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SOME DE RHAM COHOMOLOGY GROUPS ASSOCIATED TO A SUBFOLIATION

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Abstract

In this paper we consider a (q_1, q_2)-codimensional subfoliation (F_1, F_2) on a Riemannian manifold M. We give a decomposition of the exterior derivative with respect to this subfoliation. We identify two new de Rham cohomology groups associated to (F_1, F_2). These groups are topological invariants of manifold (M, F_1, F_2).

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1 Preliminares

Foliations arise as integral manifolds of systems of differential equations. They are closed related to the dynamical systems theory and could play an important role in other science fields, like physics and biology. In the last decades the study of geometrical and toplological aspects of foliations was an interest point of research. The case of subfoliations comes naturally and it is studied by L.A. Cordero, [2]. The cohomologies of foliated manifolds are studied by I. Vaisman, [7], [8], A. El Kacimi-Alaoui, [3], X. Masa, [4] and many others.

In this paper we consider a subfoliation (F_1, F_2) on a paracompact manifold M and we study in a classical way cohomologies related to it. We follow some ideas from [8] and obtain for the exterior derivative a decomposition with respect to subfoliation. One component is satisfying *Poincaré* type lemma in two certain situations, and we find two cohomology groups associated to (F_1, F_2). For these groups we prove de Rham theorems, so they are topological invariants of (M, F_1, F_2) (Theorems 4.1, 4.2).

2 Subfoliations

For a manifold M we denote by $\Omega^0(M)$ the ring of differentiable functions on M and by $\Omega^p(M)$ the module of p-forms. For a bundle E, $\Gamma(E)$ is the set of sections of F.

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In this section, following [2], we briefly recall the notion of a (q_1, q_2)-codimensional subfoliation on a manifold.

DEFINITION 2.1. *Let M be a n-dimensional manifold and TM its tangent bundle. A (q_1, q_2)-codimensional subfoliation on M is a couple (F_1, F_2) of integrable subbundles F_k of TM of dimension $n - q_k$, $k = 1, 2$, and F_2 being at the same time a subbundle of F_1.*

EXAMPLE 2.1. *The tangent manifold of a Finsler manifold admits a $(n, 2n-1)$-subfoliation, where F_1 is the vertical bundle and F_2 is generated by the Liouville vector field, [1].*

Such a subfoliation determines two foliations on M: $\mathcal{F}_1$ a $(n - q_1)$-dimensional foliation with structural bundle F_1 and $\mathcal{F}_2$, a $(n - q_2)$-dimensional foliation with structural bundle F_2. Moreover, every leaf of $\mathcal{F}_1$ has a $d = q_2 - q_1$-codimensional foliated structure determined by F_2.

We denote by $QF_k = TM/F_k$ the transversal bundle of foliation $\mathcal{F}_k$. For a Riemannian manifold (M, g), QF_k is isomorphic with the normal bundle of F_k. We have the following decompositions:

$$TM = F_1 \oplus QF_1, \quad TM = F_2 \oplus QF_2, \quad F_1 = F_2 \oplus QF_{21}, \tag{2.1}$$

where QF_{21} is the quotient bundle F_1/F_2. We also have the isomorphism $QF_2 \cong QF_{21} \oplus QF_1$.

So, some exact sequences of vector bundles

$$0 \longrightarrow F_2 \stackrel{i}{\longrightarrow} F_1 \stackrel{\pi}{\longrightarrow} QF_{21} \longrightarrow 0, \tag{2.2}$$

$$0 \longrightarrow F_1 \stackrel{i_1}{\longrightarrow} TM \stackrel{\pi_1}{\longrightarrow} QF_1 \longrightarrow 0, \tag{2.3}$$

$$0 \longrightarrow F_2 \stackrel{i_2}{\longrightarrow} TM \stackrel{\pi_2}{\longrightarrow} QF_2 \longrightarrow 0, \tag{2.4}$$

appear in a canonical way.

3 A decomposition of the exterior derivative

Let (M, g) be a Riemannian n-dimensional manifold, and (F_1, F_2) a (q_1, q_2)-codimensional subfoliation on it. From the classical theory of foliated manifolds, there is an atlas $\{(U, \varphi)\}$ adapted to (F_1, F_2), with local adapted coordinates

$$(x^i, x^a, x^u)_{1 \le i \le q_1 < a \le q_2 < u \le n},$$

such that in every domain U, leaves of $\mathcal{F}_1$ are defined by fixing the first q_1 coordinates and the leaves of $\mathcal{F}_2$ are defined by $x^i_\alpha = const.$ and $x^a_\alpha = const.$ For two adapted local charts $(U, (x^i, x^a, x^u))$, $(\bar{U}, (\bar{x}^{i_1}, \bar{x}^{a_1}, \bar{x}^{u_1})$ which domains overlap, in $U \cap \bar{U}$, there are the following relations:

$$\frac{\partial x^i}{\partial \bar{x}^{a_1}} = \frac{\partial x^i}{\partial \bar{x}^{u_1}} = \frac{\partial x^a}{\partial \bar{x}^{u_1}} = 0.$$

The local expression of the metric g is

$$\begin{pmatrix} (g_{ij}) & (g_{ia}) & (g_{iu}) \\ (g_{bj}) & (g_{ba}) & (g_{bu}) \\ (g_{vj}) & (g_{va}) & (g_{vu}) \end{pmatrix}_{1\leq i\leq j\leq q_1<a\leq b\leq q_2<u\leq v\leq n},$$

and matrices $(g_{vu})_{u,v}$, $\begin{pmatrix} (g_{ba}) & (g_{bu}) \\ (g_{va}) & (g_{vu}) \end{pmatrix}_{a,b,u,v}$ are non-degenerated.

For an adapted chart $(U,(x^i,x^a,x^u))$, the local coordinates on the plaque $U\cap\mathcal{F}_2$ are (x^u), so the bundle F_2 is locally spanned by $(\partial_u=\frac{\partial}{\partial x^u})_{q_2<u\leq n}$. Let us denote

$$\delta_a=\pi_2(\frac{\partial}{\partial x^a}),$$

the projection of vector field $\frac{\partial}{\partial x^a}$ on the normal bundle QF_2, for every $a=\overline{q_1+1,q_2}$. Since $\delta_a-\frac{\partial}{\partial x^a}$ belongs to F_2, there are the local differential functions $t_a^u\in\Omega^0(U)$, given by $g_{au}-t_a^u g_{uv}=0$, such that

$$\delta_a=\frac{\partial}{\partial x^a}-t_a^u\frac{\partial}{\partial x^u}, \tag{3.1}$$

where we use the Einstein convention for summation.

Local coordinates on the plaque $U\cap\mathcal{F}_1$ are (x^a,x^u), so the bundle F_1 is locally spanned by $(\frac{\partial}{\partial x^a},\frac{\partial}{\partial x^u})_{q_1<a\leq q_2<u\leq n}$. Let us denote

$$\delta_i=\pi_1(\frac{\partial}{\partial x^i}),$$

the projection of $\frac{\partial}{\partial x^i}$ on the normal bundle QF_1, for every $i=\overline{1,q_1}$.

Since $\delta_i-\frac{\partial}{\partial x^i}$ belongs to F_1, there are the local differential functions $t_i^a,t_i^u\in\Omega^0(U)$ such that

$$\delta_i=\frac{\partial}{\partial x^i}-t_i^a\frac{\partial}{\partial x^a}-t_i^u\frac{\partial}{\partial x^u}. \tag{3.2}$$

The functions t_i^a,t_i^u are satisfying the orthogonality conditions $g(\delta_i,\frac{\partial}{\partial x^k})=0$, $\forall q_1<k\leq n$:

$$g_{ia}-t_i^b g_{ba}-t_i^u g_{ua}=0,\quad g_{iu}-t_i^b g_{bu}-t_i^v g_{vu}=0.$$

On the intersections of adapted charts $(U,(x^i,x^a,x^u))$, $(\bar U,(\bar x^{i_1},\bar x^{a_1},\bar x^{u_1}))$ the functions t_a^u,t_i^a,t_i^u change in the following way:

$$\bar t_{a_1}^{u_1}\frac{\partial x^u}{\partial\bar x^{u_1}}=\frac{\partial x^u}{\partial\bar x^{a_1}}+t_a^u\frac{\partial x^a}{\partial\bar x^{a_1}}, \tag{3.3}$$

$$\bar t_{i_1}^{a_1}\frac{\partial x^a}{\partial\bar x^{a_1}}=\frac{\partial x^a}{\partial\bar x^{i_1}}+t_i^a\frac{\partial x^i}{\partial\bar x^{i_1}}, \tag{3.4}$$

$$\bar t_{i_1}^{a_1}\frac{\partial x^u}{\partial\bar x^{a_1}}+\bar t_{i_1}^{u_1}\frac{\partial x^u}{\partial\bar x^{u_1}}=\frac{\partial x^u}{\partial\bar x^{i_1}}+t_i^u\frac{\partial x^i}{\partial\bar x^{i_1}}, \tag{3.5}$$

since we have

$$\bar{\delta}_{a_1} = \frac{\partial x^a}{\partial \bar{x}^{a_1}}\delta_a, \quad \bar{\delta}_{i_1} = \frac{\partial x^i}{\partial \bar{x}^{i_1}}\delta_i.$$

We obtained in this way the local basis

$$\{\delta_i, \delta_a, \partial_u\}, \tag{3.6}$$

of TM, adapted to (F_1, F_2), where the vector fields $\{\delta_i\}_i$ spanned a complementary distribution to the structural distribution of $\mathcal{F}_1$ in TM, and $\{\delta_i, \delta_a\}_{i,a}$ spanned a complementary distribution to the structural distribution of $\mathcal{F}_2$ in TM.

Let $\{dx^i, \omega^a, \theta^u\}$ be the adapted cobasis, dual of (3.6). By a straightforward computation, we obtain:

$$\omega^a = dx^a + t_i^a dx^i, \tag{3.7}$$

$$\theta^u = dx^u + t_a^u dx^a + (t_i^u + t_i^a t_a^u) dx^i,$$

or, equivalent,

$$\theta^u = dx^u + t_a^u \omega^a + t_i^u dx^i. \tag{3.8}$$

The relations (3.3), (3.4), (3.5) show that on the intersections of adapted charts $(U, (x^i, x^a, x^u))$, $(\bar{U}, (\bar{x}^{i_1}, \bar{x}^{a_1}, \bar{x}^{u_1})$,

$$\bar{\omega}^{a_1} = \frac{\partial \bar{x}^{a_1}}{\partial x^a}\omega^a; \quad \bar{\theta}^{u_1} = \frac{\partial \bar{x}^{u_1}}{\partial x^u}\theta^u.$$

Ones can see that now, locally, we have

$$\begin{array}{lcl} \mathcal{F}_1 & : & \qquad dx^i = 0, \\ \mathcal{F}_2 & : & dx^i = 0, \quad \omega^a = 0. \end{array}$$

We also obtain the relations:

$$d\omega^a = \delta_j t_i^a dx^j \wedge dx^i + \delta_b t_i^a \omega^b \wedge dx^i + \partial_u t_i^a \theta^u \wedge dx^i, \tag{3.9}$$

$$d\theta^u = \delta_j t_i^u dx^j \wedge dx^i + (\delta_b t_i^u - \delta_i t_b^u)\omega^b \wedge dx^i + \delta_b t_a^u \omega^b \wedge \omega^a + \partial_v t_j^u \theta^v \wedge dx^j + \\ + \partial_v t_a^u \theta^v \wedge \omega^a + t_a^u d\omega^a,$$

$$\begin{aligned} d\theta^u = (\delta_i t_j^u - t_a^u \delta_j t_i^a) dx^i \wedge dx^j + (\delta_i t_a^u - \delta_a t_i^u - t_b^u \delta_a t_i^b) dx^i \wedge \omega^a - \\ -(\partial_v t_i^u + t_a^u \partial_v t_i^a) dx^i \wedge \theta_v + \delta_b t_a^u \omega^b \wedge \omega^a - \partial_v t_a^u \omega^a \wedge \theta^v. \end{aligned} \tag{3.10}$$

On the foliated manifold $(M, \mathcal{F}_1)$, [7], a (p, s)-form is a $(p+s)$-form ω on M such that $\omega(X_1, ...X_{p+s})$ could be non-zero only if exactly p arguments are sections of QF_1 and s arguments are vector fields from $\Gamma(F_1)$. We denote by $\Omega^{p,s}_{\mathcal{F}_1}(M)$ the module of (p, s)-forms on $(M, \mathcal{F}_1)$.

On the foliated manifold $(M, \mathcal{F}_2)$, [7], a (t, r)-form is a $(t+r)$-form ω on M such that $\omega(X_1, ...X_{t+r})$ could be non-zero only if exactly t arguments are sections of QF_2 and r arguments are vector fields from $\Gamma(F_2)$. We denote by $\Omega^{t,r}_{\mathcal{F}_2}(M)$ the module of (t, r)-forms on $(M, \mathcal{F}_2)$.

DEFINITION 3.1. *We call a (p,q,r)-form of $(M,\mathcal{F}_1,\mathcal{F}_2)$ a $(p+q+r)$-form ω on M such that could be non-zero only if exaclty p of its arguments are from $\Gamma(QF_1)$, q of its arguments are from $\Gamma(QF_{21})$ and r arguments are from $\Gamma(F_2)$.*

We denote by $\Omega^{p,q,r}(M)$ the module of (p,q,r)-form of the manifold (M,F_1,F_2).

REMARK 3.1. *A (p,q,r)-form belongs also to $\Omega^{p,q+r}_{\mathcal{F}_1}(M)$ and to $\Omega^{p+q,r}_{\mathcal{F}_2}(M)$, at the same time. Moreover, we have*

$$\Omega^{p,s}_{\mathcal{F}_1}(M)=\oplus_{q+r=s}\Omega^{p,q,r}(M),\quad \Omega^{t,r}_{\mathcal{F}_2}(M)=\oplus_{p+q=t}\Omega^{p,q,r}(M).$$

General theory of foliated manifolds, [7], assures that the exterior derivative d of M admits the following decompositions:

$$d=d^{\mathcal{F}_1}_{10}+d^{\mathcal{F}_1}_{2,-1}+d^{\mathcal{F}_1}_{01},\quad d=d^{\mathcal{F}_2}_{10}+d^{\mathcal{F}_2}_{2,-1}+d^{\mathcal{F}_2}_{01},\tag{3.11}$$

where

$$d^{\mathcal{F}_1}_{10}:\Omega^{p,s}_{\mathcal{F}_1}(M)\to\Omega^{p+1,s}_{\mathcal{F}_1}(M),\quad d^{\mathcal{F}_1}_{2,-1}:\Omega^{p,s}_{\mathcal{F}_1}(M)\to\Omega^{p+2,s-1}_{\mathcal{F}_1}(M),$$
$$d^{\mathcal{F}_1}_{01}:\Omega^{p,s}_{\mathcal{F}_1}(M)\to\Omega^{p,s+1}_{\mathcal{F}_1}(M),$$
$$d^{\mathcal{F}_2}_{10}:\Omega^{t,r}_{\mathcal{F}_2}(M)\to\Omega^{t+1,r}_{\mathcal{F}_2}(M),\quad d^{\mathcal{F}_2}_{2,-1}:\Omega^{t,r}_{\mathcal{F}_2}(M)\to\Omega^{t+2,r-1}_{\mathcal{F}_2}(M),$$
$$d^{\mathcal{F}_2}_{01}:\Omega^{t,r}_{\mathcal{F}_2}(M)\to\Omega^{t,r+1}_{\mathcal{F}_2}(M).$$

Now, let ω be a (p,q,r)-form of $(M,\mathcal{F}_1,\mathcal{F}_2)$. Its local form with respect to local cobasis (dx^i,ω^a,θ^u), in a local adapted chart with domain V is

$$\omega=\omega_{IAU}dx^I\wedge\omega^A\wedge\theta^U,$$

where $\omega_{IAU}\in\Omega^0(V)$ and we denoted $I=(i_1,...,i_p)$, $A=(a_1,...,a_q)$, $U=(u_1,...,u_r)$,

$$1\le i_1<i_2<...<i_p\le q_1<a_1<...<a_q\le q_2<u_1<...<u_r\le n,$$
$$dx^I=dx^{i_1}\wedge...\wedge dx^{i_p},\quad \omega^A=\omega^{a_1}\wedge...\wedge\omega^{a_q},\quad \theta^U=\theta^{u_1}\wedge...\wedge\theta^{u_r}.$$

We compute

$$\begin{array}{c} d\omega=\delta_i\omega_{IAU}dx^i\wedge dx^I\wedge\omega^A\wedge\theta^U+\delta_a\omega_{IAU}\omega^a\wedge dx^I\wedge\omega^A\wedge\theta^U+\\ +\partial_u\omega_{IAU}\theta^u\wedge dx^I\wedge\omega^A\wedge\theta^U+\\ +\sum_{k=1}^q(-1)^{p+k-1}\omega_{IAU}dx^I\wedge\omega^{a_1}\wedge...\wedge d\omega^{a_k}\wedge...\wedge\omega^{a_q}\wedge\theta^U+\\ +\sum_{k=1}^r(-1)^{p++qk-1}\omega_{IAU}dx^I\wedge\omega^A\wedge\theta^{u_1}\wedge...\wedge d\theta^{u_k}\wedge...\wedge\theta^{u_r}. \end{array}\tag{3.12}$$

Using in (3.12) relations (3.9), (3.10), we obtain the decomposition of the exterior derivative d into eight operators:

$$d=d_{1,0,0}+d_{2,-1,0}+d_{1,-1,1}+d_{0,1,0}+d_{2,0,-1}+d_{1,1,-1}+d_{0,2,-1}+d_{0,0,1},\tag{3.13}$$

with

$$d_{\alpha,\beta,\gamma} : \Omega^{p,q,r}(M) \to \Omega^{p+\alpha,q+\beta,r+\gamma}(M), \quad \alpha+\beta+\gamma=1, \quad \alpha,\beta,\gamma \in \{\pm 1,0,2\}.$$

The local form of these operators could be determined by a straightforward computation. For example,

$$d_{0,0,1}\omega = \frac{1}{(q-1)!}(-1)^{p+q}\delta^{uu_1\ldots u_r}_{v_1v_2\ldots v_{r+1}}\delta_u\omega_{IAU}dx^I\wedge\omega^A\wedge\theta^{v_1}\wedge\ldots\wedge\theta^{v_{r+1}},$$

$$d_{1,0,0}\omega = \frac{1}{(p-1)!}\delta^{ii_1\ldots i_p}_{j_1j_2\ldots j_{p+1}}(\delta_i\omega_{IAU} - \frac{1}{q!}\sum_{k=1}^{q}\delta^{a_1\ldots a_{k-1}ba_{k+1}\ldots a_q}_{b_1b_2\ldots b_q}\omega_{IAU}\delta_b t^{a_k}_i -$$

$$-\frac{1}{r!}\sum_{j=1}^{r}\omega_{IAU}\delta^{u_1\ldots u_{j-1}vu_{j+1}\ldots u_r}_{v_1v_2\ldots v_r}(\partial_v t^{u_j}_i + t^{u_j}_a\partial_v t^a_i))dx^{j_1}\wedge\ldots\wedge dx^{j_{p+1}}\wedge\omega^{b_1}\wedge\ldots\omega^{b_q}\wedge$$

$$\wedge\theta^{v_1}\wedge\ldots\wedge\theta^{v_r},$$

$$d_{1,-1,1}\omega = \frac{(-1)^{p+q}}{(p+1)!(q-1)!(r+1)!}\delta^{ji_1\ldots i_p}_{j_1\ldots j_{p+1}}\delta^{uu_1\ldots u_r}{}_{v_1\ldots v_{r+1}}\omega_{Iab_1\ldots b_{q-1}U}\partial_u t^a_j$$

$$dx^{j_1}\wedge\ldots\wedge dx^{j_{p+1}}\wedge\omega^{b_1}\wedge\ldots\wedge\omega^{b_{q-1}}\wedge\theta^{v_1}\wedge\ldots\wedge\theta^{v_{r+1}}.$$

Hence, the restrictions to $\Omega^{p,q,r}(M)$ of operators from (3.11) are:

$$d^{\mathcal{F}_1}_{01}|_{\Omega^{p,q,r}(M)} = d_{0,1,0} + d_{0,2,-1} + d_{0,0,1}, \tag{3.14}$$

$$d^{\mathcal{F}_1}_{10}|_{\Omega^{p,q,r}(M)} = d_{1,0,0} + d_{1,-1,1} + d_{1,1,-1}, \quad d^{\mathcal{F}_1}_{2,-1}|_{\Omega^{p,q,r}(M)} = d_{2,-1,0} + d_{2,0,-1},$$

$$d^{\mathcal{F}_2}_{01}|_{\Omega^{p,q,r}(M)} = d_{1,-1,1} + d_{0,0,1}, \tag{3.15}$$

$$d^{\mathcal{F}_2}_{10}|_{\Omega^{p,q,r}(M)} = d_{1,0,0} + d_{0,1,0} + d_{2,-1,0}, \quad d^{\mathcal{F}_2}_{2,-1}|_{\Omega^{p,q,r}(M)} = d_{1,1,-1} + d_{0,2,-1} + d_{2,0,-1}.$$

It is well-known, [7] that $d^{\mathcal{F}_1}_{01}$, $d^{\mathcal{F}_2}_{01}$ are the exterior derivatives on the leaves of $(M,\mathcal{F}_1)$ and $(M,\mathcal{F}_2)$, respectively. They are satisfying

$$(d^{\mathcal{F}_1}_{01})^2 = 0, \quad (d^{\mathcal{F}_2}_{01})^2 = 0,$$

which give us

$$d^2_{0,0,1} = 0. \tag{3.16}$$

Moreover, for every $\omega \in \Omega^{p,q,r}(M)$, we have

$$d^{\mathcal{F}_2}_{01}\omega \equiv d_{0,0,1}\omega(mod \quad dx^1,\ldots,dx^{q_1}), \tag{3.17}$$

$$d^{\mathcal{F}_2}_{2,-1}\omega \equiv d_{0,2,-1}\omega(mod \quad dx^1,\ldots,dx^{q_1}), \quad d^{\mathcal{F}_2}_{10}\omega \equiv d_{0,1,-1}\omega(mod \quad dx^1,\ldots,dx^{q_1}),$$

so in every leaf $\mathcal{L}$ of $\mathcal{F}_1$, relation (3.14) is exactly the decomposition of the exterior derivative $d^{\mathcal{F}_1}_{01}$ of foliated manifold $(\mathcal{L},\mathcal{F}_2|_{\mathcal{L}})$.

4 Two de Rham Cohomology groups associated to $d_{0,0,1}$

The relation (3.16) gives the semiexact sequence of shaves

$$0 \to \Phi^{p,q} \xrightarrow{i} \Omega^{p,q,0} \xrightarrow{d_{0,0,1}} \Omega^{p,q,1} \xrightarrow{d_{0,0,1}} \dots . \xrightarrow{d_{0,0,1}} \Omega^{p,q,n-q_2} \to 0, \qquad (4.1)$$

where $\Omega^{p,q,r}$ is the sheaf of germs of (p,q,r)-forms, $\Phi^{p,q}$ is the sheaf of germs of $(p,q,0)$-forms ω which are satisfying $d_{0,0,1}\omega = 0$ and i is the canonical inclusion.

In this section we shall prove that operator $d_{0,0,1}$ satisfies a *Poincaré* type lemma, in the case $p = 0$ or $q = 0$, using the similar property of the leafwise derivatives $d_{01}^{\mathcal{F}_1}$, $d_{01}^{\mathcal{F}_2}$.

PROPOSITION 4.1. *For every $\omega \in \Omega^{0,q,r}(M)$ such that $d_{0,0,1}\omega = 0$ in a neighborhood U, there is $\theta \in \Omega^{0,q,r-1}(U)$ with $\omega = d_{0,0,1}\theta$ in U.*

Proof. Let ω be a $(0,q,r)$ form with $d_{0,0,1}\omega = 0$ in U. Using Remark 3.1 we have $\omega \in \Omega_{\mathcal{F}_2}^{q,r}(M)$ and by (3.17) it results

$$d_{01}^{\mathcal{F}_2}\omega \equiv 0 (mod \quad dx^1, ..., dx^{q_1}).$$

Since $d_{01}^{\mathcal{F}_2}$ satisfies a *Poincaré* type lemma, [7], there is $\theta \in \Omega_{\mathcal{F}_2}^{q,r-1}(U)$ with

$$\omega \equiv d_{01}^{\mathcal{F}_2}\theta (mod \quad dx^1, ..., dx^{q_1}),$$

in U.

By Remark 3.1, there are s, t naturals such that $s + t = q$ and $\theta \in \Omega^{s,t,r-1}(U)$. From relation (3.15) we have

$$\omega \equiv d_{1,-1,1}\theta + d_{0,0,1}\theta (mod \quad dx^1, ..., dx^{q_1}). \qquad (4.2)$$

Taking into account that $\omega \in \Omega^{0,q,r}(M)$, $d_{1,-1,1}\theta \in \Omega^{s+1,t-1,r}(U)$, $d_{0,0,1}\theta \in \Omega^{s,t,r}(U)$, if we identify the $(0,q,r)$-form from the both members of (4.4), it results $s = 0$ and, in U,

$$\omega = d_{0,0,1}\theta.$$

□

We denote by $Z^{0,q,r}(M)$ the space of forms $\omega \in \Omega^{0,q,r}(M)$, with $d_{0,0,1}\omega = 0$ and we call the *de Rham* $(0,q,r)$*-cohomology group* of M the quotient group

$$H^{0,q,r}(M) = \frac{Z^{0,q,r}(M)}{d_{0,0,1}\Omega^{o,q,r-1}(M)}.$$

Proposition 4.1 says that the sequence

$$0 \to \Phi^{0,q} \xrightarrow{i} \Omega^{0,q,0} \xrightarrow{d_{0,0,1}} \Omega^{0,q,1} \xrightarrow{d_{0,0,1}} \dots . \xrightarrow{d_{0,0,1}} \Omega^{0,q,n-q_2} \to 0, \qquad (4.3)$$

is a fine resolution of the sheaf $\Phi^{0,q}$. Using now a well-known theorem of algebraic topology, (see [5] Theorem 3.5, p205), we obtain:

THEOREM 4.1. *The de Rham $(0,q,r)$-cohomology group of M is isomorphic with the r-dimensional Čech cohomology group of M, with coefficients in the sheaf $\Phi^{0,q}$:*

$$H^{0,q,r}(M) \approx \mathcal{H}^r(M, \Phi^{0,q}).$$

REMARK 4.1. *1. By the above theorem, $H^{0,q,r}(M)$ is a topological invariant of the manifold M.*

2. Ones can see that the restriction of the sequence (4.3) to a leaf $\mathcal{L}$ of the foliation $\mathcal{F}_1$ is a fine resolution of the sheaf of germs of basic q-forms in the foliated manifold $(\mathcal{L}, \mathcal{F}_2)$.

In order to find another topological invariant of $(M, \mathcal{F}_1, \mathcal{F}_2)$, let $\Phi^{p,0}$ be the sheaf of germs of $(p,0,0)$-forms ω with $d_{0,0,1}\omega = 0$.

PROPOSITION 4.2. *For every $\omega \in \Omega^{p,0,r}(M)$ such that $d_{0,0,1}\omega = 0$ in a neighborhood U, there is $\theta \in \Omega^{p,0,r-1}(U)$ with $\omega = d_{0,0,1}\theta$ in U.*

Proof. Let ω be a $(p,0,r)$ form with $d_{0,0,1}\omega = 0$ in U. Every $\omega \in \Omega^{p,0,r}$ belongs also to $\Omega^{p,r}_{\mathcal{F}_1}(M)$, (see Remark 3.1). By (3.14) we have

$$d^{\mathcal{F}_1}_{01}\omega = d_{0,1,0}\omega + d_{0,2,-1}\omega + d_{0,0,1}\omega,$$

so we can see

$$d^{\mathcal{F}_1}_{01}\omega \equiv 0 (mod \quad \omega^{q_1+1}, ..., \omega^{q_2}).$$

Since $d^{\mathcal{F}_1}_{01}$ satisfies a *Poincaré* type lemma, [7], there is $\theta \in \Omega^{p,r-1}_{\mathcal{F}_1}(U)$ with

$$\omega \equiv d^{\mathcal{F}_1}_{01}\theta (mod \quad \omega^{q_1+1}, ..., \omega^{q_2}),$$

in U.

By Remark 3.1, there are s, t naturals such that $s+t = r-1$ and $\theta \in \Omega^{p,s,t}(U)$. From relation (3.14) we have

$$\omega \equiv d_{0,1,0}\theta + d_{0,2,-1}\theta + d_{0,0,1}\theta (mod \quad \omega^{q_1+1}, ..., \omega^{q_2}).$$

Taking into account that $\omega \in \Omega^{p,0,r}(M)$, $d_{0,1,0}\theta \in \Omega^{p,s+1,t}(U)$, $d_{0,2,-1}\theta \in \Omega^{p,s+2,t-1}(U)$ and $d_{0,0,1}\theta \in \Omega^{p,s,t+1}(U)$, if we identify the $(p,0,r)$-components from the both members, it results $s = 0$ and, in U,

$$\omega = d_{0,0,1}\theta (mod \quad \omega^{q_1+1}, ..., \omega^{q_2}).$$

We obtain $\omega = d_{0,0,1}\theta + \sum_{i=q_1+1}^{q_2} \beta_i \wedge \omega^i$. Taking again into account that ω is a $(p,0,r)$-form and θ is a $(p,0,r-1)$-form, it results $\omega = d_{0,0,1}\theta$ in U. □

We denote by $Z^{p,0,r}(M)$ the space of forms $\omega \in \Omega^{p,0,r}(M)$, with $d_{0,0,1}\omega = 0$ and we call the *de Rham $(p,0,r)$-cohomology group* of M the quotient group

$$H^{p,0,r}(M) = \frac{Z^{p,0,r}(M)}{d_{0,0,1}\Omega^{p,0,r-1}(M)}.$$

Proposition 4.2 says that the sequence

$$0 \to \Phi^{p,0} \stackrel{i}{\to} \Omega^{p,0,0} \stackrel{d_{0,0,1}}{\to} \Omega^{p,0,1} \stackrel{d_{0,0,1}}{\to} .... \stackrel{d_{0,0,1}}{\to} \Omega^{p,0,n-q_2} \to 0, \qquad (4.4)$$

is a fine resolution of the sheaf $\Phi^{p,0}$. Using now the same well-known theorem of algebraic topology, (see [5] Theorem 3.5, p205), we obtain:

THEOREM 4.2. *The de Rham $(p,0,r)$-cohomology group of M is isomorphic with the r-dimensional Čech cohomology group of M, with coefficients in the sheaf $\Phi^{p,0}$:*

$$H^{p,0,r}(M) \approx \mathcal{H}^r(M, \Phi^{p,0}).$$

REMARK 4.2. *Ones can see that $H^{p,0,r}(M)$ is a subgroup of the de Rham (p,r)-cohomology group of $(M, \mathcal{F}_2)$, since $Z^{p,0,r}(M) \subseteq Z^{p,r}_{\mathcal{F}_2}(M)$ and the inclusion map $i_r : \Omega^{p,0,r}(M) \to \Omega^{p,r}_{\mathcal{F}_2}(M)$ satisfies $i_r \circ d_{0,0,1} = d^{\mathcal{F}_2}_{01} \circ i_{r-1}$.*

Indeed, for $\omega \in Z^{p,0,r}(M)$, we have $\omega \in Z^{p,r}_{\mathcal{F}_2}(M)$ and $d^{\mathcal{F}_2}_{01}\omega = d_{0,0,1}\omega$, since $d_{1,-1,1}$ is not defined on $\Omega^{p,0,r}(M)$. For any $\theta \in \Omega^{p,0,r-1}(M)$,

$$i_r(d_{0,0,1}\theta) = d_{0,0,1}\theta = d^{\mathcal{F}_2}_{01}\theta.$$

The map i_r induces a group morphism

$$i_{r,*} : H^{p,0,r}(M) \to H^{p,r}_{\mathcal{F}_2}(M), \quad i_{r,*}[\omega] = [i_r(\omega)]_{\mathcal{F}_2}, \quad \forall[\omega] \in H^{p,0,r}(M).$$

This morphism is injective because for every $[\omega], [\lambda] \in H^{p,0,r}(M)$ such that $i_r(\omega) - i_r(\lambda) = d^{\mathcal{F}_2}_{01}\theta$ for some $\theta \in \Omega^{p,r-1}(M)$, it reults

$$\omega - \lambda = d_{0,0,1}\theta + d_{1,-1,1}\theta,$$

which gives $\theta \in \Omega^{p,0,r-1}(M)$ and $\omega - \lambda = d_{0,0,1}\theta$.

REMARK 4.3. *The cohomology group $H^{0,q,r}(M)$ is not included in $H^{q,r}_{\mathcal{F}_2}(M)$ because for $\omega \in Z^{0,q,r}(M)$, we have $d^{\mathcal{F}_2}_{01}\omega = d_{1,-1,1}\omega$, since $d_{1,-1,1}$ is defined on $\Omega^{0,q,r}(M)$.*

5 The case of Finsler manifolds

Let (M, F) be a n-dimensional Finsler manifold and G be the Sasaki-Finsler metric on its slit tangent manifold TM^0. The vertical bundle VTM^0 of TM^0 is the tangent (structural) bundle to vertical foliation F_V determined by the fibers of $\pi : TM^0 \to M$. If $(x^i, y^i)_{i=\overline{1,n}}$ are local coordinates on TM^0, then VTM^0 is locally spanned by $\{\frac{\partial}{\partial y^i}\}_i$. A canonical transversal (also called horizontal) distribution is constructed, [1], using a nonlinear connection G^i, so there exists on TM^0 a n distribution HTM^0 locally spanned by the vector fields

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - G^j_i \frac{\partial}{\partial y^j}, \quad (\forall) i = \overline{1,n}. \qquad (5.5)$$

The Riemannian metric G on TM^0 is satisfying

$$G(\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^j})=G(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j})=g_{ij},\quad G(\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^j})=0,\quad (\forall)i,j. \tag{5.6}$$

The local basis $\{\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^i}\}_i$ is called *adapted* to vertical foliation F_V and we have the decomposition

$$TTM^0=HTM^0\oplus VTM^0. \tag{5.7}$$

Now, let Z be the vertical Liouville vector field on TM^0,

$$Z=y^i\frac{\partial}{\partial y^i}, \tag{5.8}$$

which is globally defined, and let L be the space of line fields spanned by Z. We call this space *the Liouville distribution* on TM^0. The complementary orthogonal distributions to L in VTM^0 and TTM^0 are denoted by L' and $L^{\perp}$, respectively. It is proved, [1], that the both distributions L' and $L^{\perp}$ are integrable and we also have the decomposition

$$VTM^0=L'\oplus L. \tag{5.9}$$

Hence we can identify the $(n,2n-1)$ subfoliation (VTM^0,L) of the Riemannian manifold (TM^0,G). Here, $F_1=VTM^0$, $QF_1=HTM^0$, $F_2=L$, $QF_2=L^{\perp}$, $QF_{21}=L'$.

As we already seen, the local basis in TTM^0 is $\{\frac{\delta}{\delta x^i},\frac{\partial}{\partial y^i}\}_{i=\overline{1,n}}$, where the vertical bundle is locally spanned by $\{\frac{\partial}{\partial y^i}\}_{i=\overline{1,n}}$. The dual cobasis is $\{dx^i,\delta y^i=dy^i+G^i_jdx^j\}$.

We consider the vertical vector fields $X_i=\frac{\partial}{\partial y^i}-t_iZ$, with $G(X_i,Z)=0,\forall i=\overline{1,n}$. Since $G(Z,Z)=F^2$, we have $t_i=\frac{1}{F}\frac{\partial F}{\partial y^i}$. A local basis adapted to decomposition (5.9) is $\{\delta_a,Z\}_{a=\overline{n+1,2n-1}}$, where $\{\delta_a,\}_{a=\overline{n+1,2n-1}}$ are $(n-1)$ independent vector fields from $\{X_1,X_2,...,X_n\}$. So, we obtained the local basis $\{\frac{\delta}{\delta x^i},\delta_a,Z\}$ in TTM^0, adapted to subfoliation (VTM^0,L). The vertical global 1-form $\theta_0=t_i\delta y^i$ satisfies $\theta_0(Z)=1$ and $\theta_0(\delta_a)=0$, so we have the dual cobasis $\{dx^i,\omega^a,\theta_0\}$. The exact sequence (4.3) becomes

$$0\to\Phi^{0,q}\stackrel{i}{\to}\Omega^{0,q,0}\stackrel{d_{0,0,1}}{\to}\Omega^{0,q,1}\stackrel{d_{0,0,1}}{\to}0, \tag{5.10}$$

and $\Phi^{0,q}$ is the sheaf of germs of $(0,q,0)$-forms $\omega=f_{a_1...a_q}\omega^{a_1}\wedge\omega^{a_2}\wedge...\wedge\omega^{a_q}$ with $f_{a_1...a_q}\in\Omega^0(TM^0)$ such that $Zf_{a_1...a_q}=0$.

The sequence (4.4) becomes

$$0\to\Phi^{p,0}\stackrel{i}{\to}\Omega^{p,0,0}\stackrel{d_{0,0,1}}{\to}\Omega^{p,0,1}\stackrel{d_{0,0,1}}{\to}0, \tag{5.11}$$

where $\Phi^{p,0}$ is the sheaf of germs of $(p,0,0)$-forms $\omega=f_{i_1i_2...i_p}dx^{i_1}\wedge dx^{i_2}\wedge...\wedge dx^{i_p}$ with $f_{i_1i_2...i_p}\in\Omega^0(TM^0)$ such that $Zf_{i_1i_2...i_p}=0$.

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