jh} \left\{\Omega^{ij}_{rm}\left(S^{rh} - \frac{g^{rh} S'}{2(n-1)}\right)\right\}, \tag{4.15}$$

where: $\mathcal{K}_{mj} = \mathcal{K}_m{}^i{}_{ji}, \mathcal{K} = g^{mj}\mathcal{K}_{mj}, \underset{(\alpha)}{\mathcal{P}}{}_{mj} = \underset{(\alpha)}{\mathcal{P}}{}_m{}^i{}_{ji}, \underset{(\alpha)}{\mathcal{P}} = g^{mj} \underset{(\alpha)}{\mathcal{P}}{}_{mj}, \ (\alpha = \overline{1, k-1})$ $\underset{(\alpha\beta)}{\mathcal{S}}{}_{mj} = \underset{(\alpha\beta)}{\mathcal{S}}{}_m{}^i{}_{ji}, \underset{(\alpha\beta)}{\mathcal{S}} = g^{mj} \underset{(\alpha\beta)}{\mathcal{S}}{}_{mj}, (\alpha \leq \beta, \alpha, \beta = \overline{1, k-1}), S^{ij} = S_m{}^{ijm}, S = g_{ij} S^{ij}$.

In order to find other invariants of group $\overset{ms}{\mathcal{T}}_N$, let us consider the transformation formulas of the torsion $d-$tensor fields by a transformation $t(\sigma, \underset{(\alpha)}{\tau}, v)$: $D\Gamma(N) \longrightarrow \bar{D}\Gamma(N)$,
$(\alpha = 1, ..., k-1)$, of metrical semisymmetric $N-$linear connections of the space $H^{(k)n}$ corresponding to the same nonlinear connection N, given by (4.1).

Proposition 4.1. *By a transformation* (4.1) *of metrical semisymmetric* $N-$ *linear connections, corresponding to the same nonlinear connection* N : $t(\sigma, \underset{(\alpha)}{\tau}, v)$: $D\Gamma(N) \longrightarrow \bar{D}\Gamma(N)$, $(\alpha = 1, ..., k-1)$, *the torsion tensor fields:* $\underset{(01)}{r}{}^i{}_{jh}$, $\underset{(\alpha\beta)}{B}{}^i{}_{jh}$, $\underset{(\alpha)}{B}{}_{ijh}$, $\underset{(0)}{B}{}^h{}_{ij}$, $\underset{(\alpha\beta)}{B}{}_{ijh}$, $\underset{(\alpha)}{B}{}_i{}^{jh}$, $\underset{(0\alpha)}{R}{}^i{}_{jh}$, $T^i{}_{jh}$, $\underset{(0)}{R}{}_{ijh}$, $\underset{(\alpha)}{S}{}^i{}_{jh}$, $S_i{}^{jh}$, $\underset{(\alpha\beta)}{\overset{(\gamma)}{C}}{}^i{}_{jh}$, $\underset{(\alpha 1)}{C}{}_i{}^{jh}$,..., $\underset{(\alpha,k-1)}{C}{}_i{}^{jh}$, $(\alpha \le \beta, \alpha, \beta, \gamma = 1, ..., k-1)$ *are transformed as follows:*

$$\begin{cases} \underset{(0\alpha)}{\bar{r}}{}^i{}_{jh} = \underset{(0\alpha)}{r}{}^i{}_{jh}, \underset{(\alpha\beta)}{\bar{B}}{}^i{}_{jh} = \underset{(\alpha\beta)}{B}{}^i{}_{jh}, \underset{(0)}{\bar{B}}{}^h{}_{ij} = \underset{(0)}{B}{}^h{}_{ij}, \underset{(\alpha)}{\bar{B}}{}_{ijh} = \underset{(\alpha)}{B}{}_{ijh} \\ \underset{(\alpha\beta)}{\bar{B}}{}_{ijh} = \underset{(\alpha\beta)}{B}{}_{ijh}, \underset{(\alpha)}{\bar{B}}{}_i{}^{jh} = \underset{(\alpha)}{B}{}_i{}^{jh}, (\alpha \le \beta, \alpha, \beta = 1, ..., k-1), \\ \underset{(0\alpha)}{\bar{R}}{}^i{}_{jh} = \underset{(0\alpha)}{R}{}^i{}_{jh}, (\alpha = 1, ..., k-1), \bar{R}_{ijh} = R_{ijh}, \\ \underset{(\alpha\beta)}{\overset{(\bar{\gamma})}{C}}{}^i{}_{jh} = \underset{(\alpha\beta)}{\overset{(\gamma)}{C}}{}^i{}_{jh}, (\alpha \le \beta, \alpha, \beta = 1, ..., k-1), \\ \underset{(\alpha 1)}{\bar{C}}{}_i{}^{jh} = \underset{(\alpha 1)}{C}{}_i{}^{jh}, ..., \underset{(\alpha,k-1)}{\bar{C}}{}_i{}^{jh} = \underset{(\alpha,k-1)}{C}{}_i{}^{jh}, (\alpha = 1, ..., k-1), \\ \bar{T}^i{}_{jh} = T^i{}_{jh} + \frac{1}{2}\mathcal{A}_{jh}\left\{\sigma_j \delta^i_h\right\}, \\ \underset{(\alpha)}{\bar{S}}{}^i{}_{jh} = \underset{(\alpha)}{S}{}^i{}_{jh} + \frac{1}{2}\mathcal{A}_{jh}\left\{\underset{(\alpha)}{\tau}{}_i \delta^i_h\right\}, \\ \bar{S}_i{}^{jh} = S_i{}^{jh} + \frac{1}{2}\mathcal{A}_{jh}\left\{v^j \delta^h_i\right\} \end{cases} \tag{4.16}$$

We denote with:

$$\underset{(\alpha\beta)}{t}{}^i{}_{jh} = \mathcal{A}_{jh}\left\{\frac{\delta \underset{(\alpha)}{N}{}^i{}_j}{\delta y^{(\beta)h}}\right\}, (\alpha, \beta = 1, ..., k-1),$$
$$t^i_{jh} = \mathcal{A}_{jh}\left\{\frac{\delta N_{jh}}{\delta p_i}\right\}, \underset{(\alpha)}{t}{}_j{}^{ih} = \mathcal{A}_{jh}\left\{\frac{\delta \underset{(\alpha)}{N}{}^i{}_j}{\delta p_h}\right\}, (\alpha = 1, ..., k-1), \tag{4.17}$$

$$\left\{\begin{array}{l}
\underset{(\alpha\beta)}{t}{}^{*}{}_{ijh}=\Sigma_{ijh}\left\{g_{im}\underset{(\alpha\beta)}{t}{}^{m}{}_{jh}\right\},\ \underset{(\alpha)}{t}{}^{*h}{}_{ij}=\Sigma_{ijh}\left\{g_{im}\underset{(\alpha)}{t}{}_{j}{}^{mh}\right\},\\
t^{*}{}_{ijh}=\Sigma_{ijh}\left\{g_{im}t^{m}{}_{jh}\right\},(\alpha\leq\beta,\alpha,\beta=1,...,k-1),\\
\underset{(0\alpha)}{r}{}^{*}{}_{ijh}=\Sigma_{ijh}\left\{g_{im}\underset{(0\alpha)}{r}{}^{m}{}_{jh}\right\},(\alpha=1,...,k-1),\\
T^{*}{}_{ijh}=\Sigma_{ijh}\left\{g_{im}T^{m}{}_{jh}\right\},\\
\underset{(0\alpha)}{R}{}^{*}{}_{ijh}=\Sigma_{ijh}\left\{g_{im}\underset{(0\alpha)}{R}{}^{m}{}_{jh}\right\},(\alpha=1,...,k-1),R^{*i}_{jh}=\Sigma_{ijh}\left\{g^{im}R_{mjh}\right\},\\
\overset{(\gamma)}{\underset{(\alpha\beta)}{C}}{}^{*}{}_{ijh}=\Sigma_{ijh}\left\{g^{im}\overset{(\gamma)}{\underset{(\alpha\beta)}{C}}{}^{m}{}_{jh}\right\},(\alpha\leq\beta,\alpha,\beta=1,...,k-1),
\end{array}\right. \tag{4.18}$$

$$\left\{\begin{array}{l}
\underset{(\alpha 1)}{C}{}^{*h}_{\ ij}=\Sigma_{ijh}\left\{g_{im}\underset{(\alpha 1)}{C_{j}}{}^{mh}\right\},...,\underset{(\alpha,k-1)}{C}{}^{*h}_{\ ij}=\Sigma_{ijh}\left\{g_{im}\underset{(\alpha,k-1)}{C_{j}}{}^{mh}\right\},(\alpha=\overline{1,k-1}),\\
\underset{(\alpha\beta)}{B}{}^{*}{}_{ijh}=\Sigma_{ijh}\left\{g_{im}\underset{(\alpha\beta)}{B}{}^{m}{}_{jh}\right\},(\alpha\leq\beta,\alpha,\beta=\overline{1,k-1}),\\
\overset{(1)}{\underset{(\alpha)}{B}}{}^{*i}_{\ jh}=\Sigma_{ijh}\left\{g^{im}\underset{(\alpha)}{B}{}_{mjh}\right\},(\alpha=\overline{1,k-1}),\\
\underset{(0)}{B}{}^{*}{}_{ijh}=\Sigma_{ijh}\left\{g_{im}\underset{(0)}{B}{}^{m}{}_{jh}\right\},\ \underset{(\alpha\beta)}{B}{}^{*i}_{\ jh}=\Sigma_{ijh}\left\{g^{im}\underset{(\alpha\beta)}{B}{}_{mjh}\right\}(\alpha\leq\beta,\alpha,\beta=\overline{1,k-1}),\\
\overset{(2)}{\underset{(\alpha)}{B}}{}^{*h}_{\ ij}=\Sigma_{ijh}\left\{g_{im}\underset{(\alpha)}{B_{j}}{}^{mh}\right\},(\alpha=1,...,k-1),\\
\underset{(\alpha)}{S}{}^{*}{}_{ijh}=\Sigma_{ijh}\left\{g_{im}\underset{(\alpha)}{S}{}^{m}{}_{jh}\right\},(\alpha=\overline{1,k-1}),S^{*ijh}=\Sigma_{ijh}\left\{g^{im}S_{m}{}^{jh}\right\},\\
H^{*}{}_{ijh}=\Sigma_{ijh}\left\{\mathcal{A}_{jh}\left\{g_{jm}H^{m}{}_{jh}\right\}\right\},\\
\underset{(\alpha)}{C}{}^{*}{}_{ijh}=\Sigma_{ijh}\left\{\mathcal{A}_{jh}\left\{g_{jm}\underset{(\alpha)}{C}{}^{m}{}_{ih}\right\}\right\},(\alpha=\overline{1,k-1}),\\
C^{*ijh}=\Sigma_{ijh}\left\{\mathcal{A}_{jh}\left\{g^{jm}C_{m}{}^{ih}\right\}\right\},
\end{array}\right.$$

where $\Sigma_{ijh}\{...\}$ *denotes the cyclic summation, and with:*

$$\begin{cases} \overset{(1)}{K}_{ijh} = g_{im}T^{m}{}_{jh} - \mathcal{A}_{jh}\left\{g_{hm}H^{m}{}_{ij}\right\}, \\ \underset{(\alpha)}{\overset{(2)}{K}}_{ijh} = g_{im}\underset{(\alpha)}{S}{}^{m}{}_{jh} - \mathcal{A}_{jh}\left\{g_{hm}\underset{(\alpha)}{C}{}^{m}_{ij}\right\}, (\alpha = 1, ..., k-1), \\ \underset{(\alpha\beta)}{\overset{\overset{(2)}{(\gamma)}}{K}}{}_{ijh} = \mathcal{A}_{jh}\left\{g_{hm}\underset{(\alpha\beta)}{\overset{(\gamma)}{C}}{}^{m}_{ij}\right\}, \\ \underset{(\alpha\beta)}{\overset{\overset{(3)}{(\gamma)}}{K}}{}_{ijh} = g_{mj}\underset{(\alpha\beta)}{\overset{(\gamma)}{C}}{}^{m}_{ih} + g_{im}\underset{(\alpha\beta)}{\overset{(\gamma)}{C}}{}^{m}_{jh}, (\alpha \le \beta, \alpha, \beta, \gamma = 1, ..., k-1), \\ \overset{(3)}{K}{}^{ijh} = g^{im}S_{m}{}^{jh} - \mathcal{A}_{jh}\left\{g^{hm}C^{ij}_{m}\right\}, \\ \underset{(\alpha\beta)}{\overset{(1)}{\mathcal{S}}}{}_{ijh} = \mathcal{A}_{ij}\left\{g_{im}\underset{(\alpha\beta)}{B}{}^{m}_{jh}\right\}, \\ \underset{(\alpha\beta)}{\overset{(2)}{\mathcal{S}}}{}_{ijh} = \mathcal{A}_{jh}\left\{g_{mj}\mathcal{A}_{ih}\left\{\underset{(\alpha\beta)}{B}{}^{m}_{ih}\right\}\right\}, (\alpha \le \beta, \alpha, \beta, \gamma = 1, ..., k-1). \end{cases} \tag{4.19}$$

Remark 4.2. It is noted that $\underset{(\alpha\beta)}{t}{}^{*}{}_{ijh}$, $\underset{(\alpha)}{t}{}^{*}{}_{ij}{}^{h}, t^{*}{}_{ijh}, \underset{(0\alpha)}{r}{}^{*}{}_{ijh}, T^{*}{}_{ijh}, R^{*i}_{jh}, \underset{(\alpha)}{S}{}^{*}{}_{ijh}, S^{*ijh}$ are alternate $\overset{(1)}{K}_{ijh}, \underset{(\alpha)}{\overset{(2)}{K}}_{ijh}, \underset{(\alpha\beta)}{\overset{\overset{(2)}{(\gamma)}}{K}}{}_{ijh}, \overset{(3)}{K}{}^{ijh}, \underset{(\alpha\beta)}{\overset{(2)}{\mathcal{S}}}{}_{ijh}, H^{*}{}_{ijh}, \underset{(\alpha)}{C}{}^{*}{}_{ijh}, C^{*ijh}$ are alternate with respect to: j, h, and $\underset{(\alpha\beta)}{\overset{(1)}{\mathcal{S}}}{}_{ijh}$ are alternate with respect to i, j.

Theorem 4.4. *The tensor fields:* $\underset{(0\alpha)}{r}{}^{i}{}_{jh}$, $\underset{(\alpha\beta)}{B}{}^{i}{}_{jh}$, $\underset{(\alpha)}{B}{}_{ijh}$, $\underset{(0)}{B}{}^{h}{}_{ij}$, $\underset{(\alpha\beta)}{B}{}_{ijh}$, $\underset{(\alpha)}{B}{}_{i}{}^{jh}$, $\underset{(0\alpha)}{R}{}^{i}{}_{jh}$, R_{ijh}, $\underset{(\alpha\beta)}{\overset{(\gamma)}{C}}{}^{i}{}_{jh}$, $\underset{(\alpha1)}{C}{}_{i}{}^{jh}, \ldots, \underset{(\alpha,k-1)}{C}{}_{i}{}^{jh}$, $\underset{(\alpha\beta)}{t}{}^{i}{}_{jh}, t^{i}_{jh}, \underset{(\alpha)}{t}{}^{ih}{}_{j}$, $\underset{(\alpha\beta)}{t}{}^{*}{}_{ijh}$, $\underset{(\alpha)}{t}{}^{*h}{}_{ij}, t^{*}{}_{ijh}$, $\underset{(0\alpha)}{r}{}^{*}{}_{ijh}, T^{*}{}_{ijh}, \underset{(0\alpha)}{R}{}^{*}{}_{ijh}$, $\underset{(\alpha\beta)}{\overset{(\gamma)}{C}}{}^{*}{}_{ijh}$, R^{*i}_{jh}, $\underset{(\alpha1)}{C}{}^{*h}_{ij}, , \ldots, \underset{(\alpha,k-1)}{C}{}^{*h}_{ij}$, $\underset{(\alpha\beta)}{B}{}^{*}{}_{ijh}$, $\underset{(\alpha)}{\overset{(1)}{B}}{}^{*i}_{jh}$, $\underset{(0)}{B}{}^{*}{}_{ijh}$, $\underset{(\alpha\beta)}{B}{}^{*i}_{jh}$, $\underset{(\alpha)}{\overset{(2)}{B}}{}^{*h}_{ij}$, $\underset{(\alpha)}{S}{}^{*}{}_{ijh}, S^{*ijh}$, $\overset{(1)}{K}_{ijh}, \underset{(\alpha)}{\overset{(2)}{K}}_{ijh}$, $\underset{(\alpha\beta)}{\overset{\overset{(2)}{(\gamma)}}{K}}{}_{ijh}$, $\underset{(\alpha\beta)}{\overset{\overset{(3)}{(\gamma)}}{K}}{}_{ijh}, \overset{(3)}{K}{}^{ijh}$, $\underset{(\alpha\beta)}{\overset{(1)}{\mathcal{S}}}{}_{ijh}$, $\underset{(\alpha\beta)}{\overset{(2)}{\mathcal{S}}}{}_{ijh}$ *are invariants of the group* $\overset{ms}{\mathcal{T}}_{N}$.

Proof. By means of transformations of the torsion given in (4.16) and using the notations: (4.17), (4.18) and (4.19), from (4.1) we obtain the results by direct calculation. □

Theorem 4.5. *Between the invariants in Theorem 4.4 there exist the following relations:*

$$\Sigma_{ijh}\left\{\overset{(1)}{K}_{ijh}\right\} = T^{*}{}_{ijh} + H^{*}{}_{ijh} = 0,\ \Sigma_{ijh}\left\{\underset{(\alpha)}{\overset{(2)}{K}}{}_{ijh}\right\} = \underset{(\alpha)}{S}{}^{*}{}_{ijh} + \underset{(\alpha)}{C}{}^{*}{}_{ijh} = 0,$$

$$\Sigma_{ijh}\underset{(\alpha\beta)}{\overset{(2)\,(\gamma)}{K}}{}_{ijh} = \underset{(\alpha\beta)}{\overset{(\gamma)}{C}}{}^{*}_{ijh} - \underset{(\alpha\beta)}{\overset{(\gamma)}{C}}{}^{*}_{ihj}, (\alpha \le \beta, \alpha, \beta, \gamma = 1, ..., k-1),$$

$$\Sigma_{ijh}\left\{\overset{(3)}{K}{}^{ijh}\right\} = S^{*ijh} + C^{*ijh} = 0, \Sigma_{ijh}\underset{(\alpha\beta)}{\overset{(3)\,(\gamma)}{K}}{}_{ijh} = \underset{(\alpha\beta)}{\overset{(\gamma)}{C}}{}^{*}_{ijh} + \underset{(\alpha\beta)}{\overset{(\gamma)}{C}}{}^{*}_{ihj},$$

$$(\alpha \le \beta, \alpha, \beta, \gamma = 1, ..., k-1),$$

$$\Sigma_{ijh}\left\{\underset{(\alpha\beta)}{\overset{(1)}{\mathcal{S}}}{}_{ijh}\right\} = \underset{(\alpha\beta)}{B}{}^{*}{}_{ijh} - \underset{(\alpha\beta)}{B}{}^{*}{}_{jih}, (\alpha \le \beta, \alpha, \beta, \gamma = 1, ..., k-1),$$

$$\Sigma_{ijh}\left\{\underset{(\alpha\beta)}{\overset{(2)}{\mathcal{S}}}{}_{ijh}\right\} = \mathcal{A}_{ih}\left\{\underset{(\alpha\beta)}{B}{}^{*}{}_{jih}\right\} - \mathcal{A}_{ij}\left\{\underset{(\alpha\beta)}{B}{}^{*}{}_{hij}\right\}, (\alpha \le \beta, \alpha, \beta = 1, ..., k-1),$$

$$\Sigma_{ij}\left\{\underset{(\alpha\beta)}{\overset{(3)\,(\gamma)}{K}}\right\} = 0, (\alpha \le \beta, \alpha, \beta, \gamma = 1, ..., k-1). \tag{4.20}$$

Proof. Using notations (4.18), (4.19), relations (4.1) and the definitions of the torsion $d-$tensor fields given in [5] - p.161, we obtain the results. □

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