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A NEW CLASS OF DIVISORS: THE EXPONENTIAL SEMIPROPER DIVISORS

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Abstract

The aim of this paper is to present the notion of *exponential semiproper divisor* and to study some properties of arithmetical functions which use exponential semiproper divisors. We also investigate the maximal order and the minimal order of these arithmetical functions.

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1 Introduction

First we enumerate several types of divisors found in some papers on Number Theory.

In [17] R. Vaidyanathaswamy introduced the notion of *block-factor* in the following way: a divisor d of n is a block-factor when $\left(d, \frac{n}{d}\right) = 1$. Later, E. Cohen [1] introduced the current terminology for a block-factor, namely, *the unitary divisor*. In 1966, M. V. Subbarao and L. J. Warren [11] introduced the *unitary perfect numbers* satisfying $\sigma^*(n) = 2n$, where $\sigma^*(n)$ denotes the sum of the unitary divisors of n. Let $\tau^*(n)$ denote the number of unitary divisors of n, which is, in fact, the number of the squarefree divisors of n.

F. Mertens, in [4], proved the relation

$$\sum_{n\leq x}\tau^*(n) = \frac{x}{\zeta(2)}\left(\log x + 2\gamma - 1 - \frac{2\zeta'(2)}{\zeta(2)}\right) + S_2(x), \text{ where } S_2(x) = O\left(x^{\frac{1}{2}}\log x\right). \quad (1)$$

A. A. Gioia and A. M. Vaidya [2] showed that $S_2(x) = O\left(x^{\frac{1}{2}}\right)$.
R. Sitaramachandrarao and D. Suryanarayana [9] found the following result:

$$\sum_{n\leq x}\sigma^*(n) = \frac{\pi^2 x^2}{12\zeta(3)} + O\left(x\log^{\frac{2}{3}} x\right). \quad (2)$$

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We recall that the notion of *exponential divisor* was introduced by M. V. Subbarao in [10] in the following way: d is said to be an *exponential divisor* (or *e-divisor*) of $n = p_1^{a_1}...p_r^{a_r} > 1$, if $d = p_1^{b_1}...p_r^{b_r}$, where $b_i|a_i$ for any $1 \le i \le r$. A series of results related to the exponential divisors are given in more papers: [3,8,13,14].

N. Minculete and L. Tóth in [5] presented some properties of the arithmetical functions which use *exponential unitary divisors* or *e-unitary divisors* of $n = p_1^{a_1}...p_r^{a_r} > 1$, if $d = p_1^{b_1}...p_r^{b_r}$, where b_i is a unitary divisor of a_i, so $\left(b_i, \dfrac{a_i}{b_i}\right) = 1$, for any $1 \le i \le r$.

2 Main result

We now introduce a new class of divisors. Let n be a positive integer, such that $n = p_1^{a_1}...p_r^{a_r} > 1$ and the arithmetical function $\gamma(n) = p_1p_2...p_r$, which is called the "core" of n.

A divisor d of n, so that $\gamma(d) = \gamma(n)$ and $\left(\dfrac{d}{\gamma(n)}, \dfrac{n}{d}\right) = 1$ will be called an *exponential semiproper divisor* or an *e-semiproper divisor* of n.
As an example, we consider the number $n = 2^6 \cdot 3^4$; then the e-semiproper divisors of n are the following:

$$2 \cdot 3,\ 2^6 \cdot 3,\ 2 \cdot 3^4,\ 2^6 \cdot 3^4.$$

Let $\tau^{(e)s}(n)$ denote the number of the e-semiproper divisors of n, and $\sigma^{(e)s}(n)$ denote the sum of the e-semiproper divisors of n. We note $d|_{(e)s}n$. By convention, 1 is an exponential semiproper divisor of itself, so that $\sigma^{(e)s}(1) = \tau^{(e)s}(1) = 1$. We notice that 1 is not an e-semiproper divisor of $n > 1$, the smallest e-semiproper divisor of n is $\gamma(n)$ and the greatest e-semiproper divisor is n.

Any e-semiproper divisor d of n is written as $d = \gamma(n) \cdot d'$, where d' is a unitary divisor of $\dfrac{n}{\gamma(n)}$. Therefore, the number of the e-semiproper divisors of n is $\tau^*\left(\dfrac{n}{\gamma(n)}\right)$ and the sum of the e-semiproper divisors of n is $\gamma(n) \cdot \sigma^*\left(\frac{n}{\gamma(n)}\right)$, so we have the following relations:

$$\tau^{(e)s}(n) = \tau^*\left(\frac{n}{\gamma(n)}\right),\ \sigma^{(e)s}(n) = \gamma(n) \cdot \sigma^*\left(\frac{n}{\gamma(n)}\right). \tag{3}$$

We observe that if the integer $d = p_1^{b_1}...p_r^{b_r}$ is an exponential semiproper divisor of $n = p_1^{a_1}...p_r^{a_r} > 1$, then $b_i \in \{1, a_i\}$, for any $1 \le i \le r$. Among the divisors of n defined in this way there is the improper divisor n and the others (if there are) are the proper divisors of n. This creates a connection between the exponents as the improper divisors and the proper divisors of n chosen from the exponential divisors of n, suggesting a hybrid concept, namely, the exponential semiproper divisor. Hence, according to the things mentioned above, we have

$$\tau^{(e)s}(p^a) = \begin{cases} 1, & for\ a = 1 \\ 2, & for\ a \ge 2, \end{cases} \tag{4}$$

so, p is an e-semiproper divisor of p, and e-semiproper divisors of $p^a (a \geq 2)$ are p and p^a, which means that

$$\sigma^{(e)s}(p^a) = \begin{cases} p, \ for \ a = 1 \\ p^a + p, \ for \ a \geq 2. \end{cases} \tag{5}$$

We remark also that the e-semiproper divisors of n are among the e-unitary divisors of n and the e-unitary divisors of n are among the e-divisors of n, so it is easy to see that

$$\tau^{(e)s}(n) \leq \tau^{(e)*}(n) \leq \tau^{(e)}(n) \text{ and } \sigma^{(e)s}(n) \leq \sigma^{(e)*}(n) \leq \sigma^{(e)}(n), \tag{6}$$

where $\tau^{(e)}$ is the number of exponential divisors of n, $\sigma^{(e)}$ is the sum of exponential divisors of n, $\tau^{(e)*}$ is the number of exponential unitary divisors of n and $\sigma^{(e)*}$ is the sum of exponential unitary divisors of n. It is obvious that the arithmetical functions $\tau^{(e)s}$ and $\sigma^{(e)s}$ are multiplicative and we have

$$\tau^{(e)s}(n) = 2^t, \ \sigma^{(e)s}(n) = p_1...p_u \prod_{i=u+1}^{r} (p_i^{a_i} + p_i), \tag{7}$$

where $n = p_1...p_u p_{u+1}^{a_{u+1}}...p_r^{a_r}$, with $a_i \geq 2$ for any $i \in \{u+1, ..., r\}$ and $t = r - u$, so, t is the number of the exponents in the prime factorization of n which are ≥ 2.

If n is square-free, then $\tau^{(e)s}(n) = 1$ and $\sigma^{(e)s}(n) = n$.

Similar to the exponential unitary convolution, we introduce the *exponential semiproper convolution (e-semiproper convolution)* of arithmetical functions, which is defined by

$$(f *_{(e)s} g)(n) = \sum_{\substack{b_1 c_1 = a_1 \\ b_1, c_1 \in \{1, a_1\}}} ... \sum_{\substack{b_r c_r = a_r \\ b_r, c_r \in \{1, a_r\}}} f(p_1^{b_1}...p_r^{b_r}) g(p_1^{c_1}...p_r^{c_r}) \tag{8}$$

The e-semiproper convolution is commutative, associative and has the identity element $\overline{\mu}$, where $\overline{\mu}(1) = 1$ and

$$\overline{\mu}(p^a) = \begin{cases} 1, \ for \ a = 1 \\ 0, \ for \ a \geq 2. \end{cases} \tag{9}$$

It easy to see that that $\overline{\mu}$ is a multiplicative function. Furthermore, a function f has an inverse with respect to the e-semiproper convolution iff $f(1) \neq 0$ and $f(p_1...p_k) \neq 0$, for any distinct primes $p_1, ..., p_k$.

The inverse with respect to the e-semiproper convolution of the constant 1 function is denoted by μ_s. The arithmetical function μ_s is given by $\mu_s(1) = 1$ and for $n > 1$, we have

$$\mu_s(p^a) = \begin{cases} 1, \ for \ a = 1 \\ -1, \ for \ a \geq 2 \end{cases} \tag{10}$$

Hence, we obtain the identity

$$\mu_s *_{(e)s} \mu_s = \mu_s \cdot \tau^{(e)s}. \tag{11}$$

In [6], we meet the regular convolutions of Narkiewicz-type, and here we observe that the e-semiproper convolution is a special case of these.

For the maximal order of the function $\tau^{(e)s}$, we have

Theorem 1.

$$\lim_{n\to\infty}\sup\frac{\log\tau^{(e)s}(n)\log\log n}{\log n}=\frac{\log 2}{2}. \tag{12}$$

Proof. We use the following general result given in [12]: Let F be a multiplicative function with $F(p^a)=f(a)$ for every prime powers p^a, where f is positive and satisfying $f(n)=O(n^\beta)$ for some fixed $\beta>0$. then

$$\lim_{n\to\infty}\sup\frac{\log F(n)\log\log n}{\log n}=\sup_m\frac{\log f(m)}{m}.$$

Take $F(n)=\tau^{(e)s}(n)$, which is a multiplicative function, and

$$f(a)=\begin{cases}1, & for\ a=1\\ 2, & for\ a\geq 2.\end{cases}$$

But $f(n)=O(1)=O(n^0)$, it follows that

$$\lim_{n\to\infty}\sup\frac{\log\tau^{(e)s}(n)\log\log n}{\log n}=\sup_m\frac{\log f(m)}{m}=\sup_m\frac{\log 2}{m}=\frac{\ln 2}{2},$$

therefore, we obtain the result of the statement.

Theorem 2.

$$\sum_{n\leq x}\tau^{(e)s}(n)=\frac{15}{\pi^2}x+Ax^{\frac{1}{2}}+O\left(x^{\frac{1}{3}+\epsilon}\right), \tag{13}$$

for every $\epsilon>0$, where A is a constant, and the Dirichlet series of $\tau^{(e)s}(n)$ is

$$\sum_{n=1}^{\infty}\frac{\tau^{(e)s}(n)}{n^t}=\frac{\zeta(t)\zeta(2t)}{\zeta(4t)},\ \text{for Re}t>1. \tag{14}$$

Proof. L. Tóth in [15, Theorem, p. 2] proved the following general result:
Let f be a complex valued multiplicative arithmetic function such that

a) $f(p)=f(p^2)=...=f(p^{l-1}),\ f(p^l)=f(p^{l+1})=k$, for every prime p, where $l,k\geq 2$ are fixed integers and

b) there are constants $C,\ m>0$, such that $|f(p^a)|\leq Ca^m$ for every prime p and every $a\geq l+2$.

Then, for $t\in\mathbb{C}$,
i)

$$\sum_{n=1}^{\infty}\frac{f(n)}{n^t}=\zeta(t)\cdot\zeta^{k-1}(lt)\cdot V(t),\ \text{for Re}t>1$$

where the Dirichlet series $V(t) = \sum_{n=1}^{\infty} \frac{v(n)}{n^t}$ is absolutely convergent for Re$t > \frac{1}{l+2}$, and $v = f * \mu * \mu_l^{(k-1)}$ is a multiplicative function such that $v(1) = 1$, $v(p) = v(p^2) = ... = v(p^{l+1}) = 0$ and $v(p^a) = \sum_{j\geq 0}(-1)^j \begin{pmatrix} k-1 \\ j \end{pmatrix} (f(p^{a-jl}) - f(p^{a-jl-1}))$ for $a = kl$.

ii)

$$\sum_{n\leq x} f(n) = C_f x + x^{\frac{1}{l}} P_{f,k-2}(\log x) + O(x^{u_{k,l}+\epsilon}),$$

for every $\epsilon > 0$, where $P_{f,k-2}$ is a polynomial of degree $k-2$, $u_{k,l} = \frac{2k-1}{3+(2k-1)l}$ and

$$C_f := \prod_p \left(1 + \sum_{a=l} \frac{f(p^a) - f(p^{a-1})}{p^a}\right),$$

where the arithmetical function μ_l is given by $\mu_l(1) = 1$ and for $n > 1$, we have

$$\mu_l(p^a) = \begin{cases} -1, & \text{if } a = l \\ 0, & \text{otherwise} \end{cases} \tag{15}$$

and for an integer $h \geq 1$ let the function $|mu_l^{(h)}$ be defined in terms of the Dirichlet convolution by

$$\mu_l^{(h)} = \mu_l * \mu_l * ... * \mu_l.$$

For the arithmetic function $f(n) = \tau^{(e)s}(n)$, take $l = 2$ and $k = 2$, because $\tau^{(e)s}(p) = 1$, $\tau^{(e)s}(p^2) = \tau^{(e)s}(p^3) = 2$, and for every $a \geq 2$, we have

$$|\tau^{(e)s}(p^a)| = 2 \leq Ca^m,$$

where C and m are two constants. Therefore, the conditions from Tóth's theorem are satisfied, so it follows the relation

$$\sum_{n\leq x} \tau^{(e)s}(n) = C_f x + x^{\frac{1}{2}} P_{f,0}(\log x) + O(x^{u_{2,2}+\epsilon}).$$

But $C_f := \prod_p \left(1 + \sum_{a=l} \frac{f(p^a) - f(p^{a-1})}{p^a}\right)$, so

$$C_f = \prod_p \left(1 + \sum_{a=2} \frac{\tau^{(e)s}(p^a) - \tau^{(e)s}(p^{a-1})}{p^a}\right)$$
$$= \prod_p \left(1 + \frac{1}{p^2} + \sum_{a=3} \frac{\tau^{(e)s}(p^a) - \tau^{(e)s}(p^{a-1})}{p^a}\right) = \prod_p \left(1 + \frac{1}{p^2}\right) = \frac{\zeta(2)}{\zeta(4)} = \frac{15}{\pi^2}.$$

We obtain that $u_{2,2} = \frac{1}{3}$, and $P_{f,0}$ is a constant, which is denoted by A. Therefore, the proof of relation (14) is complete.
Let $v(p) = v(p^2) = v(p^3) = 0$ and
$v(p^a) = \sum_{j\geq 0}(-1)^j \left(\begin{array}{c} 1 \\ j \end{array} \right)(\tau^{(e)s}(p^{a-jl}) - \tau^{(e)s}(p^{a-jl-1})) = \tau^{(e)s}(p^a) - \tau^{(e)s}(p^{a-1}) - \tau^{(e)s}(p^{a-2}) + \tau^{(e)s}(p^{a-3}) = 0$, if $a \geq 5$, and for $a = 4$ we have $v(p^4) = -1$. Therefore, we obtain $v(p^4) = -1$, and $v(p^a) = 0$ for any $a \neq 4$. But the Dirichlet series $V(t) = \sum_{n=1}^{\infty} \frac{v(n)}{n^t}$ is absolutely convergent for Re$t > \frac{1}{4}$ and is equal to $\prod_{p\ prim} \left(1 - \frac{1}{p^{4t}}\right) = \frac{1}{\zeta(4t)}$, so $V(t) = \frac{1}{\zeta(4t)}$, thus, relation (14) is true.

Theorem 3. *For any integer $r \geq 1$, there are the following relations:*

$$\sum_{n=1} \frac{[\tau^{(e)s}(n)]^r}{n^t} = \zeta(t)\zeta^{2^r-1}(2t)\left[2 - 2^r + \frac{(2^r-1)}{\zeta(4t)}\right],\ for\ \mathrm{Re}t > 1, \tag{16}$$

and

$$\sum_{n=1}^{\infty}[\tau^{(e)s}(n)]^r = A_r x + x^{\frac{1}{2}}P_{f,2^r-2}(\log x) + O(x^{u_r+\epsilon}), \tag{17}$$

for every $\epsilon > 0$, where $P_{f,2^r-2}$ is a polynomial of degree $2^r - 2$, $u_r = \frac{2^{r+1}-1}{2^{r+2}+1}$ and

$$A_r := \prod_p \left(1 + \frac{2^r - 1}{p^2}\right).$$

Proof. In case $f(n) = [\tau^{(e)s}(n)]^r$, with $r \geq 1$, we apply Tóth's Theorem for $l = 2,\ k = 2^r$ and we obtain the relations of statement.

We mention that a number n is an *exponential semiproper perfect* number if we have

$$\sigma^{(e)s}(n) = 2n.$$

If m is a squarefree number and n is an exponential semiproper perfect number so that $(m,n) = 1$, then mn is exponential semiproper perfect, because

$$\sigma^{(e)s}(m,n) = \sigma^{(e)s}(m) \cdot \sigma^{(e)s}(n) = m \cdot 2n = 2mn.$$

The first e-semiproper perfect numbers until 1000 are the following:

$$36,\ 180,\ 252,\ 396,\ 468,\ 612,\ 684,\ 684,\ 828.$$

There is an infinity of e-semiproper perfect numbers.
The number $9539712 = 2^6 \cdot 3^2 \cdot 7^2 \cdot 13^2$ is an e-unitary perfect number, but it is not e-semiproper perfect.

Theorem 4. *There are no odd e-semiproper perfect numbers.*

Proof. It is similar to [5, Theorem 6]. Suppose that $n = p_1^{a_1}...p_r^{a_r}$ is an odd e-semiproper perfect number, so we have

$$\sigma^{(e)s}(p_1^{a_1})...\sigma^{(e)s}(p_r^{a_r}) = 2p_1^{a_1}...p_r^{a_r}. \tag{18}$$

We can assume that $a_i \geq 2$, for any $i \in \{1, ..., r\}$, because if $a_i = 1$ for an i, then $\sigma^{(e)s}(p_i) = p_i$ and we can simplify with p_i in relation (17), so relation (17) becomes $(p_1^{a_1} + p_1)...(p_r^{a_r} + p_r) = 2p_1^{a_1}...p_r^{a_r}$. Therefore, we have $(p_1^{a_1-1} + 1)...(p_r^{a_r-1} + 1) = 2p_1^{a_1-1}...p_r^{a_r-1}$, which means that $r = 1$. Consequently, we deduce the relation

$$p_1^{a_1-1} + 1 = 2p_1^{a_1-1},$$

which implies $a_1 = 1$, which is a contradiction. Thus, the demonstration ends.

Remark 1. The number n is an e-semiproper perfect number if and only if $\dfrac{n}{\gamma(n)}$ is a unitary perfect number.

Theorem 5.

$$\liminf_{n\to\infty} \frac{\sigma^{(e)s}(\sigma(n))}{n} = 1, \tag{19}$$

where $\tau(n)$ is the number of the divisors of n and $\sigma(n)$ is the sum of the divisors of n.

Proof. Since $n \leq \sigma^{(e)s}(n) \leq \sigma(n)$ for any $n \geq 1$, we apply Theorem 5 form [7].

Theorem 6. *For every $n \geq 1$, there is the following:*

$$\tau(n) \leq \sqrt{n\gamma(n)} \leq \frac{\sigma^{(e)s}(n)}{\tau^{(e)s}(n)}. \tag{20}$$

Proof. For $n = 1$ we have $\tau(1) = 1 = \sqrt{1\gamma(1)} = 1 = \frac{\sigma^{(e)s}(1)}{\tau^{(e)s}(1)}$.
For $n = p_1p_2...p_up_{u+1}^{a_{u+1}}...p_r^{a_r} > 1$, we deduce the inequality

$$p_1p_2...p_up_{u+1}^{\frac{a_{u+1}+1}{2}}...p_r^{\frac{a_r+1}{2}} \leq p_1p_2...p_u \prod_{j=u+1}^{r} \left(\frac{p_j^{a_j} + p_j}{2}\right) =$$

$$= \frac{1}{2^{r-u}}p_1p_2...p_u \prod_{j=u+1}^{r} (p_j^{a_j} + p_j) = \frac{\sigma^{(e)s}(n)}{\tau^{(e)s}(n)}.$$

But, we have the equality $p_1p_2...p_up_{u+1}^{\frac{a_{u+1}+1}{2}}...p_r^{\frac{a_r+1}{2}} = \sqrt{n\gamma(n)}$. Therefore, we obtain the inequality

$$\sqrt{n\gamma} \leq \frac{\sigma^{(e)s}(n)}{\tau^{(e)s}(n)}.$$

We show first that
$$\sqrt{p^a\gamma(p^a)} \geq \tau(p^a),$$
so $p^{\frac{a+1}{2}} \geq a+1$, which is true, because $p^{\frac{a+1}{2}} \geq 2^{\frac{a+1}{2}} \geq a+1$, for any $a \geq 1$.
Using the fact that the arithmetical function τ and γ are multiplicative, it follows that
$$\sqrt{n\gamma(n)} \geq \tau(n), \text{ for any } n \geq 1.$$
Thus, the demonstration is complete.

Remark 2. By simple calculation it is easy to see that
$$\frac{n+\gamma(n)}{2} \geq \frac{\sigma^{(e)s}(n)}{\tau^{(e)s}(n)} \geq \frac{\sigma^*(n)}{\tau^*(n)} \geq \frac{\sigma(n)}{\tau(n)} \geq \sqrt{n}, \text{ for any } n \geq 1. \tag{21}$$

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