Bulletin of the *Transilvania* University of Braşov • Vol 7(56), No. 1 - 2014
Series III: Mathematics, Informatics, Physics, 13-24

ON BOUNDED CARTAN TORSION OF SPECIAL FINSLER (α, β)-METRICS

A. R. KAVYASHREE, Mallikarjuna Y. KUMBAR and S. K. NARASIMHAMURTHY[1],

Abstract

In this paper, we proved that some special Finsler (α, β)-metrics have bounded Cartan torsion. Further, we find the relation between the norm of Cartan and the mean Cartan torsion for the class of (α, β)-metrics.

2000 *Mathematics Subject Classification:* 53B40
Key words: Finsler space, (α, β)-metrics, norm of Cartan, bounded Cartan torsion, mean Cartan torsion.

1 Introduction

Cartan torsion is one of the most fundatmental non-Riemannian quantities. It was first introduced by P. Finsler[1] and emphasized by E. Cartan[2], who measured a departure from a Riemannian manifold. Finsler metrics are Riemannian metrics on a manifold M without quadratic restriction. They give Minkowski norms instead of inner products on each tangent space T_xM.

More precisely, a Finsler metric is Riemannian if and only if it has vanishing Cartan torsion. Intuitively the norm of Cartan torsion is farther from zero, than this Finsler manifold. In 1957, J. Nash[3] proved that any n-dimensional Riemannian manifold can be isometrically imbedded into a higher dimensional Euclidean space. So the natural question arises here whether in Finsler geometry, every Finsler manifold can be isometrically imbedded into a Minkowski space?. However, the answer is affirmative. In 1997[4], Z.Shen proved that Finsler manifold with unbounded Cartan torsioncould not be isometrically imbedded into any Minkowski space. For Finsler manifolds, the problem under certain circumstances was considered by Burago-Iranov, Gu and Ingarden([5][6][7][8]). Then the norm of Cartan torsion plays an important role for the study of immersion theory in

[1]Corresponding author, Department of P.G. Studies and Research in Mathematics, *Kuvempu* University, Shankarghatta - 577 451. Shimoga, Karnataka, INDIA, e-mail: nmurthysk@hotmail.com

Finsler geometry. For the Finsler metric F, one can define the norm of the Cartan torsion C as follows,

$$\| C \| = Sup_{F(y)=1, v\neq 0} \frac{| C_y(v,v,v) |}{[g_y(v,v)]^{3/2}}. \tag{1}$$

The bound for two dimensional Randers metrics is verified by B. Lackey and Z. Shen[11] proved that the Cartan torsion of Randers metrics on a manifold M of dimension $n \geq 3$ is uniformly bounded by $\frac{3}{\sqrt{2}}$. Recently A. Teyabi and H. Sadeghi[12] have studied a relation between the norm of Cartan and mean Cartan torsion of Finsler metrics defined by a Riemannian metric and 1-form on manifold M. They proved that generalized Kropina metrics $F = \frac{\alpha^{m+1}}{\beta^m}(m \neq 0)$ and generalized Randers metric $F = (c_1\alpha^2 + 2c_2\alpha\beta + c_3\beta^2)$, where c_1, c_2, c_3 were real constants have bounded Cartan torsion. It turns out that every C-reducible Finsler metric has bounded Cartan torsion.

A natural task for us is to find other Finsler metrics which have bounded Cartan torsion. In this paper, we find two more subclasses of (α, β)-metrics which have bounded Cartan torsion. Then, we give a relation between the norm of Cartan and the mean Cartan torsion for the class of (α, β)-metrics with related examples.

2 Preliminaries

A Finsler metric on a manifold M is a C^∞ function on TM_0 having the following properties:
(i) $F(x, y) \geq 0$ for any $y \in T_xM$ and $F(x, y) = 0$ if and only if $y = 0$;
(ii) $F(x, \lambda y) = \lambda F(x, y)$ for any $y \in T_xM$ and $\lambda > 0$;
(iii) For any tangent vector $y \in T_xM$, the following bilinear symmetric form g_y on T_xM is positive definite,

$$g_y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}[F^2(y + su + tv)] \mid_{s,t=0}, u, v \in T_xM.$$

Riemannian metric is the special case that at each point $x \in M$ the fundamental tensor g_y is independent of the tangent vector $y \in T_xM_0$. To measure the non-Riemannian feature of F, define $C_y : T_xM \otimes T_xM \otimes T_xM \to \Re$ by

$$C_y(u, v, w) := \frac{1}{2}\frac{d}{dt}[g_{y+tw}(u, v)] \mid_{t=0}, \quad u, v, w \in T_xM.$$

This trilinear symmetric form on pullback bundle π^*TM is called *Cartan torsion*. E. Cartan got this quantity when he introduced his metric-compatible connection. Obviously, F is Riemannian metric if and only if $C_y = 0$.

For $y \in T_xM_0$, define mean Cartan torsion I_y by $I_y(u) = I_i(y)u^i$, where $I_i = g^{ij}C_{ijk}$. In 1953, Deicke[13] proved that F is Riemannian if and only if the mean Cartan torsion $I_y = 0$ for any $y \in T_xM_0$. The bound of Cartan torsion C at a point $x \in M$ is defined by

$$\parallel C \parallel_x = Sup_{y,u \in T_xM} \frac{F(x,y)|C_y(u,u,u)|}{[g_y(u,u)]^{3/2}}$$

and the bound of Cartan torsion on M is defined by :

$$\parallel C \parallel = Sup_{x \in M} \parallel C \parallel_x .$$

Let (M, F) be a Finsler manifold. For $y \in T_xM_0$, define the *Matsumoto torsion* $M_y : T_xM \otimes T_xM \otimes T_xM \to \Re$ by $M_y(u,v,w) := M_{ijk}(y)u^iv^jw^k$, where

$$M_{ijk} := C_{ijk} - \frac{1}{n+1}\{I_ih_{jk} + I_jh_{ik} + I_kh_{ij}\},$$

and $h_{ij} := g_{ij} - \frac{1}{F^2}g_{ip}y^pg_{jq}y^q$ is the angular metric. A Finsler metric F is said to be C-reducible, if $M_y = 0$. M. Matsumoto[14] proved that every Randers metric satisfies $M_y = 0$. Later on, Matsumoto-*Hojo* proved that the converse is true too.

Lemma 1. *A Finsler metric F on a manifold of dimension $n \geq 3$ is a Randers metric if and only if the Matsumoto torsion vanishes.*

Let $r(t) : [0,1] \longrightarrow M$ be a piecewise C^∞ curve on a Finsler manifold (M, F). We can define the length of $r(t)$ by

$$L(r) := \int_0^1 F(r(t), r'(t))dt.$$

The geodesic curves on a smooth manifold M are characterized by the second order differential equations

$$\frac{d^2r}{dt^2} + 2G^i(r(t), r'(t)) = 0,$$

where the local functions $G^i = G^i(x,y)$ are called the spray coefficients of F, and given by in a local coordinate[17]:

$$G^i(x,y) := \frac{1}{4}g^{il}\{\frac{\partial^2F^2}{\partial x^k \partial y^l}y^k - \frac{\partial F^2}{\partial x^l}\}.$$

A positive complete Finsler manifold means that every geodesic $r(t)$, where $t \in [0,1]$ can be extended to $(0, \infty)$. For a tangent vector $y \in T_xM_0$, in local coordinate define a tensor on pullback bundle π^*TM , $B_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ by

$$B_y(u,v,w) := B^i_{jkl}(y)u^jv^kw^l\frac{\partial}{\partial x^i},$$

where $u = u^i\frac{\partial}{\partial x^i}\mid_x, v = v^i\frac{\partial}{\partial x^i}\mid_x, w = w^i\frac{\partial}{\partial x^i}\mid_x$, and

$$B^i_{jkl} := \frac{\partial^3G^i}{\partial y^j\partial y^k\partial y^l}.$$

A Finsler metric is called a Berwald metric if B=0. This is equivalent to its spray coefficients G^i to be quadratic in y at every point $x \in M$. Riemannian metric is Berwaldian because in this case $G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^jy^k$, where Γ^i_{jk} are Christoffel symbols.

3 Bounded Cartan Torsion Of Special Finsler (α, β)-metrics

3.1. Bounded Carton Torsion for the metric $F = \sqrt[3]{c_1\alpha^2\beta + c_2\beta^3}$:

In this section, we consider the special (α, β)-metric $F = \sqrt[3]{c_1\alpha^2\beta + c_2\beta^3}$ where $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric, $\beta = b_i(x)y^i$ is an 1-form on manifold M and $c_1, c_2 > 0$ are real constants and we prove the following theorem:

Theorem 1. *Let $F = \sqrt[3]{c_1\alpha^2\beta + c_2\beta^3}$ be a special (α, β)-metric where, $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric, $\beta = b_i(x)y^i$ is an 1-form on manifold M and $c_1, c_2 > 0$ are real constants. Then F has bounded Cartan torsion.*

Proof. Let us first consider the case of $dimM = 2$. There exist local orthonormal coframes ω_1, ω_2 of Riemannian metric α. So α^2 can be written as

$$\alpha^2 = \omega_1^2 + \omega_2^2.$$

If we denote $\alpha = \sqrt{a_{ij}y^iy^j}$, where $y = \sum_{i=1}^2 y^ie_i$ and e_i is the dual frame of ω_i then $a_{ij} = \delta_{ij}$ and $a^{ij} = \delta^{ij}$. Adjust coframe ω_1, ω_2 properly such that

$$\beta = k\omega_1.$$

Then $b_1 = k$ and $b_2 = 0$ where $\beta = \sum_{i=1}^2 b_iy^i$. Hence

$$\|\beta\|_\alpha = \sqrt{a^{ij}b_ib_j} = k.$$

For an arbitrary tangent vector $y = ue_1 + ve_2 \in T_pM$, we can obtain that

$$\alpha(p, y) = \sqrt{u^2 + v^2}, \ \beta(p, y) = ku,$$
$$F(p, y) = \sqrt[3]{c_1(u^2 + v^2).ku + c_2(ku)^3}.$$

Assume that $y^\perp$ satisfies:

$$g_y(y, y^\perp) = 0, \ g_y(y^\perp, y^\perp) = F^2(p, y). \tag{2}$$

Obviously $y^\perp$ is unique because the metric is non-degenerate. The frame $\{y, y^\perp\}$ is called the Berwald frame. Let

$$y = rcos(\theta)e_1 + rsin(\theta)e_2.$$

i.e.,

$$u = rcos(\theta), v = rsin(\theta)$$

plugging the above expression into (2) and computing by Maple program [*See Appendix − I*] we obtain,

$$y^\perp = \frac{r(-\sqrt{2}c_1\sin(\theta)\cos(\theta), \ \frac{1}{\sqrt{2}}((2c_1 + 3c_2k^2)\cos(\theta)^2 + c_1))}{\sqrt{((4c_1 + 3c_2k^2)\cos(\theta)^2 - c_1)c_1}}. \tag{3}$$

By the definition of the bound of Cartan torsion, it is easy to show that for the Berwald frame $(y, y^{\perp})$,

$$\| C \|_p = Sup_{y \in T_pM_0} \xi(p, y),$$

where,

$$\xi(p, y) = \frac{F(p, y) \mid C_y(y^{\perp}, y^{\perp}, y^{\perp}) \mid}{\mid g_y(y^{\perp}, y^{\perp}) \mid^{3/2}}.$$

Again computing by Maple program $[See\ Appendix - III(i)]$, we obtain

$$\xi(p, y) = \frac{1}{\sqrt{2}} \mid \frac{c_1 sin(\theta)((8c_1 + 9c_2k^2)cos(\theta)^2 + c_1)}{[(4c_1 + 3c_2k^2)cos(\theta)^2 - c_1]^{\frac{3}{2}}} \mid .$$

Define two functions on $[0, 1] \times [-1, 1]$ by following,

$$f(k, x) = (4c_1 + 3c_2k^2)x^2 - c_1,$$

$$g(k, x) = \frac{1}{\sqrt{2}} \frac{c_1\sqrt{1 - x^2}((8c_1 + 9c_2k^2)x^2 + c_1)}{f(x, y)^{\frac{3}{2}}}.$$

Hence

$$\| C \|_p = Max_{0 \le \theta \le 2\pi} \mid g(k, cos\theta) \mid . \tag{4}$$

For a fixed $k = k_0$ $(k_0 \in [0, 1])$, we have

$$\frac{\partial}{\partial x} f(k_0, x) = 2(4c_1 + 3c_2k_0^2)x.$$

So from $\frac{\partial}{\partial x} f(k_0, x) = 0$ gives, $x = 0$. $i.e., x \in [-1, 1]$.
By simple computation, we get,

$$f(k_0, -1) = f(k_0, 1) = 3(c_1 + c_2k_0^2) > 0, \quad c_1, c_2 > 0\ and\ k_0 \in [0, 1].$$

Therefore $f(k, x) > 0$, then $g(k, x)$ is continuous in $[0, 1] \times [-1, 1]$ and has an upper bound.

In general for higher dimensions, the definition of the Cartan torsion is bound at $p \in M$ is

$$\| C \|_p = Sup_{y,u \in T_pM} \frac{F(p, y) \mid c_y(u, u, u) \mid}{\mid g_y(u, u) \mid^{\frac{3}{2}}}.$$

Considering the plane $P = span(u, y)$, from the above conclusion we obtain $\| C \|_p$ is bounded. Furthermore, the bound is independent of the plane $P \subset T_pM$ and the point $p \in M$. Hence the Cartan torsion is also bounded. This completes the proof. □

3.2. Bounded Carton Torsion for the metric $F = c_1\beta + \frac{\alpha^2}{\beta}$:

In this section, we consider the special (α, β)-metric $F = c_1\beta + \frac{\alpha^2}{\beta}$ where, $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric, $\beta = b_i(x)y^i$ is a 1-form on manifold M and $c_1 > 0$ be real constant, then we prove the following theorem:

Theorem 2. *Let $F = c_1\beta + \frac{\alpha^2}{\beta}$ be a special (α, β)-metric where, $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric, $\beta = b_i(x)y^i$ is a 1-form on manifold M and $c_1 > 0$ be real constant. Then F has bounded Cartan torsion.*

Proof. Let us first consider the case of $dimM = 2$, for an arbitrary tangent vector $y = ue_1 + ve_2 \in T_pM$, we can obtain that

$$\alpha(p, y) = \sqrt{u^2 + v^2}, \quad \beta(p, y) = ku,$$

$$F(p, y) = c_1ku + \frac{(u^2 + v^2)}{ku}.$$

By using the Maple program [$See\ Appendix - II$], we get

$$y^{\perp} = \frac{r(-\sqrt{2}c_1 \sin(\theta)\cos(\theta),\ \frac{1}{\sqrt{2}}(2c_1cos(\theta)^2 + 2cos(\theta)^2 - 1)}{\sqrt{c_1k^2cos(\theta)^2 + 1}}. \tag{5}$$

Again computing by Maple program [$See\ Appendix - III(ii)$], we obtain

$$\xi(p, y) = \frac{3}{\sqrt{2}} \mid \frac{sin\theta}{\sqrt{c_1k^2cos\theta^2 + 1}} \mid .$$

Define two functions on $[0, 1] \times [-1, 1]$ by following

$$f(k, x) = c_1k^2x^2 + 1,$$

$$g(k, x) = \frac{3}{\sqrt{2}} \mid \frac{\sqrt{1 - x^2}}{f(k, x)^{\frac{1}{2}}} \mid .$$

Hence

$$\| C \|_p = Max_{0\leq\theta\leq2\pi} \mid g(k, cos\theta) \mid . \tag{6}$$

For a fixed $k = k_0$ $(k_0 \in [0, 1])$, we have,

$$\frac{\partial}{\partial x} f(k_0, x) = 2c_1xk^2.$$

So from $\frac{\partial}{\partial x} f(k_0, x) = 0$, gives $x = 0$. $i.e., x \in [-1, 1]$.
By simple computation we get,

$$f(k_0, 1) = f(k_0, -1) = c_1k_0^2 + 1 > 0, \quad since\ \ c_1 > 0\ and\ k_0 \in [0, 1]$$

Therefore $f(k, x) > 0$, then $g(k, x)$ is continuous in $[0, 1] \times [-1, 1]$ and has an upper bound. For higher dimensions, it is similar to the above case. □

4 Examples

In this section, we will discuss some related examples and link our theorems to the results in [18] and also we use the following theorem proved by A. Teyabi and H. Sadeghi[12]:

Theorem 3. *Let $F = \alpha\phi(s)$ be a non-Riemannian (α,β)-metric on a manifold M of dimension $n \geq 3$. Then norm of Cartan and mean Cartan torsion of F satisfy the following relation*

$$\| C \| = \sqrt{\frac{3p^2 + 6pq + (n-1)q^2}{n+1}} \| I \| . \tag{7}$$

where $p = p(x,y)$ and $q = q(x,y)$ are scalar functions on TM satisfying $p + q = 1$ and given by following

$$p = \frac{n+1}{aA}[s(\phi\phi'' + \phi'\phi') - \phi\phi'], \tag{8}$$

$$a = \phi(\phi - s\phi'), \tag{9}$$

$$A = (n-2)\frac{s\phi''}{\phi - s\phi'} - (n+1)\frac{\phi'}{\phi} - \frac{(b^2 - s^2)\phi''' - 3s\phi''}{(b^2 - s^2)\phi'' + \phi - s\phi'}. \tag{10}$$

The Cartan tensor of an (α,β)-metric is given by the following

$$C_{ijk} := \frac{p}{1+n}\{I_k h_{ij} + I_i h_{jk} + I_j h_{ki}\} + \frac{q}{\| I \|^2} I_i I_j I_k. \tag{11}$$

where $p = p(x,y)$ and $q = q(x,y)$ are scalar functions on TM satisfying $p + q = 1$ and p is defined by (8).

Example 4.1: For a Finsler metric $F = \sqrt[3]{c_1\alpha^2\beta + c_2\beta^3}$ on a manifold M, we have

$$\phi = \sqrt[3]{c_1 s + c_2 s^3}$$

Then we get the following

$$a = \frac{2(c_1 s)}{\sqrt[3]{c_1 s + c_2 s^3}},$$

$$A = \frac{-2c_1[kb^2 - 4nc_1 s^2]}{3(c_1 s + c_2 s^3)lb^2 - 4c_1 s^2},$$

where

$$k = (n-3)c_1 - 3(n+1)c_2 s^2, \qquad l = c_1 - 3c_2 s^2.$$

Thus,

$$p = \frac{(n+1)(lb^2 - 4c_1 s^2)}{kb^2 - 4nc_1 s^2}, \tag{12}$$

where

$$k = (n-3)c_1 - 3(n+1)c_2 s^2, \qquad l = c_1 - 3c_2 s^2.$$

Then we get the following

Corollary 1. *Let $F = \sqrt[3]{c_1\alpha^2\beta + c_2\beta^3}$ be a special (α, β)-metric where $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric, $\beta = b_i(x)y^i$ is a 1-form on manifold M and $c_1, c_2 > 0$ are real constants. Then the relation between the norm of Cartan and the mean Cartan torsion of F satisfies (7), where p is given by (12).*

Example 4.2: For a Finsler metric $F = c_1\beta + \frac{\alpha^2}{\beta}$ on a manifold M, we have

$$\phi = c_1 s + \frac{1}{s}$$

Then we get the following

$$a = \frac{2(c_1 s^2 + 1)}{s^2},$$

$$A = \frac{2(n+1)}{s(c_1 s^2 + 1},$$

Thus,

$$p = 1. \tag{13}$$

Similar to corollary (1) we get the following

Corollary 2. *Let $F = c_1\beta + \frac{\alpha^2}{\beta}$ be a special (α, β)-metric where $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric, $\beta = b_i(x)y^i$ is a 1-form on manifold M and $c_1 > 0$ is a real constant. Then the relation between the norm of Cartan and the mean Cartan torsion of F satisfies (7), where p is given by (13).*

Remark: For the (α, β)-metric $F = c_1\beta + \frac{\alpha^2}{\beta}$, if dim $n = 2$ and $p = 1$ then, from (7) we get that the norm of Cartan is the same as the mean Cartan torsion i.e., $\| C \| = \| I \|$.

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Appendix – I

In this section we provide the Maple program which we used to prove Theorem 1

$> restart;$
$> with(linalg):$
$> F := sqrt[3]((c[1]*u^2+v^2)*(k*u)+c[2]*(k*u)^3):$
$> g := simplify(1/2*hessain(F^2,[u,v])):$
$> gr := simplify(subs(u=cos(theta), v=sin(theta), g)):$
$> y := vector(2,[r*cos(theta), r*sin(theta)]);$

$$y := [u = rcos(\theta), v = rsin(\theta)]$$

$> yp := vector(2) :$
$> eq := simplify(evalm(transpose(y) * gr * yp)) = 0 :$
$> x := solve(eq, yp[1]);$

$$x = -\frac{yp_2 sin(\theta) c[1] cos(\theta)}{2c[1]cos(\theta)^2 + 3c[2]k^2 cos(\theta)^2 + c[1]}$$

$> ny := simplify(r^2.subs(u = cos(theta), v = sin(theta), F^2));$

$$ny := r^2(k(c[2]k^2 cos(\theta)^2 + c[1])cos(\theta))^{2/3}$$

$> yp[1] := -2sin(\theta)cos(\theta)c[1] :$
$> yp[2] := 2cos(\theta)^2 c[1] + 3c[2]k^2 cos(\theta)^2 + c[1] :$
$> nyp := simplify(evalm(transpose(yp) * gr * yp)) :$
$> lambda := simplify(sqrt(r^2 * nyp/ny)/r) :$
$> yp[1] := yp[1]/lambda :$
$> yp[2] := yp[2]/lambda :$
$> print(yp);$

$$[-\frac{sin(\theta)cos(\theta)c[1]\sqrt{2}r}{\sqrt{(4c[1]cos(\theta)^2 - c[1] + 3c[2]k^2 cos(\theta)^2)c[1]}},$$
$$\frac{(2c[1]cos(\theta)^2 + 3c[2]k^2 cos(\theta)^2 + c[1])\sqrt{2}r}{2\sqrt{(4c[1]cos(\theta)^2) - c[1] + 3c[2]k^2 cos(\theta)^2 c[1]}}]$$

Appendix – II

Maple program to prove Theorem 2

$> restart;$
$> with(linalg) :$
$> F := c[1] * (k * u) + \frac{(u^2+v^2)}{(k*u)} :$
$> g := simplify(1/2 * hessain(F^2, [u, v]))$:
$> gr := simplify(subs(u = cos(theta), v = sin(theta), g)) :$
$> y := vector(2, [r * cos(theta), r * sin(theta)]);$

$$y := [u = rcos(\theta), v = rsin(\theta)]$$

$> yp := vector(2) :$
$> eq := simplify(evalm(transpose(y) * gr * yp)) = 0 :$
$> x := solve(eq, yp[1]);$

$$x = -\frac{yp_2 sin(\theta)cos(\theta)2}{c[1]cos(\theta)^2 k^2 + 2cos(\theta)^2 - 1}$$

$> ny := simplify(r^2.subs(u = cos(theta), v = sin(theta), F^2));$

$$ny := \frac{r^2(k^4 c[1]^2 cos(\theta)^4 + 2c[1]k^2 cos(\theta)^2 + 1)}{cos(\theta)^2 k^2}$$

$> yp[1] := -2sin(\theta)cos(\theta)$:
$> yp[2] := cos(\theta)^2 c[1]k^2 + 2cos(\theta)^2 - 1$:
$> nyp := simplify(evalm(transpose(yp) * gr * yp))$:
$> lambda := simplify(sqrt(r^2 * nyp/ny)/r)$:
$> yp[1] := yp[1]/lambda$:
$> yp[2] := yp[2]/lambda$:
$> print(yp)$;

$$[-\frac{sin(\theta)cos(\theta)\sqrt{2}r}{\sqrt{c[1]cos(\theta)^2k^2+1}}, \frac{(c[1]cos(\theta)^2k^2+2cos(\theta)^2-1)\sqrt{2}r}{2\sqrt{c[1]cos(\theta)^2k^2+1}}]$$

Appendix – III

The method of computation

Step 1: Solve the equation $g_y(y, y^{\perp}) = 0$.

$$(x, yp_{[2]}) = \left(\frac{yp_{[2]}yp_{[1]}}{yp_{[2]}}, yp_{[2]}\right)$$

and $yp = (yp_{[1]}, yp_{[2]})$ is a particular solution.

Step 2: Assume that $y^{\perp} = \frac{1}{\lambda}yp$ is the satisfied solution. Notice that

$$g_y(y^{\perp}, y^{\perp}) = F^2(y) = ny$$

Then we get

$$\lambda = \sqrt{\frac{nyp}{ny}}$$

which nyp is defined by

$$nyp := g_y(yp, yp)$$

Step 3: Plug these results into $y^{\perp}$, we get the Finsler frame $(y, y^{\perp})$.

i) Computation of $\xi(p, y)$ for the metric $F = \sqrt[3]{c_1\alpha^2\beta + c_2\beta^3}$

$> nyp = simplify(evalm(transpose(yp) * gr * yp))$;

$$nyp = \frac{r^2kcos(\theta)(c_2k^2cos(\theta)^2+c_1)}{(k(c_2k^2cos(\theta)^2+c_1)cos(\theta))^{1/3}}$$

$> bc = factor(abs(simplify(r^2 * subs(t = 0, q = 0, p = 0, diff(subs(u = cos(theta) + t * yp[1]/r + q * yp[1]/r + p * yp[1]/r, v = sin(theta) + t * yp[2]/r + q * yp[2]/r + p * yp[2]/r, F^2/4), [t, q, p])))/nyp))$

$$bc = \frac{sin(\theta)c_1(8c_1cos(\theta)^2+c_1+9c_2k^2cos(\theta)^2)}{\sqrt{2c_1}(4c_1cos(\theta)^2-c_1+3c_2k^2cos(\theta)^2)^{3/2}}$$

.

ii) Computation of $\xi(p, y)$ for the metric $F = c_1\beta + \frac{\alpha^2}{\beta}$

$> nyp = simplify(evalm(transpose(yp) * gr * yp));$

$$nyp = \frac{r^2(c_1^2 k^4 cos(\theta)^4 + 2c_1 k^2 cos(\theta)^2 + 1)}{k^2 cos(\theta)^2}$$

$> bc = factor(abs(simplify(r^2 * subs(t = 0, q = 0, p = 0, diff(subs(u = cos(theta) + t * yp[1]/r + q * yp[1]/r + p * yp[1]/r, v = sin(theta) + t * yp[2]/r + q * yp[2]/r + p * yp[2]/r, F^2/4), [t, q, p])))/nyp))$

$$bc = \frac{3}{\sqrt{2}} \frac{sin(\theta)}{\sqrt{c_1 k^2 cos(\theta)^2 + 1}}$$

.