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EQUATION GEODESIC IN A TWO-DIMENSIONAL FINSLER SPACE WITH SPECIAL (α, β)-METRIC

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Abstract

In the year 1997 and 1998 Matsumoto And Park obtained the equations of geodesic in a two-dimensional Randers, Kropina and Matsumoto space. In 2011, Chaubey, Prasad and Tripathi obtained the equation of geodesic for a more general (α, β)-metric as compared to Randers, Kropina and Matsumoto mertric. In the continuation of the above paper, here we have found out the equation of geodesic for the well known metric $L = \alpha + \frac{\beta^2}{\alpha}$, $L = \frac{\beta^2}{\alpha}$ and special Matsumoto metric $L = L = \frac{\beta^2}{(\beta - \alpha)}$. The main results are illustrated in the different figures.

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1 Indroduction

In 1997 Matsumoto and Park [6] obtained the equation of geodesic in two-dimensional Finsler spaces with the Randers metric $(L = \alpha + \beta)$ and the Kropina metric $(L = \frac{\alpha^2}{\beta})$ whereas in 1998 they [7] obtained the equation of geodesic in two-dimensional Finsler space with the slope metrics, i.e. Matsumoto metric given by $(L = \frac{\alpha^2}{(\alpha - \beta)})$. In 2011 Chaubey, Prasad and Tripathi [2] obtained the equation of geodesic for a more general (α, β)-metric $(L = \frac{k_1\alpha^2 + k_2\alpha\beta + k_3\beta^2}{a_1\alpha + a_2\beta}$ where $a's$ and $k's$ are constants) by considering β as an infinitesimal of degree one and neglecting infinitesimals of degree more or equal to two they obtained the geodesics of two-dimensional Finsler space in the form $y'' = f(x, y, y')$, where (x, y) are co-ordinate of two-dimensional Finsler space.

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In the present paper we have shown that under the same conditions, the geodesic of the two-dimensional space with following metrics:

$$L = \alpha + \frac{\beta^2}{\alpha} \tag{1}$$

$$L = \frac{\beta^2}{\alpha} \tag{2}$$

and

$$L = \frac{\beta^2}{(\beta - \alpha)} \tag{3}$$

All the above three metrics are studied in detail by the authors of the papers [1, 3, 4, 5].

2 Preliminaries

We consider a two-dimensional Finsler space $F^2 = (M^2, L(x, y))$ with the (α, β)-metric [6, 7] where $\alpha = \sqrt{a_{ij}(x)x^i x^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is one form on M^2. The space $F^2 = (M^2, \alpha)$ is said to be a Riemannian space associated to F^2.

Matsumoto and Park [6, 7] constructed the problem on the following consideration :

(I). The underlying manifold M^2 is thought of as a surface S of the ordinary 3-space with an orthonormal co-ordinate system $X^\alpha, \alpha = 1, 2, 3$, which by the parametric equation $X^\alpha = X^\alpha(x^1, x^2)$. Then S is equipped with the induced Riemannian metric α. Thus two tangent vector fields $B_i, i = 1, 2$, are given with the components $B_i^\alpha = \frac{\partial X^\alpha}{\partial x^i}$ and then $a_{ij} = \sum_\alpha B_i^\alpha B_j^\alpha$. Let $N = N^\alpha$ be the unit normal vector to S.

An isothermal co-ordinate system $x^i = (x, y)$ in S may be referred in which α is of the form $\alpha = aE$, where $a = a(x, y)$ is a positive-valued function and $E = \sqrt{\dot{x}^2 + \dot{y}^2}$. Then the Christoffel symbols $\gamma^i_{jk}(x, y)$ of S in x^i are given by $(\gamma^1_{11}, \gamma^1_{12}, \gamma^1_{22}; \gamma^2_{11}, \gamma^2_{12}, \gamma^2_{22}) = (\frac{a_x}{a}, \frac{a_y}{a}, -\frac{a_x}{a}, -\frac{a_y}{a}, \frac{a_x}{a}, \frac{a_y}{a})$. We shall denote by $(;)$ the covariant differentiation with respect to Christoffel symbols in R^2.

(II). Let $B = B^\alpha$ be a constant vector field in the ambient 3-space and put

$$B = b^i B_i + b^0 N \tag{4}$$

along S. Then the tangential component of B gives the linear form

$$\beta = b_i \dot{x}^i, \qquad b_i = a_{ij} b^j \tag{5}$$

The Gauss-Weingarten derivation formulae lead from (4) to

$$B_{;j} = (b^i_{;j}B_i + b^i H_{ij} N) + (b^0_{;}N - b^0 H^i_j B_i)$$

where H_{ij} is the second fundamental tensor of S and $H_{ij} = a_{ik}H^k_j$. From $B_{;j} = 0$, we get $b^i_{;j} = b^0 H^i_j$, that is

$$b_{i;j} = b^0 H_{ij} \tag{6}$$

Consequently we have $b_{i;j} = b_{j;i}$ that is $b_{1y} = b_{2x}$ and hence b_i is a gradient vector field in S.

(III.) The linear form β was originally to be induced one by the Finslerian surface S due to the earth's gravity [6]. Hence, it is assumed here that the constant vector field B is parallel to the X^3-axis, i.e. $B^a = (0, 0, -G)$, $G = const. > 0$. Therefore (4) gives $G^2 = a_{ij}b^i b^j + (b^0)^2$. Since $(a_{11}, a_{12}, a_{22}) = (a^2, 0, a^2)$, we have

$$(\tfrac{G}{a})^2 = (b^1)^2 + (b^2)^2 + (\tfrac{b^0}{a})^2$$

We shall regard the quantity $\frac{G}{a}$ as an infinitesimal of degree one, and neglect the infinitesimal of degree more or equal to two. It is natural from the above that b^1, b^2 and $\frac{b^0}{a}$ are also those of degree one. Further (6) shows that $\frac{\beta_{;0}}{a} = \frac{b_{i;j}\dot{x}^i\dot{x}^j}{a}$ may be regarded as an infinitesimal of degree one. Consequently

$$\lambda = \frac{\beta}{a^2}, \qquad \mu = \frac{\gamma}{a^2}, \qquad \nu = \frac{\beta_{;0}}{a} \tag{7}$$

are infinitesimals of degree one where $\gamma = b_1\dot{y} - b_2\dot{x}$.

Thus we have summarized all the above three conditions as:

I. α is the induced Riemannian metric in a surface S and, in particular $\alpha = aE$.

II. β is the linear form in $\dot{x}^i$, induced from a constant vector field (0, 0, -G) by (4) and (5).

III. λ, μ and ν of (7) are regarded as infinitesimals of degree one and infinitesimals of degree more or equal to two are neglected.

3 Geodesics of the special (α, β)-metric

Matsumoto and Park [6] obtained the differential equation of the geodesic in an isothermal co-ordinate system $(x^i) = (x, y)$ for the (α, β)-metric is as follows:

$$(L_\alpha + aEw\gamma^2)Ri(C) - \beta_{;0}a^2 w\gamma - L_\beta(b_1 y - b_2 x) = 0 \tag{8}$$

where $w = \frac{L_{\alpha\alpha}}{\beta^2} = -\frac{L_{\alpha\beta}}{\alpha\beta} = \frac{L_{\beta\beta}}{\alpha^2}$ and

$$Ri(C) = \tfrac{a(\dot{x}\ddot{y} - \dot{y}\ddot{x})}{E^3} + \tfrac{(a_x\dot{y} - a_y\dot{x})}{E}$$

It is remarked that the equation Ri(C)=0 gives the geodesic of the associated Riemannian space.

Now according to the above contribution, equation (8) may be written for the metric $L = \alpha + \frac{\beta^2}{\alpha}$ in the form

$$(1 - \tfrac{a^2\lambda^2}{E^2} + \tfrac{2a^2\mu^2}{E^2})Ri(C) = \tfrac{2a^2\mu\nu}{E^3}$$

Let us neglect the infinitesimals of degree more or equal to two. Then we have

$$Ri(C) = \frac{2a^2\mu\nu}{E^3} \tag{9}$$

Therefore, on our construction, we obtain the approximate equation of geodesics in the form

$$y^{''} = \frac{2\beta^*_{;0}\gamma^*}{a^2} - \frac{E^{*2}(a_x y^{'} - a_y)}{a} \tag{10}$$

where

$$y^{'} = \frac{dy}{dx}, \qquad E^* = \sqrt{1 + y^{'2}}, \qquad \gamma^* = b_1 y^{'} - b_2 \tag{11}$$

$$\beta^*_{;0} = b_{1;1} + (b_{1;2} + b_{2;1})y^{'} + b_{2;2}(y^{'})^2$$

Next, if we take the metric (2) then the differential equation (8) of geodesic is written as

$$(-\tfrac{a^2\lambda^2}{E^2} + \tfrac{2a^2\mu^2}{E^2})Ri(C) = \tfrac{a^2\mu\nu}{E^3}$$

Let us neglect the infinitesimals of degree more or equal to two. Then we have

$$Ri(C) = \frac{\mu\nu}{E(2\mu^2 - \lambda^2)} \tag{12}$$

Therefore on our construction, we obtain the approximate equation of geodesics in the form

$$y^{''} = \frac{\beta^*_{;0}\gamma^* E^{*2}}{(2b_1^2 - b_2^2)(y^{'})^2 + (2b_2^2 - b_1^2) - 6b_1 b_2 y^{'}} - \frac{E^{*2}(a_x y^{'} - a_y)}{a} \tag{13}$$

where $\beta^*_{;0}, \gamma^*, E^*$ and $y^{'}$ are given in 11.

Next, if we take the metric (3) then the differential equation (8) of geodesic is written as

$$(\lambda^2(1 - \tfrac{a\lambda}{E}) + 2\mu^2)Ri(C) = \tfrac{\mu\nu}{E}$$

Let us neglect the infinitesimals of degree more than two. Then we have

$$Ri(C) = \frac{\mu\nu}{E(2\mu^2 + \lambda^2)} \tag{14}$$

Therefore on our construction, we obtain the approximate equation of geodesics in the form

$$y^{''} = \frac{\beta^*_{;0}\gamma^* E^{*2}}{(2b_1^2 + b_2^2)(y^{'})^2 + (2b_2^2 + b_1^2) - 2b_1 b_2 y^{'}} - \frac{E^{*2}(a_x y^{'} - a_y)}{a} \tag{15}$$

where $\beta^*_{;0}, \gamma^*, E^*$ and $y^{'}$ are given in 11.

4 Some Examples

In the following we shall use the notation as follows:

$$(X^a) = (X, Y, Z), \qquad (x^i) = (x, y)$$

Example 1: We consider the circular cylinder $S : X^2 + Z^2 = 1, \quad Y = y$, which is also written as

$$S : X = \cos x, \qquad Y = y, \qquad Z = \sin x$$

Then we get

$$B_1 = (-\sin x, 0, \cos x), \qquad B_2 = (0, 1, 0), \qquad N = (\cos x, 0, \sin x)$$

$$(a_{11}, a_{12}, a_{22}) = (1, 0, 1), \qquad (b^1, b^2, b^0) = (G\cos x, 0, G\sin x)$$

Consequently we have

$$\alpha^2 = dx^2 + dy^2, \qquad \beta = -G\cos x\ dx$$

Therefore (10) gives the approximate differential equation of geodesic for the metric (1) in the given condition of above example as

$$y^{''} + \tan x\ y^{'} = 0 \tag{16}$$

which has the solution

$$y = A\sin x + B \tag{17}$$

where A and B are constants of integration.

Further from (13) the approximate differential equation of geodesic for the metric (2) in the given condition of Example 1 is given by

$$y^{''} + \tan x \frac{(1 + y^{'2})}{2y^{'2} - 1} = 0 \tag{18}$$

solving the above equation with the help of Mathematica software we have

$$y = \int_1^x [f_1^{-1}\{A + \log(\cos u)\}]du + B \tag{19}$$

where $f_1(t) = 2t - 3\tan^{-1} t$.

Again from (15) the approximate differential equation of geodesic for the metric (3) in the given condition of Example 1 is given by

$$y^{''} + \tan x \frac{(1 + y^{'2})}{2y^{'2} + 1} = 0 \tag{20}$$

solving the above equation with the help of Mathematica software we have

$$y = \int_1^x [f_2^{-1}\{A + \log(\sec u)\}]du + B \tag{21}$$

where $f_2(t) = 2t - \tan^{-1} t$.

Next we are interested in revolution surfaces the axis of which is parallel to the constant vector field B. Such a surface S is given by,

$$X = g(u)\cos y, \qquad Y = g(u)\sin y, \qquad Z = f(u)$$

Denoting (u, y) by (x^i), we have

$$B_1 = (g^{'}\cos y,\ \ g^{i}\sin y,\ \ f^{'}), \qquad B_2 = (-g\sin y,\ \ g\cos y,\ \ 0)$$

$$N = \frac{(-f^{'}\cos y,\ -f^{'}\sin y,\ g^{'})}{F}, \qquad F = \sqrt{f'^2 + g'^2}$$

$$(a_{11},\ a_{12},\ a_{22}) = (F^2,\ 0,\ G^2), \quad (b^1,\ b^2,\ b^0) = (-\tfrac{Gf^{'}}{F},\ 0,\ -\tfrac{Gg^{'}}{F}),$$

$$(b_1,\ b_2) = (Gf^{'},\ 0)$$

Consequently we get

$$\alpha^2 = F^2 du^2 + g^2 dy^2, \qquad \beta = -Gf^{'}du$$

We need an isothermal co-ordinate system, if we take

$$x = \int \frac{F}{g} du \tag{22}$$

Then we obtain

$$\alpha^2 = g(u)^2(dx^2 + dy^2), \qquad \beta = -G\frac{f^{'}g}{F} \tag{23}$$

Example 2: We shall deal with the sphere, surface of constant curvature +1 : $g(u) = \cos u$ and $f(u) = \sin u$. Then F = 1 and (22) gives

$$x = \int \tfrac{1}{\cos u} du = \tfrac{1}{2}\log \tfrac{1+\sin u}{1-\sin u}$$

Then $\frac{1+\sin u}{1-\sin u} = e^{2x}$ implies $\frac{1}{\cos u} = \cosh u$, hence $du = \frac{dx}{\cosh x}$. Consequently (23) leads to

$$\alpha^2 = \tfrac{1}{\cosh^2 x}(dx^2 + dy^2), \qquad \beta = -\tfrac{G}{\cosh^2 x}dx$$

Therefore (1) gives the approximate differential equation of geodesics in the form

$$y^{''} = \tanh x(1 - \frac{2G^2}{\cosh^2 x})(y^{'} + (y^{'})^3) \tag{24}$$

The solution of the above equation with the help of Mathematica software is given by

$$y = \int_1^x \frac{e^{\frac{4e^{2t}}{(1+e^{2t})^2}+A-t}(1+e^{2t})}{\sqrt{1-e^{2\{\frac{4e^{2t}}{(1+e^{2t})^2}+A-t+\log(1+e^{2t})\}}}}dt \tag{25}$$

Again (2) gives the approximate differential equation of geodesics in the form

$$y^{''} = \tanh x \frac{(y^{'}-1)(2y^{'}+1)(y^{'2}+1)}{2y^{'2}-1} \tag{26}$$

The solution of the above equation with the help of Mathematica software is given by

$$y = \int_1^x [f_4^{-1}\{A+\log(\cosh t)\}]du + B \tag{27}$$

where $f_4(t) = \frac{2\tan^{-1}u}{10} + \frac{1}{6}(\log(1-u)) + \frac{2}{15}\log(1+2u) - \frac{3}{20}\log(1+u^2)$.

Again (3) gives the approximate differential equation of geodesics in the form

$$y^{''} = \tanh x \frac{(y^{'}-1)(2y^{'}+1)(y^{'2}+1)}{2y^{'2}+1} \tag{28}$$

The solution of the above equation with the help of Mathematica software is given by

$$y = \int_1^x [f_5^{-1}\{A+\log(\cosh t)\}]du + B \tag{29}$$

where $f_5(t) = \frac{3\tan^{-1}u}{10} + \frac{1}{2}(\log(1-u)) + \frac{2}{5}\log(1+2u) - \frac{1}{20}\log(1+u^2)$.

5 Results and Discussions

On the basis of the above calculations we have following important propositions:

Proposition 1. *The solution of equation of the geodesic for the Finsler metric (1) in a circular cylinder* $S: X^2+Z^2=1,\ \ Y=y$ *is given by equation (17).*

Proposition 2. *The solution of equation of the geodesic for the Finsler metric (2) in a circular cylinder* $S: X^2+Z^2=1,\ \ Y=y$ *is given by equation (19).*

Proposition 3. *The solution of equation of the geodesic for the Finsler metric (3) in a circular cylinder* $S: X^2+Z^2=1,\ \ Y=y$ *is given by equation (21).*

Proposition 4. *The solution of equation of the geodesic for the Finsler metric (1) in a sphere, surface of constant curvature* $+1: g(u)=\cos u$ *and* $f(u)=\sin u$ *is given by equation (25).*

Proposition 5. *The solution of equation of the geodesic for the Finsler metric (2) in a sphere, surface of constant curvature* $+1 : g(u) = \cos u$ *and* $f(u) = \sin u$ *is given by equation (27).*

Proposition 6. *The solution of equation of the geodesic for the Finsler metric (3) in a sphere, surface of constant curvature* $+1 : g(u) = \cos u$ *and* $f(u) = \sin u$ *is given by equation (29).*

As it can be observed from all of the above solutions of equation of geodesics in Propositions 5.1 to 5.6, the nature of the solution is governed a lot by the first constant of integration A, whereas the second constant of integration B is just a shifting parameter. Therefore, the behavior of the curves has been plotted for different values of A and taking B=0 without a loss of generality. As the analytic solutions in Propositions 5.1 to 5.6 are complex in nature, the plots have been drawn using Mathematica 7.0.

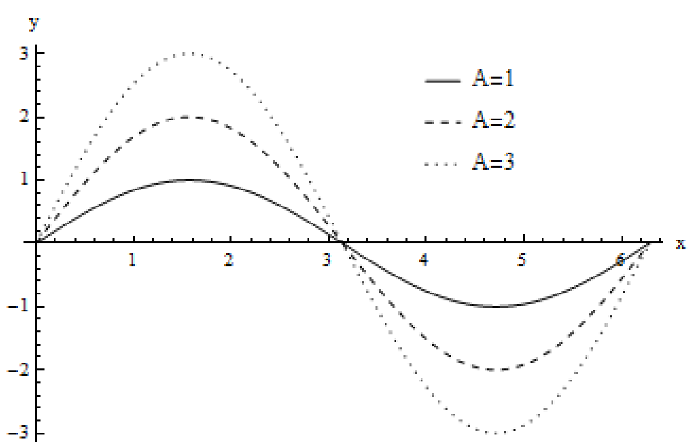

Fig. 1 The solution of the equation of geodesic for the Finsler metric (1) in a circular cylinder $S : X^2 + Z^2 = 1, \; Y = y$ behaves like sine curve.

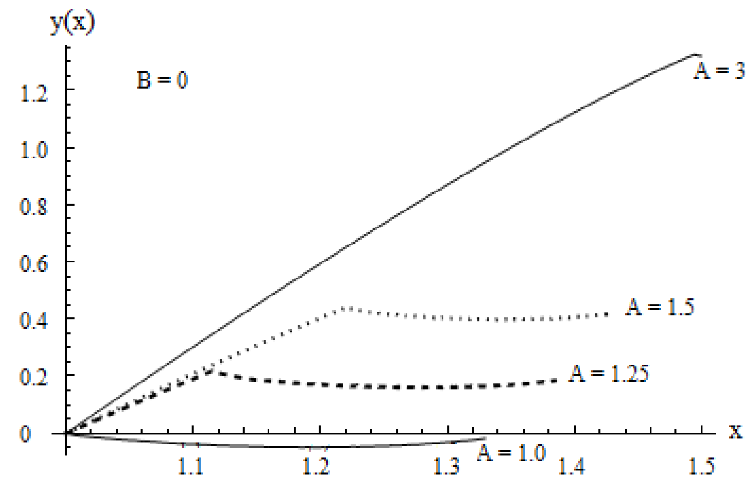

Fig. 2 The solution of the equation of geodesic for the Finsler metric (2) in a circular cylinder $S : X^2 + Z^2 = 1,\;\; Y = y$ behaves like the above figure.

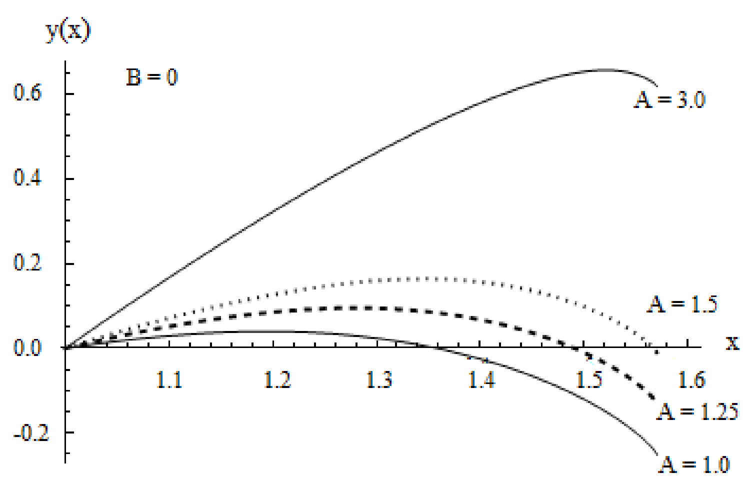

Fig. 3 The solution of the equation of geodesic for the Finsler metric (3) in a circular cylinder $S : X^2 + Z^2 = 1,\;\; Y = y$ behaves like the above figure.

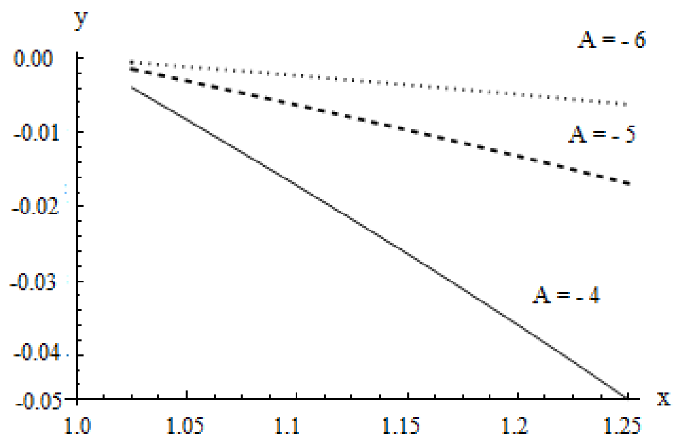

Fig. 4 The solution of the equation of geodesic for the Finsler metric (1) in the sphere, surface of constant curvature $+1 : g(u) = \cos u$ and $f(u) = \sin u$ and at G=1, behaves like the above figure.

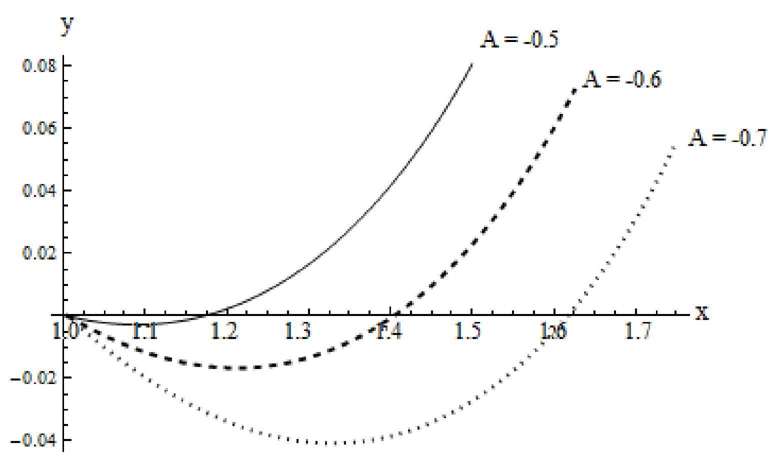

Fig. 5 The solution of the equation of geodesic for the Finsler metric (2) in the sphere, surface of constant curvature $+1 : g(u) = \cos u$ and $f(u) = \sin u$ behaves like the above figure.

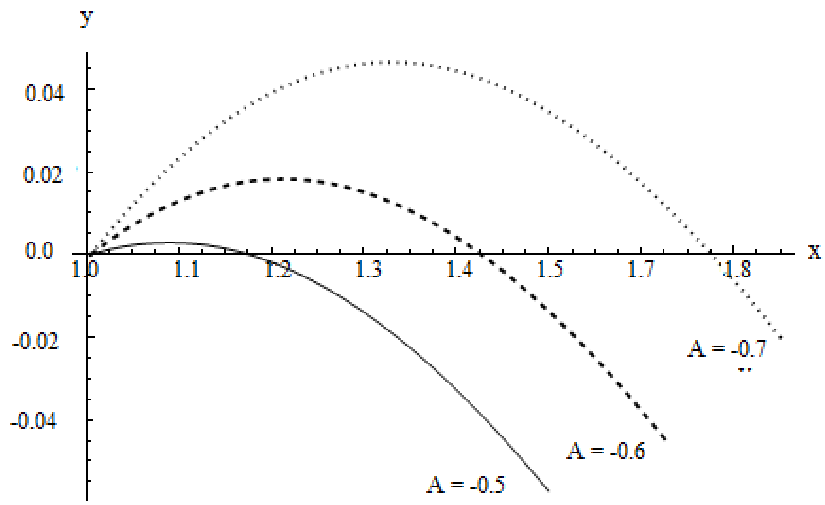

Fig. 6 The solution of the equation of geodesic for the Finsler metric (3) in the sphere, surface of constant curvature $+1 : g(u) = \cos u$ and $f(u) = \sin u$ behaves like the above figure.

References

[1] Antonelli, P. L., Ingarden, R. S. and Matsumoto, M.: *The theory of sprays and Finsler spaces with applications in physics and biology*, Kluwer Academic Publishers, Dordrecht, Boston, London, 1993.

[2] Chaubey, V. K., Prasad, B. N. and Tripathi, D. D.: *Equations of geodesic for a (α, β)-metric in a two-dimensional Finsler space*, J. Math. Comput. Sci., **3** (2013), No. 3, 863-872.

[3] Mishra, Padmdeo: *A study of inverse problem in special Finsler spaces*, Ph. D. thesis, D.D.U. Gorakhpur University, Gorakhpur, India 2007, 90-106.

[4] Matsumoto, M.: *A slope of mountain is a Finsler surface with respect to a time measure*, J. Math. Kyoto Univ., **29**, (1989), 17-25.

[5] Matsumoto, M.: *Foundation of Finsler geometry and special Finsler spaces*, Kaisesisha Press, Otsu, Japan, 1986.

[6] Matsumoto, M. and Park, H. S.: *Equations of geodesics in two-dimensional Finsler spaces with (α, β)-metric*, Rev. Roum. Pures. Appl., **42**, (1997), 787-793.

[7] Matsumoto, M. and Park, H. S.: *Equations of geodesics in two-dimensional Finsler spaces with (α, β)-metric-II*, Tensor, N. S., **60**, (1998), 89-93.