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ON THE CHEBYSHEV APPROXIMATION OF A FUNCTION WITH TWO VARIABLES

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Abstract

An approach to find an approximation polynomial of a function with two variables based on the two dimensional discrete Fourier transform is presented. The approximation polynomial is expressed through Chebyshev polynomials. An uniform convergence result is given.

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1 Introduction

The purpose of the paper is to present some aspects about the construction of an approximation polynomial for a function with two variables. The approximation polynomial is expressed through Chebyshev polynomials. Throughout this paper the n-th Chebyshev polynomial is defined as $T_n(x) = \cos(n \arccos x),\ x \in [-1, 1]$ and $n \in \mathbb{N}$.

Constructing an approximation polynomial of a function with the corresponding applications is the subject of the *Chebfun* software, presented in details in [4], [2]. The *Chebfun2* part of the software deals with the construction of an approximation polynomial of a function with two variables. According to [5], [6], to this end a method based on Gaussian elimination as a low rank function approximation is used.

In *Chebfun* the approximation polynomial of a function with one variable is obtained using one dimensional discrete Fourier transform. The approach of this paper will use a two dimensional discrete Fourier transform.

In spectral methods the Chebyshev polynomials are often used. The same form of the approximation polynomial is used in [1], [9], too.

After recalling some formulas on the Fourier series for a function with two variables and the two dimensional discrete Fourier transform an algorithm to obtain an approximation polynomial of a function with two variables and a convergence result are presented. A Lagrange type interpolation problem for a function with two variables is studied. Two applications are mentioned: a numerical integration formula on a rectangle and a numerical computation of the partial derivatives.

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2 Two dimensional Fourier series

Let $f : \mathbb{R}^2 \to R$ be a continuous periodical function in each variable with the period 2π. The Fourier series attached to the function is [8], t.3

$$f(x,y) \sim \sum_{n,m=0}^{\infty} (a_{n,m} \cos nx \cos my + b_{n,m} \cos nx \sin my +$$

$$+ c_{n,m} \sin nx \cos my + d_{n,m} \sin nx \sin my)$$

with the coefficients given by

$$a_{0,0} = \frac{1}{4\pi^2} \iint_\Omega f(x,y) \mathrm{d}x\mathrm{d}y \qquad a_{n,m} = \frac{1}{\pi^2} \iint_\Omega f(x,y) \cos nx \cos my \mathrm{d}x\mathrm{d}y$$

$$a_{n,0} = \frac{1}{2\pi^2} \iint_\Omega f(x,y) \cos nx \mathrm{d}x\mathrm{d}y \qquad b_{n,m} = \frac{1}{\pi^2} \iint_\Omega f(x,y) \cos nx \sin my \mathrm{d}x\mathrm{d}y$$

$$a_{0,m} = \frac{1}{2\pi^2} \iint_\Omega f(x,y) \cos my \mathrm{d}x\mathrm{d}y \qquad c_{n,m} = \frac{1}{\pi^2} \iint_\Omega f(x,y) \sin nx \cos my \mathrm{d}x\mathrm{d}y$$

$$b_{0,m} = \frac{1}{2\pi^2} \iint_\Omega f(x,y) \sin my \mathrm{d}x\mathrm{d}y \qquad d_{n,m} = \frac{1}{\pi^2} \iint_\Omega f(x,y) \sin nx \sin my \mathrm{d}x\mathrm{d}y$$

$$c_{n,0} = \frac{1}{2\pi^2} \iint_\Omega f(x,y) \sin nx \mathrm{d}x\mathrm{d}y$$

where $\Omega = [0, 2\pi]^2$.

The complex form of the Fourier series is

$$\sum_{n.m \in \mathbb{Z}} \gamma_{n,m} e^{inx+imy}$$

with

$$\gamma_{0,0} = a_{0,0}$$

$$\gamma_{n,0} = \frac{1}{2}(a_{n,0} - ic_{n,0}) \qquad \gamma_{-n,0} = \frac{1}{2}(a_{n,0} + ic_{n,0})$$

$$\gamma_{0,m} = \frac{1}{2}(a_{0,m} - ib_{0,m}) \qquad \gamma_{0,-m} = \frac{1}{2}(a_{0,m} + ib_{0,m})$$

$$\gamma_{n,m} = \frac{1}{4}(a_{n,m} - ib_{n,m} - ic_{n,m} - d_{n,m}) \qquad \gamma_{-n,m} = \frac{1}{4}(a_{n,m} - ib_{n,m} + ic_{n,m} + d_{n,m})$$

$$\gamma_{n,-m} = \frac{1}{4}(a_{n,m} + ib_{n,m} - ic_{n,m} + d_{n,m}) \qquad \gamma_{-n,-m} = \frac{1}{4}(a_{n,m} + ib_{n,m} + ic_{n,m} - d_{n,m})$$

or

$$\gamma_{m,n} = \frac{1}{4\pi^2} \iint_\Omega f(x,y) e^{-inx-imy} \mathrm{d}x\mathrm{d}y, \quad \forall n,m \in \mathbb{Z}. \tag{1}$$

If the function is even in any variable then the $b_{n,m}, c_{n,m}, d_{n,m}$ coefficients are all zero.

We shall suppose that the convergence conditions of the Fourier series to $f(x,y)$ are fulfilled (the function has bounded first order partial derivatives in Ω and in a neighborhood of (x,y) there exists $\frac{\partial^2 f}{\partial x \partial y}$, or $\frac{\partial^2 f}{\partial y \partial x}$, which is continuous in (x,y), cf. [8], t.3, 697).

3 Two dimensional discrete Fourier transform

Let be the infinite matrix $(x_{k,j})_{k,j\in\mathbb{Z}}$ with the periodicity properties $x_{k+p,j} = x_{k,j},\ x_{k,j+q} = x_{k,j},\ \forall k,j \in \mathbb{Z}$. The discrete Fourier transform construct another infinite matrix $(y_{r,s})_{r,s\in\mathbb{Z}}$ with an analog periodicity properties defined by

$$y_{r,s} = \sum_{k=0}^{p-1}\sum_{j=0}^{q-1} x_{k,j} e^{-i\frac{2\pi kr}{p}} e^{-i\frac{2\pi js}{q}},$$

for $r \in \{0,1,\ldots,p-1\}$ and $s \in \{0,1,\ldots,q-1\}$.

The complexity to compute the pq numbers with the discrete fast Fourier transform algorithm is $pq\log_2 pq$.

As an application, if the Fourier series coefficients (1) are computed using the trapezoidal rule for each of the iterated integrals then:

$$\gamma_{n,m} = \frac{1}{4\pi^2}\iint_{\Omega} f(x,y)e^{-inx-imy}\mathrm{d}x\mathrm{d}y = \frac{1}{4\pi^2}\int_0^{2\pi} e^{-inx}\left(\int_0^{2\pi} f(x,y)e^{-imy}\mathrm{d}y\right)\mathrm{d}x$$

$$\approx \frac{1}{4\pi^2}\int_0^{2\pi} e^{-inx}\left(\frac{2\pi}{q}\sum_{j=0}^{q-1} f(x,\frac{2\pi j}{q})e^{-im\frac{2\pi j}{q}}\right)\mathrm{d}x =$$

$$= \frac{1}{2\pi q}\sum_{j=0}^{q-1} e^{-im\frac{2\pi j}{q}}\int_0^{2\pi} f(x,\frac{2\pi j}{q})e^{-inx}\mathrm{d}x \approx \frac{1}{pq}\sum_{k=0}^{p-1}\sum_{j=0}^{q-1} f(\frac{2\pi k}{p},\frac{2\pi j}{q})e^{-in\frac{2\pi k}{p}}e^{-im\frac{2\pi j}{q}}.$$

Thus the Fourier coefficients $(\gamma_{n,m})$ may be computed applying the discrete Fourier transform to $\left(f(\frac{2\pi k}{p},\frac{2\pi j}{q})\right)_{k\in\{0,1,\ldots,p-1\},j\in\{0,1,\ldots,q-1\}}$.

If the function f is even in any variable then there is an alternative to compute the coefficients $a_{r,s}$, introduced in the previous section, based on the discrete cosine transform

$$y_{r,s} = \sum_{k=0}^{p-1}\sum_{j=0}^{q-1} x_{k,j}\cos{(k+\frac{1}{2})}\frac{r\pi}{p}\cos{(j+\frac{1}{2})}\frac{s\pi}{q},$$

for $r \in \{0,1,\ldots,p-1\}$ and $s \in \{0,1,\ldots,q-1\}$ and the Gauss quadrature formula

$$\int_{-1}^{1}\frac{\varphi(x)}{\sqrt{1-x^2}}\mathrm{d}x \approx \frac{\pi}{n}\sum_{k=1}^{n}\varphi(x_k),$$

where $x_k = \cos{(k+\frac{1}{2})}\frac{\pi}{n},\ k \in \{0,1,\ldots,n-1\}$ are the roots of the Chebyshev polynomial $T_n(x)$.

Applying this formula to compute the coefficients $a_{r,s}$ it results

$$a_{r,s} = \frac{4}{pq}\sum_{k=0}^{p-1}\sum_{j=0}^{q-1} f\left(\cos{(k+\frac{1}{2})}\frac{\pi}{p},\cos{(j+\frac{1}{2})}\frac{\pi}{q}\right)\cos{(k+\frac{1}{2})}\frac{r\pi}{p}\cos{(j+\frac{1}{2})}\frac{s\pi}{q},$$

namely the discrete cosine transform applied to $\left(f(\cos{(k+\frac{1}{2})}\frac{\pi}{p},\cos{(j+\frac{1}{2})}\frac{\pi}{q})\right)_{k\in\{0,1,\ldots,p-1\},j\in\{0,1,\ldots,q-1\}}$.

4 The Chebyshev series

Considering a continuous two real variables function $f(x,y),\ x,y\in[-1,1]$, the attached Chebyshev series is

$$f(x,y)\sim\sum_{n,m=0}^{\infty}\alpha_{n,m}T_n(x)T_m(y) \tag{2}$$

where

$$\alpha_{0,0}=\tfrac{1}{\pi^2}\int_{-1}^{1}\int_{-1}^{1}\tfrac{f(x,y)}{\sqrt{1-x^2}\sqrt{1-y^2}}\mathrm{d}x\mathrm{d}y \qquad \alpha_{n,0}=\tfrac{2}{\pi^2}\int_{-1}^{1}\int_{-1}^{1}\tfrac{f(x,y)T_n(x)}{\sqrt{1-x^2}\sqrt{1-y^2}}\mathrm{d}x\mathrm{d}y$$

$$\alpha_{0,m}=\tfrac{2}{\pi^2}\int_{-1}^{1}\int_{-1}^{1}\tfrac{f(x,y)T_m(y)}{\sqrt{1-x^2}\sqrt{1-y^2}}\mathrm{d}x\mathrm{d}y \qquad \alpha_{n,m}=\tfrac{4}{\pi^2}\int_{-1}^{1}\int_{-1}^{1}\tfrac{f(x,y)T_n(x)T_m(y)}{\sqrt{1-x^2}\sqrt{1-y^2}}\mathrm{d}x\mathrm{d}y$$

Changing $x=\cos t, y=\cos s$, the coefficient $\alpha_{n,m}$ will be

$$\alpha_{n,m}=\frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi}f(\cos t,\cos s)\cos nt\cos ms\mathrm{d}t\mathrm{d}s= \tag{3}$$

$$=\frac{1}{\pi^2}\int\int_{\Omega}f(\cos t,\cos s)\cos nt\cos ms\mathrm{d}t\mathrm{d}s.$$

Analogous formulas may be obtained for $\alpha_{0,0},\alpha_{n,0}$ and $\alpha_{0,m}$, too. Thus the coefficients of the Chebyshev series are the coefficients of the Fourier series of the function $\varphi(t,s)=f(\cos t,\cos s)$.

If function f has second order derivatives then the Fourier series attached to φ converges to φ and consequently

$$f(x,y)=\sum_{n,m=0}^{\infty}\alpha_{n,m}T_n(x)T_m(y),\quad x,y\in[-1,1]. \tag{4}$$

The polynomial

$$f_{n,m}(x,y)=\sum_{k=0}^{n}\sum_{j=0}^{m}\alpha_{k,j}T_k(x)T_j(y)$$

is called the Chebyshev approximation polynomial of function $f(x,y)$ in the square $[-1,1]^2$.

The parameters n,m are determined adaptively to satisfy the inequalities $|\alpha_{k,j}|<tol(=10^{-15}$, machine precision), for $k>n$ and $j>m$. This is the goal of the algorithm 1. The coefficients whose absolute value are less then tol are eliminated and the remained coefficients are stored as a sparse matrix.

The $f_{n,m}(x,y)$ polynomial may be obtained with the least square method as the solution of the optimization problem

$$\min_{\lambda_{k,j}}\int_{-1}^{1}\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}\sqrt{1-y^2}}\left(f(x,y)-\sum_{k=0}^{n}\sum_{j=0}^{m}\lambda_{k,j}T_k(x)T_j(y)\right)^2\mathrm{d}x\mathrm{d}y.$$

Due to the Parseval equality

$$\alpha_{0,0}^2+\frac{1}{2}\sum_{n=1}^{\infty}\alpha_{n,0}^2+\frac{1}{2}\sum_{m=1}^{\infty}\alpha_{0,m}^2+\frac{1}{4}\sum_{n,m=1}^{\infty}\alpha_{n,m}^2=\frac{1}{\pi^2}\int_{-1}^{1}\int_{-1}^{1}\frac{f^2(x,y)}{\sqrt{1-x^2}\sqrt{1-y^2}}\mathrm{d}x\mathrm{d}y$$

the quality of the approximation polynomial may be evaluated by

$$\frac{1}{\pi^2}\int_{-1}^{1}\int_{-1}^{1}\frac{f^2(x,y)}{\sqrt{1-x^2}\sqrt{1-y^2}}-\left(\alpha_{0,0}^2+\frac{1}{2}\sum_{k=0}^{n}\alpha_{k,0}^2+\frac{1}{2}\sum_{j=0}^{m}\alpha_{0,j}^2+\frac{1}{4}\sum_{k=1}^{n}\sum_{j=1}^{m}\alpha_{k,j}^2\right). \tag{5}$$

Algorithm 1 Algorithm to compute the Chebyshev approximation polynomial

procedure CHEBFUN2(f)
  $n \leftarrow 8$
  $tol \leftarrow 10^{-15}$
  $sw \leftarrow true$
  while sw **do** ▷ The approximation polynomial is determined adaptively
    $m \leftarrow 2n$
    $x, y \leftarrow \cos\frac{2k\pi}{m},\ k = 0 : m-1$
    $z \leftarrow f(x,y)$
    $g \leftarrow FFT(z)/m^2$
    $a \leftarrow 4\Re g(1:n, 1:n)$
    $a(1,1) \leftarrow a(1,1))/4$
    $a(1, 2:n) \leftarrow a(1, 2:n)/2$
    $a(2:n, 1) \leftarrow a(2:n, 1)/2$
    if $|a(i-1:i, 1:n)| < tol$ & $|a(1:n, i-1:i)| < tol$ **then**
      $sw \leftarrow false$
    else
      $n \leftarrow 2n$
    end if
  end while
  for $i = 1:n$ **do** ▷ Removal of negligible coefficients
    for $j = 1:n$ **do**
      if $|a(i,j)| < tol$ **then**
        $a(i,j) \leftarrow 0$
      end if
    end for
  end for
  return a
end procedure

The value of the polynomial $f_{n,m}$ in a point (x, y) may be computed adapting the Clenshaw algorithm, [9], but we find that the evaluation of the expression $f_{n,m}(x,y) = V_n'(x)A_{n,m}V_m(y)$, where $V_\nu(s) = (T_0(s), T_1(s), \ldots, T_\nu(s))'$ and $A =$

$(a_{k,j})_{k=0:n,j=0:m}$ is more efficient within a matrix oriented software. V' denotes the transpose of vector V. The complexity order of both algorithms is $O(nm)$. This evaluation does not take into account the computation of coefficients $a_{k,j}$.

5 The Chebyshev series of partial derivatives

We assume that function $f(x,y)$ has first order continuous partial derivatives and series (4), there is required to find the coefficients $(b_{n,m})_{n,m\in\mathbb{N}}$ such that

$$\frac{\partial f(x,y)}{\partial x} = \sum_{n,m=0}^{\infty} b_{n,m}T_n(x)T_m(y). \tag{6}$$

Using the equalities

$$T_1'(x) = T_0(x), \qquad T_2'(x) = 4T_1(x)$$

and

$$\frac{1}{2}\left(\frac{T_{n+1}'(x)}{n+1} - \frac{T_{n-1}'(x)}{n-1}\right) = T_n(x), \qquad n > 1.$$

(6) may be written as

$$\frac{\partial f(x,y)}{\partial x} = \sum_{m=0}^{\infty}\left(\sum_{n=0}^{\infty} b_{n,m}T_n(x)\right)T_m(y) =$$

$$= \sum_{m=0}^{\infty}\left(b_{0,m}T_1'(x) + \frac{b_{1,m}}{2}\frac{T_2'(x)}{2} + \sum_{k=2}^{\infty}\frac{b_{k,m}}{2}\left(\frac{T_{k+1}'(x)}{k+1} - \frac{T_{k-1}'(x)}{k-1}\right)\right)T_m(y) =$$

$$= \sum_{m=0}^{\infty}\left((b_{0,m} - \frac{b_{2,m}}{2})T_1'(x) + \sum_{k=2}^{\infty}\frac{1}{2k}(b_{k-1,m} - b_{k+1,m})T_k'(x)\right)T_m(y) =$$

$$= \sum_{m=0}^{\infty}\left(\sum_{k=1}^{\infty}\alpha_{k,m}T_k'(x)\right)T_m(y).$$

Identifying the coefficients of $T_k'(x)$ the obtained linear algebraic system is

$$\left\{\begin{array}{lcl} b_{0,m} - \frac{b_{2,m}}{2} & = & \alpha_{1,m} \\ \frac{1}{2k}(b_{k-1,m} - b_{k+1,m}) & = & \alpha_{k,m}, \qquad k \geq 2, \qquad m \in \mathbb{N}. \end{array}\right. \tag{7}$$

Summing the above equalities for $k = n+1, n+3, n+5, \ldots$ it results

$$b_{n,m} = 2\left((n+1)a_{n+1,m} + (n+3)a_{n+3,m} + (n+5)a_{n+5,m} + \ldots\right) \qquad \forall\ n,m \in \mathbb{N}.$$

In the same way it is deduced that the coefficients of the series

$$\frac{\partial f(x,y)}{\partial y} = \sum_{n,m=0}^{\infty} c_{n,m}T_n(x)T_m(y)$$

verifies the relations

$$\begin{cases} c_{n,0} - \frac{c_{n,2}}{2} & = \alpha_{n,1} \\ \frac{1}{2j}(c_{n,j-1} - c_{n,j+1}) & = \alpha_{n,j}, \qquad j \geq 2, \qquad n \in \mathbb{N}. \end{cases} \tag{8}$$

Let

$$\frac{\partial f^2(x,y)}{\partial x \partial y} = \frac{\partial f^2(x,y)}{\partial y \partial x} = \sum_{n,m=0}^{\infty} d_{n,m} T_n(x) T_m(y). \tag{9}$$

Because $\frac{\partial f^2(x,y)}{\partial x \partial y} = \frac{\partial}{\partial y}(\frac{\partial f(x,y)}{\partial x})$ applying (8) it results

$$\frac{1}{2j}(d_{k,j-1} - d_{k,j+1}) = b_{k,j}$$

and consequently, for $k, j > 1$,

$$\frac{1}{4kj}(d_{k-1,j-1} - d_{k-1,j+1} - d_{k+1,j-1} + d_{k+1,j+1}) = \frac{1}{2k}(b_{k-1,j} - b_{k+1,j}) = \alpha_{k,j}. \tag{10}$$

Denoting $\triangle_{k,j} = d_{k-1,j-1} - d_{k-1,j+1} - d_{k+1,j-1} + d_{k+1,j+1}$, it results $\triangle^2_{k,j} \leq 4(d^2_{k-1,j-1} + d^2_{k-1,j+1} + d^2_{k+1,j-1} + d^2_{k+1,j+1})$. From the Parseval equality corresponding to (9) it results

$$\frac{1}{4}\sum_{n,m=1}^{\infty} d^2_{n,m} \leq d^2_{0,0} + \frac{1}{2}\sum_{n=1}^{\infty} d^2_{n,0} + \frac{1}{2}\sum_{m=1}^{\infty} d^2_{0,m} + \frac{1}{4}\sum_{n,m=1}^{\infty} d^2_{n,m} \leq M^2_{1,1},$$

where $M_{1,1} \geq \max\{|\frac{\partial^2 f}{\partial x \partial y}(x,y)| : \ x, y \in [-1,1]\}$.

Consequently

$$\sum_{k=2}^{\infty}\sum_{j=2}^{\infty} \triangle^2_{k,j} \leq 16 \sum_{n,m=1}^{\infty} d^2_{n,m} \leq 64 M^2_{1,1}.$$

6 The convergence of the Chebyshev series

Using the techniques presented in [7] and [4], in some hypotheses it may be proven that the convergence in (4) is uniform in $[-1,1]^2$, for $n, m \to \infty$. First we state

Theorem 1. *If function f has second order continuous derivatives then*

$$|\alpha_{n,0}| \leq \frac{2M_{2,0}}{(n-1)^2} \quad \textit{and} \quad |\alpha_{n,1}| \leq \frac{8M_{2,0}}{\pi(n-1)^2}, \quad n > 1, \tag{11}$$

$$|\alpha_{0,m}| \leq \frac{2M_{0,2}}{(m-1)^2} \quad \textit{and} \quad |\alpha_{1,m}| \leq \frac{8M_{0,2}}{\pi(m-1)^2}, \quad m > 1, \tag{12}$$

where $M_{2,0} \geq \max\{|\frac{\partial^2 f}{\partial x^2}(x,y)| : \ x, y \in [-1,1]\}$, $M_{0,2} \geq \max\{|\frac{\partial^2 f}{\partial y^2}(x,y)| : \ x, y \in [-1,1]\}$.

Proof. The coefficient $\alpha_{n,0}$ may be written as

$$\alpha_{n,0}=\frac{2}{\pi^2}\int_0^{\pi}\left(\int_0^{\pi}f(\cos t,\cos s)\cos nt\mathrm{d}t\right)\mathrm{d}s.$$

Two partial integrations are performed in the internal integral

$$\int_0^{\pi}f(\cos t,\cos s)\cos nt\mathrm{d}t=$$

$$=\frac{1}{2n}\int_0^{\pi}\frac{\partial^2 f}{\partial x^2}(\cos t,\cos s)\sin t\left(\frac{\sin{(n-1)t}}{n-1}-\frac{\sin{(n+1)t}}{n+1}\right)\mathrm{d}t.$$

It results that

$$|\int_0^{\pi}f(\cos t,\cos s)\cos nt\mathrm{d}t|\leq\frac{\pi}{2n}M_{2,0}\left(\frac{1}{n-1}+\frac{1}{n+1}\right)\leq\frac{\pi M_{2,0}}{(n-1)^2}\tag{13}$$

and consequently $|\alpha_{n,0}|\leq\frac{2M_{2,0}}{(n-1)^2}$.

Using (13) in $\alpha_{n,1}=\frac{4}{\pi^2}\int_0^{\pi}\left(\int_0^{\pi}f(\cos t,\cos s)\cos nt\mathrm{d}t\right)\cos s\mathrm{d}s$ it results

$$|\alpha_{n,1}|\leq\frac{4}{\pi}\frac{M_{2,0}}{(n-1)^2}\int_0^{\pi}|\cos s|\mathrm{d}s=\frac{8M_{2,0}}{\pi(n-1)^2}.$$

The proof of (12) is similar. ■

Theorem 2. *If function f has second order continuous partial derivatives then $\lim_{n,m\to\infty}f_{n,m}=f$ uniformly in $[-1,1]^2$.*

Proof. From

$$f(x,y)=\sum_{n,m=0}^{\infty}\alpha_{n,m}T_n(x)T_m(y)\quad\text{and}\quad f_{n,m}(x,y)=\sum_{k=0}^{n}\sum_{j=0}^{m}\alpha_{k,j}T_k(x)T_j(y)$$

it results

$$f(x,y)-f_{n,m}(x,y)=\sum_{k=0}^{n}\sum_{j=m+1}^{\infty}\alpha_{k,j}T_k(x)T_j(y)+\sum_{k=n+1}^{\infty}\sum_{j=0}^{\infty}\alpha_{k,j}T_k(x)T_j(y).$$

Then

$$|f(x,y)-f_{n,m}(x,y)|\leq\sum_{k=0}^{n}\sum_{j=m+1}^{\infty}|\alpha_{k,j}|+\sum_{k=n+1}^{\infty}\sum_{j=0}^{\infty}|\alpha_{k,j}|=\tag{14}$$

$$=\sum_{j=m+1}^{\infty}|\alpha_{0,j}|+\sum_{j=m+1}^{\infty}|\alpha_{1,j}|+\sum_{k=2}^{n}\sum_{j=m+1}^{\infty}|\alpha_{k,j}|+$$

$$+\sum_{k=n+1}^{\infty}|\alpha_{k,0}|+\sum_{k=n+1}^{\infty}|\alpha_{k,1}|+\sum_{k=n+1}^{\infty}\sum_{j=2}^{\infty}|\alpha_{k,j}|$$

and using the Cauchy-Buniakowsky-Schwarz inequality it follows

$$(f(x,y) - f_{n,m}(x,y))^2 \leq$$

$$\leq 6 \left(\left(\sum_{j=m+1}^{\infty} |\alpha_{0,j}| \right)^2 + \left(\sum_{j=m+1}^{\infty} |\alpha_{1,j}| \right)^2 + \left(\sum_{k=2}^{n} \sum_{j=m+1}^{\infty} |\alpha_{k,j}| \right)^2 + \right.$$

$$\left. + \left(\sum_{k=n+1}^{\infty} |\alpha_{k,0}| \right)^2 + \left(\sum_{k=n+1}^{\infty} |\alpha_{k,1}| \right)^2 + \left(\sum_{k=n+1}^{\infty} \sum_{j=2}^{\infty} |\alpha_{k,j}| \right)^2 \right)$$

The following inequality holds $\sum_{i=\nu+1}^{\infty} \frac{1}{i^2} < \int_{\nu}^{\infty} \frac{\mathrm{d}x}{x^2} = \frac{1}{\nu}$.

Using the results of Theorem 1, the first, second, fourth and fifth expression are increased by

$$\begin{aligned}
\sum_{j=m+1}^{\infty} |\alpha_{0,j}| &\leq 2M_{0,2} \sum_{j=m}^{\infty} \frac{1}{j^2} < \frac{2M_{0,2}}{m-1} \\
\sum_{j=m+1}^{\infty} |\alpha_{1,j}| &\leq \frac{8M_{0,2}}{\pi} \sum_{j=m}^{\infty} \frac{1}{j^2} < \frac{8M_{0,2}}{\pi(m-1)} < \frac{4M_{0,2}}{m-1} \\
\sum_{k=n+1}^{\infty} |\alpha_{k,0}| &\leq 2M_{2,0} \sum_{k=n}^{\infty} \frac{1}{k^2} < \frac{2M_{2,0}}{n-1} \\
\sum_{k=n+1}^{\infty} |\alpha_{k,1}| &\leq \frac{8M_{2,0}}{\pi} \sum_{k=n}^{\infty} \frac{1}{k^2} < \frac{8M_{2,0}}{\pi(n-1)} < \frac{4M_{2,0}}{n-1}.
\end{aligned}$$

For the third and sixth expression we use (10) and then the Cauchy-Buniakowsky-Schwarz' inequality

$$\left(\sum_{k=2}^{n} \sum_{j=m+1}^{\infty} |\alpha_{k,j}| \right)^2 = \left(\sum_{k=2}^{n} \sum_{j=m+1}^{\infty} \frac{|\triangle_{k,j}|}{4kj} \right)^2 \leq$$

$$\leq \frac{1}{16} \left(\sum_{k=2}^{n} \sum_{j=m+1}^{\infty} \triangle_{k,j}^2 \right) \left(\sum_{k=2}^{n} \sum_{j=m+1}^{\infty} \frac{1}{k^2 j^2} \right) \leq 4M_{1,1}^2 \sum_{k=2}^{n} \frac{1}{k^2} \sum_{j=m+1}^{\infty} \frac{1}{j^2} \leq \frac{2\pi^2 M_{1,1}^2}{3m},$$

and respectively

$$\left(\sum_{k=n+1}^{\infty} \sum_{j=2}^{\infty} |\alpha_{k,j}| \right)^2 \leq \left(\sum_{k=n+1}^{\infty} \sum_{j=2}^{\infty} \frac{|\triangle_{k,j}|}{4kj} \right)^2 \leq$$

$$\leq 4M_{1,1}^2 \left(\sum_{k=n+1}^{\infty} \frac{1}{k^2} \right) \left(\sum_{j=2}^{\infty} \frac{1}{j^2} \right) \leq \frac{2\pi^2 M_{1,1}^2}{3n}.$$

Consequently

$$|f(x,y) - f_{n,m}(x,y)| \le \sqrt{6}\left(\frac{20M_{0,2}^2}{(m-1)^2} + \frac{20M_{2,0}^2}{(n-1)^2} + \frac{2\pi^2 M_{1,1}^2}{3m} + \frac{2\pi^2 M_{1,1}^2}{3n}\right)^{\frac{1}{2}} \to 0,$$

when $m, n \to \infty$. ∎

7 The Lagrange interpolation polynomial

For any grids $-1 \le x_0 < x_1 < \ldots < x_n \le 1,\ -1 \le y_0 < y_1 < \ldots < y_m \le 1$ and any $f : [-1,1]^2 \to \mathbb{R}$ the expression of the Lagrange interpolation polynomial is

$$L_{n,m}(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{m} f(x_i, y_j) lx_i(x) ly_j(y)$$

where

$$lx_i(x) = \prod_{k=0, k\neq i}^{n} \frac{x - x_k}{x_i - x_k}, \quad \text{and} \quad ly_j(x) = \prod_{l=0, l\neq j}^{m} \frac{y - y_l}{y_j - y_l}.$$

This polynomial satisfies the interpolation restrictions

$$L_{n,m}(x_k, y_l) = f(x_k, y_l), \quad \forall k \in \{0,1,\ldots,n\}, \text{ and } \forall l \in \{0,1,\ldots,m\}.$$

In the set of (n,m) degree polynomials there exists a unique interpolation polynomial.

If $x_i = \cos\frac{i\pi}{n},\ i \in \{0,1,\ldots,n\}$ and $y_j = \cos\frac{j\pi}{m},\ j \in \{0,1,\ldots,m\}$ then using the *discrete orthogonality* relations, [9],

$$\sum_{k=0}^{n} \gamma_k T_p(x_k) T_q(x_k) = \left\{ \begin{array}{lll} 0 & \text{if} & p \neq q \\ \frac{n}{2} & \text{if} & p = q \in \{1,2,\ldots,n-1\} \\ n & \text{if} & p = q \in \{0,n\} \end{array} \right. = n\alpha_p \delta_{p,q},$$

where

$$\gamma_{n,k} = \left\{ \begin{array}{lll} \frac{1}{2} & \text{if} & k \in \{0,n\} \\ 1 & \text{if} & k \in \{1,2,\ldots,n-1\} \end{array} \right. \quad \text{and } \alpha_{n,i} = \left\{ \begin{array}{lll} \frac{1}{2} & \text{if} & i \in \{1,2,\ldots,n-1\} \\ 1 & \text{if} & i \in \{0,n\} \end{array} \right.$$

the Lagrange interpolation polynomial may be written as

$$L_{n,m}(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{m} c_{i,j} T_i(x) T_j(y),$$

where $c_{i,j} = \frac{4}{nm}\gamma_{n,i}\gamma_{m,j}\sum_{k=0}^{n}\sum_{l=0}^{m}\gamma_{n,k}\gamma_{m,l} f(x_k, y_l) T_i(x_k) T_j(y_l)$.
This polynomial will be called the Lagrange-Chebyshev interpolating polynomial.

As in [4], the following statements occur:

Theorem 3. *(Aliasing of Chebyshev polynomials, [4]) For any* $n \geq 1$ *and* $0 \leq m \leq n$ *the polynomials* $T_m, T_{2n\pm m}, T_{4n\pm m}, \ldots$ *take the same values on the grid* $(\cos\frac{k\pi}{n})_{0\leq k\leq n}$.

Let $n, m \in \mathbb{N}^*$ be fixed. In $\mathbb{N}\times\mathbb{N}$ there is introduced the equivalence

$$(i_1, j_1) \sim (i_2, j_2) \quad \text{iff} \quad \begin{cases} i_1 + i_2 \,\vdots\, 2n \quad \text{or} \quad i_1 - i_2 \,\vdots\, 2n \\ \text{and} \\ j_1 + j_2 \,\vdots\, 2m \quad \text{or} \quad j_1 - j_2 \,\vdots\, 2m \end{cases}.$$

Denoting by $\widehat{(i,j)}$ the equivalence class, for fixed $k \in \{0, 1, \ldots, n\}$ and $l \in \{0, 1, \ldots, m\}$, the product $T_p(x_k)T_q(y_l)$ has the same value for any $(p, q) \in \widehat{(i,j)}$.

Theorem 4. *(Aliasing formula of Chebyshev coefficients) Let*

$$f(x, y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \alpha_{i,j} T_i(x) T_j(y)$$

and let $L_{n,m}(x, y) = \sum_{i=0}^{n}\sum_{j=0}^{m} c_{i,j} T_i(x) T_j(y)$ *be its Lagrange-Chebyshev interpolant. Then*

$$c_{i,j} = \sum_{(p,q)\in\widehat{(i,j)}} \alpha_{p,q}. \tag{15}$$

Proof. Supposing that $(c_{i,j})_{0\leq i\leq m, 0\leq j\leq m}$ are given by (15) and $\varphi(x, y) = \sum_{i=0}^{n}\sum_{j=0}^{m} c_{i,j} T_i(x) T_j(y)$. For any $(k, l) \in \{0, 1, \ldots, n\} \times \{0, 1, \ldots, m\}$

$$\varphi(x_k, y_l) = \sum_{i=0}^{n}\sum_{j=0}^{m} c_{i,j} T_i(x_k) T_j(y_l) = \sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{(p,q)\in\widehat{(i,j)}} \alpha_{p,q} T_i(x_k) T_j(y_l).$$

It is observed that when the indexes i, j, p, q go through their values then $(2pn \pm i, 2qm \pm j)$ go through $\mathbb{N}\times\mathbb{N}$ any two pairs are distinct. Thus, with Theorem 3,

$$\varphi(x_k, y_l) = \sum_{s=0}^{\infty}\sum_{t=0}^{\infty} \alpha_{s,t} T_s(x_k) T_t(y_l) = f(x_k, y_l).$$

The unicity of the interpolating polynomial in the set of (n, m) degree polynomials implies $L_{n,m} = \varphi$. ■

The explicit formulas corresponding to (15) are

$$
\begin{aligned}
c_{0,0} &= \sum_{p,q=0}^{\infty} \alpha_{2np,2mq} \\
c_{i,0} &= \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \alpha_{2np+i,2mq} + \sum_{p=1}^{\infty}\sum_{q=0}^{\infty} \alpha_{2np-i,2mq}, \quad i \in \{1,\ldots,n\} \\
c_{0,j} &= \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \alpha_{2np,2mq+j} + \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} \alpha_{2np,2mq-j}, \quad j \in \{1,\ldots,m\} \\
c_{i,j} &= \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \alpha_{2np+i,2mq+j} + \sum_{p=0}^{\infty}\sum_{q=1}^{\infty} \alpha_{2np+i,2mq-j} + \\
&\quad + \sum_{p=1}^{\infty}\sum_{q=0}^{\infty} \alpha_{2np-i,2mq+j} + \sum_{p=1}^{\infty}\sum_{q=1}^{\infty} \alpha_{2np-i,2mq-j}, \\
&\quad i \in \{1,\ldots,n\},\ j \in \{1,\ldots,m\}.
\end{aligned}
$$

A consequence of (15) is a relation between $L_{n,m}(x,y)$ and the approximation polynomial $f_{n,m}(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{m} \alpha_{i,j}T_i(x)T_j(y)$:

$$L_{n,m}(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{m} c_{i,j}T_i(x)T_j(y) =$$

$$= \sum_{i=0}^{n}\sum_{j=0}^{m} \left(\alpha_{i,j} + \sum_{q=1}^{\infty} \alpha_{i,2qm\pm j} + \sum_{p=1}^{\infty} \alpha_{2pn\pm i,j} + \sum_{p,q=1}^{\infty} \alpha_{2pn\pm i,2qm\pm j} \right) T_i(x)T_j(y).$$

Resetting $i := 2pn \pm i$ when $p = 1,2,\ldots$ the value of i varies from $n+1$ to ∞ and T_i becomes T_{ν_i}. T_{μ_j} is introduced analogously.

$$L_{n,m}(x,y) = f_{n,m}(x,y) + \sum_{i=0}^{n}\sum_{j=m+1}^{\infty} \alpha_{i,j}T_i(x)T_{\mu_j}(y) + \sum_{i=n+1}^{\infty}\sum_{j=0}^{m} \alpha_{i,j}T_{\nu_i}(x)T_j(y) +$$

$$+ \sum_{i=n+1}^{\infty}\sum_{j=m+1}^{\infty} \alpha_{i,j}T_{\nu_i}(x)T_{\mu_j}(y).$$

Then we find

$$|L_{n,m}(x,y) - f_{n,m}(x,y)| \leq \sum_{i=0}^{n}\sum_{j=m+1}^{\infty} |a_{i,j}| + \sum_{i=n+1}^{\infty}\sum_{j=0}^{\infty} |a_{i,j}|. \qquad (16)$$

Now we can prove the uniform convergence of the Lagrange-Chebyshev interpolation polynomials:

Theorem 5. *If function f has second order continuous partial derivatives then $\lim_{n,m\to\infty} L_{n,m} = f$ uniformly in $[-1,1]^2$.*

Proof. Using (14) and (16) we obtain

$$|f(x,y) - L_{n,m}(x,y)| \leq |f(x,y) - f_{n,m}(x,y)| + |f_{n,m}(x,y) - L_{n,m}(x,y)| \leq$$
$$\leq 2\left(\sum_{i=0}^{n}\sum_{j=m+1}^{\infty}|a_{i,j}| + \sum_{i=n+1}^{\infty}\sum_{j=0}^{\infty}|a_{i,j}|\right).$$

The rest of the proof follows the proof of Theorem 2. ∎

8 Applications

1. Integrating $f_{n,m}(x,y)$ on Ω there is obtained

$$\iint_{\Omega} f(x,y)\mathrm{d}x\mathrm{d}y \approx \iint_{\Omega} f_{n,m}(x,y)\mathrm{d}x\mathrm{d}y = 4\sum_{k=0,\ even}^{n}\sum_{j=0,\ even}^{m}\frac{\alpha_{k,j}}{(1-k^2)(1-j^2)}. \tag{17}$$

2. Computation of the first order partial derivatives. Practically, knowing the Chebyshev approximation polynomial $f_{n,m}(x,y) = \sum_{k=0}^{n}\sum_{j=0}^{m}\alpha_{k,j}T_k(x)T_j(y)$, and with the assumption that $\alpha_{k,j} \approx 0$ for $k > n > 4$, and for any $j \in \{0,1,\ldots,m\}$ the first n equations of the system (7) will be

$$\begin{cases} b_{0,j} - \frac{b_{2,j}}{2} & = & \alpha_{1,j} & \\ \frac{1}{2k}(b_{k-1,j} - b_{k+1,j}) & = & \alpha_{k,j}, & k \in \{2,3,\ldots,n-2\} \\ \frac{1}{2(n-1)}b_{n-2,j} & = & \alpha_{n-1,j} & \\ \frac{1}{2n}b_{n-1,j} & = & \alpha_{n,j} & \end{cases}$$

with the solution

$$\begin{array}{lcll} b_{n-1,j} & = & 2n\alpha_{n,j} & \\ b_{n-2.j} & = & 2(n-1)\alpha_{n-1,j} & \\ b_{k,j} & = & 2(k+1)\alpha_{k+1,j} + b_{k+2,j}, & k \in \{n-3, n-4, \ldots, 2, 1\} \\ b_{0,j} & = & \alpha_{1,j} + \frac{b_{2,j}}{2} & \end{array}.$$

Then $\frac{\partial f(x,y)}{\partial x} \approx \sum_{k=0}^{n-1}\sum_{j=0}^{m} b_{k,j}T_k(x)T_j(y)$.

The partial derivative $\frac{\partial f(x,y)}{\partial y}$ may be computed similarly.

Due to the truncation of the Chebyshev series the numerical result is influenced by the truncation error as well as by rounding errors. The automatic differentiation [3] is a method which eliminates the truncation error but it requires a specific computational environment related to the definition of the *elementary* functions (e.g. *apache commons-math3* v. 3.4).

9 Examples

Using a *Scilab* implementation the following results are obtained

1. $f(x,y) = \cos xy$ [2], Ch. 11, p. 2. The matrix of the coefficients is

0.880725579	0.	- 0.117388011	0.	0.001873213
0.	0.	0.	0.	0.
-0.117388011	0.	- 0.114883808	0.	0.002484444
0.	0.	0.	0.	0.
0.001873213	0.	0.002484444	0.	0.000603385

   The value of the indicator given by (5) is $3.97247 \cdot 10^{-10}$.

   On an equidistant grid of size 50×50 in $[0,1]^2$ the maximum absolute error is 0.000082141.

   The integral given by (17) is 3.784330902, while *Mathematica* gives 4SinIntegral[1] $\approx$ 3.78433228147.

2. $g(x) = \cos 10xy^2 + e^{-x^2}$ [2], Ch. 12, p. 6. The size of matrix of coefficients is 33×43.

   The value of the indicator given by (5) is 0.

   On an equidistant grid of size 50×50 in $[0,1]^2$ the maximum absolute error is $2.98594 \cdot 10^{-13}$.

   The integral given by (17) is 4.590369905, which is equal to that given by *Mathematica*.

10 Conclusions

An alternative to the Gaussian elimination method used in *Chebfun* software in order to construct an approximation polynomial of a function with two variables is presented.

Because the discrete Fourier transform is a common tool for the usual mathematical softwares, this approach has a simple implementation, but as a drawback, if the tolerance is the machine precision then it may require a large amount of memory.

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