

AN ASYMPTOTIC FORMULA FOR THE SEMIMARTINGALE LOCAL TIME OF REFLECTING BROWNIAN MOTION ON AN INTERVAL

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Abstract

We derive an asymptotic formula for the expected value of the difference of the semimartingale local times of the 1-dimensional reflecting Brownian motion on $[-1, 1]$ at the two ends of the interval.

As an application we derive the classical probabilistic representation of the solution of the Neumann problem for the Laplace operator in the 1-dimensional case.

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1 Introduction

The local time at the boundary of a stochastic process is an important object of study in the theory of stochastic processes, in particular for the theory and the applications of stochastic differential equations with reflection (see for example [5]). In the simplest 1-dimensional case, the celebrated Tanaka formula (2) gives at the same time a construction of the reflecting Brownian motion on $[0, \infty)$ and of the local time at the boundary, which can be identified with another remarkable process, namely (twice) the semimartingale local time at the origin. As it is known, this process is nondecreasing and unbounded, so this case is not really interesting. However, if we consider the reflecting Brownian motion X on the interval $[-1, 1]$ (by Brownian scaling we can reduce the problem of a general bounded interval to

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this case), then the local time at the boundary is related to the semimartingale local time at the two ends at the interval: $L_t^{-1}(X)$ and $L_t^{1-}(X)$ (in Proposition 1 we show that in fact it is their sum). Since $\lim_{t \rightarrow \infty} L_t^{1-}(X) = \lim_{t \rightarrow \infty} L_t^{-1}(X) = \infty$ a.s., we can ask what is the asymptotic behavior of the expected value of their difference, that is we can try to find (if it exists) the value of the following limit

$$\lim_{t \rightarrow \infty} E^x (L_t^{1-}(X) - L_t^{-1}(X)). \quad (1)$$

In the present paper we show that the above limit exists, and it is equal to $2x$, twice the starting position of the process X . As an application of this asymptotic behavior, we derive a new proof of a probabilistic representation of the solution of the Neumann problem for the Laplace operator in the 1-dimensional case (for the general result see [2], [3] and [4]).

The structure of the paper is the following. In Section 2 we introduce the definitions and the main properties of the semimartingale local time and the local time at the boundary of a continuous semimartingale. In Proposition 1 we give a connection between these two notions of local time, which is needed in the sequel.

The main result is given in Theorem 3 (Section 3), and it shows that the limit in (1) exists and equals $2x$, for all $x \in [-1, 1]$. The proof uses standard stochastic calculus arguments (e.g. Doob's optional stopping theorem) and the properties of the local time.

As an application of Theorem 3, in Section 4 we derive a new proof of the probabilistic representation of the solution of the Neumann problem for the Laplace operator in the 1-dimensional case (this result appears in [3] in the case of smooth C^3 domains, and in [2] in the more general case of Lipschitz domains).

2 Semimartingale local time

The semimartingale local time at $a \in \mathbb{R}$ for a continuous 1-dimensional semimartingale $(X_t)_{t \geq 0}$ can be defined by means of the Tanaka formula (see for example Theorem 1.2 in [6], Ch. VI), as the unique continuous non-decreasing process satisfying

$$|X_t - a| = |X_0 - a| + \int_0^t \operatorname{sgn}(X_s - a) dX_s + L_t^a(X), \quad (2)$$

$$(X_t - a)^+ = (X_0 - a)^+ + \int_0^t 1_{\{X_s > a\}} dX_s + \frac{1}{2} L_t^a(X), \quad (3)$$

and

$$(X_t - a)^- = (X_0 - a)^- - \int_0^t 1_{\{X_s \leq a\}} dX_s + \frac{1}{2} L_t^a(X), \quad (4)$$

for all $t \geq 0$, where $\operatorname{sgn}(x) = 1_{(0, \infty)}(x) - 1_{(-\infty, 0]}(x)$, $x \in \mathbb{R}$.

Equivalently (Corollary 1.9, [6] in Ch. VI), the local time can be defined by the following formulae, which give rise to the name "local time":

$$L_t^a(X) = \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_0^t 1_{[a, a+\varepsilon)}(X_s) d\langle X \rangle_s, \quad (5)$$

if X is a continuous semimartingale, and

$$L_t^a(M) = \lim_{\varepsilon \searrow 0} \frac{1}{2\varepsilon} \int_0^t 1_{(a-\varepsilon, a+\varepsilon)}(M_s) d\langle M \rangle_s, \quad (6)$$

if M is a continuous local martingale. As a consequence, it can be shown (Exercise 1.17 in [6], Ch. VI) that for a continuous semimartingale X we have

$$L_t^{a-}(X) = L_t^{-a}(-X), \quad t \geq 0, \quad a \in \mathbb{R}, \quad (7)$$

where $L_t^{a-}(X) = \lim_{\alpha \nearrow a} L_t^\alpha(X)$ denotes the left limit of the local time of X at a .

It is known (Proposition 1.3 in [6], Ch. VI) that the support of the measure $dL_t^a(X)$ is precisely the set $\{t \geq 0 : X_t = a\}$, so in particular we have

$$\int_0^\infty 1_{\mathbb{R}-\{a\}}(X_s) dL_s^a(X) = 0.$$

It is also known (Theorem 1.7 in [6], Ch. VI) that there exists a modification of the process $\{L_t^a : t \geq 0, a \in \mathbb{R}\}$ which is continuous in t and cadlag in a , which we are going to work with in the sequel. Moreover, if $X = M + V$ is the decomposition of the continuous semimartingale X into the martingale part M and the bounded variation part V , then

$$L_t^a(X) - L_t^{a-}(X) = 2 \int_0^t 1_{\{X_s=a\}} dV_s = 2 \int_0^t 1_{\{X_s=a\}} dX_s, \quad t \geq 0, \quad a \in \mathbb{R}. \quad (8)$$

The reflecting Brownian motion X on $[-1, 1]$ can be defined as the strong solution of the stochastic differential equation

$$X_t = X_0 + B_t + \frac{1}{2} \int_0^t \nu(X_s) dL_s, \quad t \geq 0, \quad (9)$$

where $(B_t)_{t \geq 0}$ is a given 1-dimensional Brownian motion starting at the origin, $(L_t)_{t \geq 0}$ is the local time of X at the boundary of $[-1, 1]$, and ν is the inner unit normal to the boundary of $[-1, 1]$, that is $\nu(-1) = 1$ and $\nu(1) = -1$. We draw attention to the normalization factor $\frac{1}{2}$ which appears in the above stochastic differential equation, since some authors do not use it, and therefore subsequent formulae differ by a factor of 2.

Note that for the reflecting Brownian motion X on $[-1, 1]$, we have two notions of local time: one is the local time $(L_t)_{t \geq 0}$ of X at the boundary of $[-1, 1]$, and the other one is the semimartingale local time $(L_t^a)_{t \geq 0}$ at a point $a \in [-1, 1]$. For further use, we prove the following connection between them.

Proposition 1. *If X is the reflecting Brownian motion on $[-1, 1]$, and $(L_t)_{t \geq 0}$, $(L_t^a(X))_{t \geq 0}$ are the local time of X at the boundary of $[-1, 1]$, respectively the semimartingale local time of X at a , then for any $t \geq 0$ we have*

$$L_t = L_t^{-1}(X) + L_t^{1-}(X). \quad (10)$$

Proof. Applying Tanaka formula (3) with $a = -1$ to X , we get

$$(X_t + 1)^+ = (x + 1)^+ + \int_0^t 1_{\{X_s > -1\}}(X_s) dX_s + \frac{1}{2}L_t^{-1}(X),$$

and since $X_t \in [-1, 1]$ for any $t \geq 0$, we obtain equivalent

$$\begin{aligned} X_t &= x + \int_0^t 1_{\{X_s > -1\}} dX_s + \frac{1}{2}L_t^{-1}(X) \\ &= x + \int_0^t 1 - 1_{\{X_s = -1\}} dX_s + \frac{1}{2}L_t^{-1}(X) \\ &= x + X_t - X_0 - \int_0^t 1_{\{X_s = -1\}} dX_s + \frac{1}{2}L_t^{-1}(X), \end{aligned}$$

which shows that

$$\begin{aligned} \frac{1}{2}L_t^{-1}(X) &= \int_0^t 1_{\{X_s = -1\}} dX_s \\ &= \int_0^t 1_{\{X_s = -1\}} dB_s + \frac{1}{2} \int_0^t 1_{\{X_s = -1\}} dL_s \\ &= \frac{1}{2} \int_0^t 1_{\{X_s = -1\}} dL_s \end{aligned} \tag{11}$$

(the stochastic integral in the last equality above is identically zero since it is a local martingale with quadratic variation process $\int_0^t 1_{\{X_s = -1\}} d\langle B \rangle_s$

$= \int_0^t 1_{\{X_s = -1\}} d\langle X \rangle_s = \int_{-\infty}^{\infty} 1_{\{a = -1\}} L_t^a da = 0$ by the occupation time formula).

Similarly, applying Tanaka formula (3) with $a = -1$ to $-X$ we get

$$(-X_t + 1)^+ = (-x + 1)^+ + \int_0^t 1_{\{-X_s > -1\}} d(-X_s) + \frac{1}{2}L_t^{-1}(-X),$$

from which we obtain

$$\begin{aligned} \frac{1}{2}L_t^{-1}(-X) &= - \int_0^t 1_{\{X_s = 1\}} dX_s \\ &= - \int_0^t 1_{\{X_s = 1\}} dB_s - \frac{1}{2} \int_0^t 1_{\{X_s = 1\}} \nu(X_s) dL_s \\ &= - \frac{1}{2} \int_0^t 1_{\{X_s = 1\}} \nu(1) dL_s \\ &= \frac{1}{2} \int_0^t 1_{\{X_s = 1\}} dL_s. \end{aligned} \tag{12}$$

Adding relations (11) and (12) we obtain

$$\begin{aligned} L_t^{-1}(X) + L_t^{1-}(X) &= L_t^{-1}(X) + L_t^{-1}(-X) \\ &= \int_0^t 1_{\{X_s = -1\}} + 1_{\{X_s = -1\}} dL_s \\ &= \int_0^t 1_{\{X_s = \pm 1\}} dL_s \\ &= L_t, \end{aligned}$$

since the support of the measure dL_t is the set $\{t \geq 0 : X_t \in \partial[-1, 1] = \{\pm 1\}\}$. \square

Remark 2. An alternate proof of the above proposition can be obtained by using relations (8) with $a = \pm 1$ (Theorem 1.6 in [6], Ch. VI), and the fact that for the reflecting Brownian X motion on $[-1, 1]$ we have $L_t^{(-1)^-}(X) = L_t^1(X) = 0$, since $X_t \in [-1, 1]$ for all $t \geq 0$.

3 An asymptotic formula for the semimartingale local time of Brownian motion

The main result is the following.

Theorem 3. If $(X_t)_{t \geq 0}$ is the reflecting Brownian motion on $[-1, 1]$ starting at $X_0 = x \in [-1, 1]$, and $(L_t^a(X))_{t \geq 0}$, $(L_t^{a-}(X))_{t \geq 0}$ denote the semimartingale local time of X at a , respectively its left limit at a , then the following asymptotic formula holds

$$\lim_{t \rightarrow \infty} E^x (L_t^{1-}(X) - L_t^{-1}(X)) = 2x. \quad (13)$$

Proof. By (7) we have $L_t^{1-}(-X) = L_t^{-1}(X)$ and $L_t^{-1}(-X) = L_t^{1-}(X)$ for all $t \geq 0$, hence replacing X by $-X$ if necessary, without loss of generality we may assume that $X_0 = x \geq 0$.

Since X is reflecting Brownian motion on $[-1, 1]$, X is the strong solution to

$$X_t = x + B_t + \frac{1}{2} \int_0^t \nu(X_s) dL_s, \quad t \geq 0, \quad (14)$$

where $(B_t)_{t \geq 0}$ a 1-dimensional Brownian motion starting at the origin, $(L_t)_{t \geq 0}$ is the local time of X at the boundary of $[-1, 1]$, and $\nu(x) = -x$, $x = \pm 1$, is the inner unit normal to the boundary of the interval $[-1, 1]$.

Consider $\tau = \inf\{t \geq 0 : X_t = 0\}$, and note that τ is a stopping time with respect to the filtration of X .

Since $X_s \geq 0$ for $s \in [0, \tau]$ (we are using here the continuity of X and the fact that $X_0 = x \geq 0$), before the time τ the process X can only hit the boundary point 1 of $[-1, 1]$ (for which $\nu(1) = -1$), and therefore from (14) we obtain

$$X_\tau = x + B_\tau + \frac{1}{2} \int_0^\tau \nu(X_s) dL_s, \quad (15)$$

or equivalent

$$X_\tau = x + B_\tau - \frac{1}{2} L_\tau. \quad (16)$$

From the definition of the stopping time τ it follows that $X_\tau = 0$, and therefore

$$\frac{1}{2} L_\tau = x + B_\tau. \quad (17)$$

From (14) and the definition of τ it follows that $X_t \leq x + B_t$ for $t \leq \tau$, and therefore

$$\tau \leq \tau_0 = \inf \{t \geq 0 : B_t + x = 0\}.$$

The process $B_t^x = x + B_t$, $t \geq 0$, is a (free) 1-dimensional Brownian motion starting at x , and it is known that its first hitting time of the origin τ_0 has expected value $(B_0^x - 0)^2 = x^2$. Combining with the above we obtain

$$E\tau \leq E\tau_0 = x^2 < \infty,$$

which shows in particular that τ is an almost surely finite stopping time.

Also, since $(B_t^2 - t)_{t \geq 0}$ is a martingale and $t \wedge \tau$ is a bounded stopping time, by Doob's optional stopping theorem we obtain

$$E(B_{t \wedge \tau}^2 - t \wedge \tau) = E(B_0^2 - 0) = 0,$$

or equivalent

$$E(B_{t \wedge \tau}^2) = E(t \wedge \tau) \leq E\tau < \infty, \quad t \geq 0,$$

and therefore

$$\sup_{t \geq 0} E(B_{t \wedge \tau}^2) \leq E\tau < \infty,$$

which shows that $(B_{t \wedge \tau})_{t \geq 0}$ is a collection of uniformly integrable random variables. Using again the fact that $(B_t)_{t \geq 0}$ is a martingale and $\tau < \infty$ a.s., from Doob's optional stopping theorem we obtain

$$EB_\tau = EB_0 = 0.$$

Combining the above with (17) we obtain $EL_\tau = 2x$, which by Proposition 1 is equivalent to

$$E^x(L_\tau^{1-}(X) - L_\tau^{-1}(X)) = 2x. \quad (18)$$

Next, conditioning on the σ -algebra \mathcal{F}_τ and using the strong Markov property of X (reflecting Brownian motion on $[-1, 1]$), we obtain

$$\begin{aligned} & E^x(L_{t+\tau}^{1-}(X) - L_{t+\tau}^{-1}(X)) = \\ &= E^x[E^x(L_{t+\tau}^{1-}(X) - L_{t+\tau}^{-1}(X)) | \mathcal{F}_\tau] \\ &= E^x[E^x((L_\tau^{1-}(X) + L_t^{1-}(X \circ \theta_\tau)) - (L_\tau^{-1}(X) + L_t^{-1}(X \circ \theta_\tau))) | \mathcal{F}_\tau] \\ &= E^x[E^x(L_\tau^{1-}(X) - L_\tau^{-1}(X) + (L_t^{1-}(X \circ \theta_\tau) - L_t^{-1}(X \circ \theta_\tau))) | \mathcal{F}_\tau] \\ &= E^x(L_\tau^{1-}(X) - L_\tau^{-1}(X)) + E^x[E^{X_\tau}(L_t^{1-}(X) - L_t^{-1}(X))] \\ &= E^x(L_\tau^{1-}(X) - L_\tau^{-1}(X)) + E^0(L_t^{1-}(X) - L_t^{-1}(X)) \\ &= 2x + E^0(L_t^{1-}(X) - L_t^{-1}(X)). \end{aligned} \quad (19)$$

Note that if X is reflecting Brownian motion on $[-1, 1]$ starting at the origin, then by symmetry so is $-X$. In particular, this shows that

$$E^0(L_t^{1-}(X) - L_t^{-1}(X)) = E^0(L_t^{1-}(-X) - L_t^{-1}(-X)), \quad t \geq 0.$$

Since $L_t^{1-}(-X) = L_t^{-1}(X)$ and $L_t^{-1}(-X) = L_t^{1-}(X)$, from the above we obtain

$$E^0(L_t^{1-}(X) - L_t^{-1}(X)) = 0, \quad t \geq 0,$$

which combined with (19) gives

$$E^x(L_{t+\tau}^{1-}(X) - L_{t+\tau}^{-1}(X)) = 2x, \quad t \geq 0.$$

Since $\tau < \infty$ a.s., passing to the limit with $t \rightarrow \infty$ in the last equality we obtain

$$\lim_{t \rightarrow \infty} E^x(L_t^{1-}(X) - L_t^{-1}(X)) = 2x,$$

concluding the proof. \square

4 An application to the Neumann problem for the Laplace operator

As an application of Theorem 3 we will derive a new proof of a probabilistic representation of the Neumann problem for the Laplace operator (Theorem 4 below) in the 1-dimensional case.

For a smooth bounded domain $D \subset \mathbb{R}^n$, consider the Neumann problem for the Laplacian

$$\begin{cases} \Delta u = 0 \text{ in } D \\ \frac{\partial u}{\partial n} = f \text{ on } \partial D \end{cases}, \quad (20)$$

where $f : \partial D \rightarrow \mathbb{R}$ is a given continuous function satisfying the (necessary) centering condition $\int_{\partial D} f(y) \sigma(dy) = 0$, σ is the surface measure on the boundary ∂D , and $n(x)$ is the outward unit normal to the boundary of D at $x \in \partial D$.

In [2], the authors extended a result of Brosamler ([3]) from the case of smooth domains, and gave the following probabilistic representation of the solution of the above problem (note that for smooth domains, a classical solution is the same as a generalized solution – see [4] for the details).

Theorem 4 ([2]). *Let D be a bounded Lipschitz domain and let $f \in B(\partial D)$ with $\int_{\partial D} f(x) \sigma(dx) = 0$. Then there is a unique generalized solution u to the Neumann boundary problem (20) satisfying the condition $\int_D u(x) dx = 0$. Furthermore, we have for each $x \in D$*

$$u(x) = \lim_{t \rightarrow \infty} \frac{1}{2} E^x \int_0^t f(X_s) dL_s, \quad (21)$$

where X is reflecting Brownian motion on D and L_t is the boundary local time for X .

Using Theorem 3, we will derive a new proof of the above result in the 1-dimensional case, as follows.

Corollary 5. For $D = [-1, 1] \subset \mathbb{R}$, the solution u of the Neumann problem (20) which satisfies $\int_{-1}^1 u(x) dx = 0$, is given by

$$u(x) = \lim_{t \rightarrow \infty} \frac{1}{2} E^x \int_0^t f(X_s) dL_s, \quad x \in [-1, 1], \quad (22)$$

where X is the reflecting Brownian motion on $[-1, 1]$ starting at $x \in [-1, 1]$ and $(L_t)_{t \geq 0}$ is the local time of X at the boundary of $[-1, 1]$.

Proof. Note that in this case the problem (20) reduces to determining the function u from $u(x) = ax + b$ (the harmonic functions in \mathbb{R}), so that $u'(1) = f(1)$ and $-u'(-1) = f(-1)$. The centering condition on f implies that $f(1) = -f(-1) = \alpha$ for some $\alpha \in \mathbb{R}$, so the solutions of the problem (20) are in this case $u(x) = \alpha x + b$, $b \in \mathbb{R}$. Finally, the condition $\int_{-1}^1 u(x) dx = 0$ implies $b = 0$, so the normalized solution of the Neumann problem (20) is $u(x) = \alpha x$, $x \in [-1, 1]$.

In order to prove claim (22), it remains to be shown that the quantity on the right equals αx , for any $x \in [-1, 1]$.

Fix an arbitrary $x \in [-1, 1]$, and let X be the reflecting Brownian motion on $[-1, 1]$ starting at x , that is a strong solution to

$$X_t = X_0 + B_t + \frac{1}{2} \int_0^t \nu(X_s) dL_s, \quad t \geq 0, \quad (23)$$

where $(B_t)_{t \geq 0}$ is a given 1-dimensional Brownian motion starting at the origin, $(L_t)_{t \geq 0}$ is the local time of X at the boundary of $[-1, 1]$, and $\nu(x) = -x = -n(x)$, $x = \pm 1$, is the inner unit normal to the boundary of $[-1, 1]$.

Using Proposition 1 and the fact that the support of the measures $dL_t^{-1}(X)$ and $dL_t^{1-}(X) = dL_t^{-1}(-X)$ are the sets $\{t \geq 0 : X_t = -1\}$, respectively $\{t \geq 0 : X_t = 1\}$, we obtain

$$\begin{aligned} \int_0^t f(X_s) dL_s &= \int_0^t f(X_s) dL_s^{-1}(X) + \int_0^t f(X_s) dL_s^{1-}(X) \\ &= \int_0^t f(-1) dL_s^{-1}(X) + \int_0^t f(1) dL_s^{1-}(X) \\ &= -\alpha \int_0^t dL_s^{-1} + \alpha \int_0^t dL_s^{1-} \\ &= \alpha (L_t^{1-} - L_t^{-1}). \end{aligned}$$

Passing to the limit with $t \rightarrow \infty$, by Theorem 3 we obtain

$$\lim_{t \rightarrow \infty} \frac{1}{2} E^x \int_0^t f(X_s) dL_s = \frac{\alpha}{2} \lim_{t \rightarrow \infty} E^x (L_t^{1-} - L_t^{-1}) = \frac{\alpha}{2} 2x = \alpha x.$$

We have shown that

$$u(x) = \alpha x = \lim_{t \rightarrow \infty} \frac{1}{2} E^x \int_0^t f(X_s) dL_s, \quad x \in [-1, 1],$$

concluding the proof. \square

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