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GENERAL ESTIMATES OF THE WEIGHTED APPROXIMATION ON INTERVAL $[0, \infty)$ USING MODULI OF CONTINUITY

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Abstract

We obtain results for weighted approximation on interval $[0, \infty)$ by general positive linear operators. Special attention is given to the weights 1 and $1/(1+x^2)$, $x \geq 0$. We give applications for the Szász-Mirakjan operators and for the Baskakov operators.

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1 Introduction

Fix a weight function ρ on interval $[0, \infty)$, that is a function which is continuous and strictly positive on this interval. We define the space $C_{\rho,\infty}[0, \infty)$ as

$$C_{\rho,\infty}[0, \infty) = \left\{ f \in C[0, \infty) | (\exists) \lim_{x \to \infty} \rho(x) f(x) \in \mathbb{R} \right\}. \tag{1}$$

On the space $C_{\rho,\infty}[0, \infty)$ we consider the norm

$$\|f\|_\rho = \sup_{x \in [0,\infty)} |f(x)| \rho(x).$$

The estimation of the degree of approximation of a function from $C_{\rho,\infty}[0, \infty)$, by a sequence of positive linear operators can be made with the aid of different weighted moduli, see for instance [6] [3], [8], [5], [7], [12] and many others. This method is possible, generally, for the larger space $C_\rho[0, \infty) = \{f \in C[0, \infty) | (\exists) M > 0 : \ |f(x)| \rho(x) \leq M, \ \forall x \geq 0\}$. In this paper we do not use this method.

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In the case of the special space $C_{\rho,\infty}[0,\infty)$, we point out that it is possible to use, in two variants, the usual moduli of first and second order. A first variant is to use estimates for operators acting on arbitrary intervals. The second variant consists in reducing the problem of approximation in the space $C_{\rho,\infty}[0,\infty)$ to the problem of approximation in the space $C[0,1]$ via a transformation. This last method of compactification was used by Bustamante [1], [2], for the problem of convergence. In this paper we are interested in obtaining quantitative results of the degree of approximation by using this transformation and to compare them with certain estimates which can be obtained directly.

To this purpose, let us denote

$$\psi(y) = \frac{y}{1-y}, \quad y \in [0,1). \tag{2}$$

The function $\psi : [0,1) \to [0,\infty)$ is a homeomorphism with inverse function

$$\psi^{-1}(x) = \frac{x}{1+x}, \quad x \in [0,\infty). \tag{3}$$

Consider the linear continuous operator $\Phi : C_{\rho,\infty}[0,\infty) \to C[0,1]$ defined by

$$\Phi(f,y) = \begin{cases} \rho(\psi(y))f(\psi(y)), & if\ y \in [0,1) \\ \lim_{x\to\infty} \rho(x)f(x), & if\ y = 1 \end{cases} \quad f \in C_{\rho,\infty}[0,\infty), \tag{4}$$

where the space $C[0,1]$ is endowed by the sup norm, $\|\cdot\|$. Operator Φ admits the inverse operator $\Phi^{-1} : C[0,1] \to C_{\rho,\infty}[0,\infty)$ given by

$$\Phi^{-1}(g,x) = \frac{g(\psi^{-1}(x))}{\rho(x)},\ g \in C[0,1],\ x \in [0,\infty). \tag{5}$$

Let $L : C_{\rho,\infty}[0,\infty) \to C_{\rho,\infty}[0,\infty)$ be a positive linear operator. Consider the linear operator $L^{\Phi} : C[0,1] \to C[0,1]$, given by:

$$L^{\Phi}(g) = (\Phi \circ L \circ \Phi^{-1})(g),\ g \in C[0,1], \tag{6}$$

which evidently is also a positive operator and satisfies the condition $\|f-L(f)\|_\rho = \|\Phi f - L^{\Phi}(\Phi f)\|$, for any $f \in C_{\rho,\infty}[0,\infty)$. Using this transformation the study of the weighted approximation by a sequence of operators $(L_n)_n$, $L_n : C_{\rho,\infty}[0,\infty) \to C_{\rho,\infty}[0,\infty)$ is reduced to the uniform approximation by the sequence of operators $(L_n^{\Phi})_n$, on the more simple space $C[0,1]$.

In order to recall some general notions consider an arbitrary interval I. Denote by $\mathcal{F}(I)$ the space of real functions on I. Denote by e_j , $j = 0,1,2,\ldots$, the functions $e_j(t) = t^j$, $(t \in I)$. For a function $g \in \mathcal{F}(I)$, and a real number $h > 0$, consider the first and the second order moduli of continuity:

$$\begin{aligned} \omega_1(g,h) &= \sup\{|g(u)-g(v)|,\ u,v \in I,\ |u-v| \le h\}, \\ \omega_2(g,h) &= \sup\{|g(u)-2g((u+v)/2)+g(v)|, u,v \in I,\ |u-v| \le 2h\}. \end{aligned}$$

In these definitions we admit the possibility that these moduli are $+\infty$. In the particular case where g belongs to $B(I)$, the space of bounded functions on I, then these moduli are sure finite.

There exist different estimates with moments and moduli of the first and the second order. See for instance [11]. In what follows we shall use only two general estimates.

First consider the estimate of Mond [9] which is a variant of the Shisha and Mond estimate [13]. Originally it was given on the interval $I = [a, b]$, but it is true for arbitrary intervals I, see [11], (Remark 1.2.5, pg. 18).

Theorem A *Let $L : V \to \mathcal{F}(I)$ be a linear positive operator, where V is a linear subspace of $\mathcal{F}(I)$ such that $e_0, e_1, e_2 \in V$ and $g \in V$. For all $y \in I$ and $h > 0$ one has*

$$\begin{aligned} |L(g,y) - g(y)| \leq & \; |g(y)||L(e_0, y) - 1| \\ & + \left(L(e_0, y) + \frac{1}{h^2} L\left((e_1 - ye_0)^2, y\right)\right) \cdot \omega_1(g,h). \qquad (7) \end{aligned}$$

The second one is an optimal estimate with the usual first and second order moduli of the first author. See [10] or [11], where this estimate is given for functionals, and in more general conditions.

Theorem B *Let $L : V \to \mathcal{F}(I)$ be a linear positive operator, where V is linear subspace of $C(I)$ such that $e_0, e_1, e_2 \in V$ and $g \in V$. Let $y \in I$ and $h > 0$, such that $h \leq (1/2)length(I)$. Then*

$$\begin{aligned} |L(g,y) - g(y)| \leq & \; |g(y)||L(e_0, y) - 1| + \frac{1}{h} \cdot |L(e_1 - ye_0, y)| \cdot \omega_1(g,h) \\ & + \left(L(e_0, y) + \frac{1}{2h^2} L\left((e_1 - ye_0)^2, y\right)\right) \cdot \omega_2(g,h). \qquad (8) \end{aligned}$$

2 Main results

Theorem 1. *Let ρ be a continuous strictly positive weight on interval $[0, \infty)$. Let $L : C_{\rho,\infty}[0, \infty) \to C_{\rho,\infty}[0, \infty)$ be a linear positive operator and $f \in C_{\rho,\infty}[0, \infty)$. For all $x \in [0, \infty)$ and $h > 0$ one has*

$$\begin{aligned} |L(f,x) - f(x)| \leq & \; |f(x)|\Big|\rho(x) L\Big(\frac{e_0}{\rho}, x\Big) - 1\Big| \\ & + \Big[L\Big(\frac{e_0}{\rho}, x\Big) + \frac{1}{h^2} L_n\Big(\frac{(\psi^{-1} - \psi^{-1}(x)e_0)^2}{\rho}, x\Big)\Big] \omega_1(\Phi f, h), \qquad (9) \\ |L(f,x) - f(x)| \leq & \; |f(x)|\Big|\rho(x) L\Big(\frac{e_0}{\rho}, x\Big) - 1\Big| \\ & + \frac{1}{h}\Big|L\Big(\frac{\psi^{-1} - \psi^{-1}(x)e_0}{\rho}, x\Big)\Big|\omega_1(\Phi f, h) \\ & + \Big[L\Big(\frac{e_0}{\rho}, x\Big) + \frac{1}{2h^2} L\Big(\frac{(\psi^{-1} - \psi^{-1}(x)e_0)^2}{\rho}, x\Big)\Big] \omega_2(\Phi f, h). \qquad (10) \end{aligned}$$

Proof. Let $f \in C_{\rho,\infty}[0,\infty)$ and $x \in [0,\infty)$. First of all let us show that the expressions given in relations (9) and (10) make sense, namely we have $\frac{e_0}{\rho} \in C_{\rho,\infty}[0,\infty)$ and $\frac{(\psi^{-1}-\psi^{-1}(x))^2}{\rho} \in C_{\rho,\infty}[0,\infty)$. Indeed, the first one follows from the limit $\lim_{t\to\infty} \rho(t)\cdot \frac{e_0}{\rho}(y) = 1$. The second one follows from the limit

$$\lim_{t\to\infty} \rho(t)\cdot \Big(\frac{(\psi^{-1}-\psi^{-1}(x))^2}{\rho}\Big)(t) = \lim_{t\to\infty} \Big(\frac{t}{1+t} - \frac{x}{1+x}\Big)^2 = \frac{1}{(1+x)^2}.$$

Let us denote $g = \Phi f$ and $y = \psi^{-1}(x)$. Note that $y \in [0,1)$. Let us consider the operator L^Φ, defined in relation (6). Using relations (4), (5), (6) and the fact that $y < 1$ we obtain

$$\begin{aligned} L^\Phi(g,y) &= \Phi(L(\Phi^{-1}g),y) = \Phi(L(f),y) = \rho(\psi(y))L(f,\psi(y)) = \rho(x)L(f,x). \\ g(y) &= \Phi(f,y) = \rho(\psi(y))f(\psi(y)) = \rho(x)f(x). \end{aligned}$$

So that we have

$$|L^\Phi(g,y) - g(y)| = \rho(x)|L(f,x) - f(x)| \tag{11}$$

Also we have

$$\begin{aligned} L^\Phi(e_0,y) &= \Phi(L(\Phi^{-1}e_0),y) = \rho(\psi(y))L\Big(\frac{e_0}{\rho},\psi(y)\Big) \\ &= \rho(x)L\Big(\frac{e_0}{\rho},x\Big), \qquad (12) \\ L^\Phi((e_1-ye_0)^j,y) &= \Phi(L(\Phi^{-1}(e_1-ye_0)^j),y) \\ &= \rho(\psi(y))L(\Phi^{-1}(e_1-ye_0)^j,y) \\ &= \rho(x)L\Big(\frac{(e_1-ye_0)^j\circ\psi^{-1}}{\rho},x\Big) \\ &= \rho(x)L\Big(\frac{(\psi^{-1}-\psi^{-1}(x)e_0)^j}{\rho},x\Big). \qquad (13) \end{aligned}$$

where j=1,2.

If we apply estimates (7), (8) for $L = L^\Phi$ and take into account relations (11), (12), (13) and devide by $\rho(x) > 0$ we obtain (9) and (10). □

In what follows we consider two important weights $\rho(x) = 1$ and $\rho(x) = 1/(1+x^2)$, $x \ge 0$. The approximation with regard to weight $\rho = e_0$ is equivalent to uniform approximation.

Theorem 2. *Let $\rho = e_0$. Let $L : C_{\rho,\infty}[0,\infty) \to C_{\rho,\infty}[0,\infty)$ be a linear positive operator and $f \in C_{\rho,\infty}[0,\infty)$. For all $x \in [0,\infty)$*

$$\begin{aligned} |L(f,x) - f(x)| &\le |f(x)|\cdot|L(e_0,x) - 1| \\ &\quad + (L(e_0,x)+1)\omega_1\left(f\circ\psi, \frac{\sqrt{L((e_1-xe_0)^2,x)}}{1+x}\right) \qquad (14) \end{aligned}$$

$$\begin{aligned}|L(f,x)-f(x)| &\le |f(x)|\cdot|L(e_0,x)-1|\\ &\quad+\sqrt{L(e_0,x)}\omega_1\left(f\circ\psi,\frac{\sqrt{L((e_1-xe_0)^2,x)}}{1+x}\right)\\ &\quad+\Big(L(e_0,x)+\frac{1}{2}\Big)\omega_2\left(f\circ\psi,\frac{\sqrt{L((e_1-xe_0)^2,x)}}{1+x}\right). \quad (15)\end{aligned}$$

Proof. Since, for $t\in[0,\infty)$ we have

$$\left(\frac{t}{1+t}-\frac{x}{1+x}\right)^2\le\frac{(t-x)^2}{(1+x)^2},$$

we obtain

$$L((\psi^{-1}-\psi^{-1}e_0)^2,x)\le\frac{L((e_1-xe_0)^2,x)}{(1+x)^2}.$$

Next, using the Cauchy-Schwarz inequality we have

$$\begin{aligned}|L(\psi^{-1}-\psi^{-1}e_0,x)| &\le \sqrt{L((\psi^{-1}-\psi^{-1}e_0)^2,x)}\sqrt{L(e_0,x)}\\ &\le \frac{\sqrt{L(e_1-xe_0)^2,x)}}{1+x}\sqrt{L(e_0,x)}.\end{aligned}$$

Applying relations (9) and (10) for $h=\sqrt{L((e_1-xe_0)^2,x)}/(1+x)$ and taking into account the relations above we obtain (14) and (15). □

Remark 1. If we apply directly Theorem A and Theorem B for $I=[0,\infty)$, $g=f$, $y=x$ and $h=\sqrt{L((e_1-xe_0)^2,x)}$ we obtain

$$\begin{aligned}|L(f,x)-f(x)| &\le |f(x)|\cdot|L(e_0,x)-1|\\ &\quad+(L(e_0,x)+1)\omega_1\left(f,\sqrt{L(e_1-xe_0)^2,x)}\right), \quad (16)\\ |L(f,x)-f(x)| &\le |f(x)|\cdot|L(e_0,x)-1|\\ &\quad+|L(e_1-xe_0,x)|\omega_1\left(f,\sqrt{L(e_1-xe_0)^2,x)}\right)\\ &\quad+\Big(L(e_0,x)+\frac{1}{2}\Big)\omega_2\left(f,\sqrt{L(e_1-xe_0)^2,x)}\right). \quad (17)\end{aligned}$$

For $h=\sqrt{L((e_1-xe_0)^2,x)}$ if we compare the moduli:

$$\begin{aligned}\omega_1\Big(f\circ\psi,\frac{h}{1+x}\Big) &= \sup\Big\{|f(u)-f(v)|,\ u,v\ge0,\ \frac{|u-v|}{(1+u)(1+v)}\le\frac{h}{1+x}\Big\}\\ \omega_1(f,h) &= \sup\{|f(u)-f(v)|,\ u,v\ge0,\ |u-v|\le h\},\end{aligned}$$

it follows that there is not a general inequality between them. It is similar for the corresponding moduli of second order. Then the same conclusion is about the estimates (14) and (15) compared with the estimates (16) and (17), respectively.

Theorem 3. *Let $\rho = e_0/(e_0+e_2)$. Let $L : C_{\rho,\infty}[0,\infty) \to C_{\rho,\infty}[0,\infty)$ be a linear positive operator and $f \in C_{\rho,\infty}[0,\infty)$. For all $x \in [0,\infty)$ one has*

$$\begin{aligned}\rho(x)|L(f,x)-f(x)| &\leq \rho(x)|f(x)|\cdot\left|\frac{L(e_0+e_2,x)}{1+x^2}-1\right| \\ &+\left[\frac{L(e_0+e_2,x)}{1+x^2}+1\right]\omega_1\left(\Phi f,\sqrt{\frac{L(e_1-xe_0)^2,x)}{(1+x^2)(1+x)^2}}\right), \quad (18)\\ \rho(x)|L(f,x)-f(x)| &\leq \rho(x)|f(x)|\cdot\left|\frac{L(e_0+e_2,x)}{1+x^2}-1\right| \\ &+\sqrt{\frac{L(e_0+e_2,x)}{1+x^2}}\omega_1\left(\Phi f,\sqrt{\frac{L((e_1-xe_0)^2,x)}{(1+x^2)(1+x)^2}}\right) \\ &+\Big[\frac{L(e_0+e_2,x)}{1+x^2}+\frac{1}{2}\Big]\omega_2\left(\Phi f,\sqrt{\frac{L((e_1-xe_0)^2,x)}{(1+x^2)(1+x)^2}}\right). \quad (19)\end{aligned}$$

Proof. We apply relations (9) and (10) multiplied by $\rho(x)$. First note that $\rho(x)L\left(\frac{e_0}{\rho},x\right) = \frac{L(e_0+e^2,x)}{1+x^2}$ and

$$\frac{(\psi^{-1}-\psi^{-1}(x)e_0)^2}{\rho}(t) = (1+t^2)\Big(\frac{t}{1+t}-\frac{x}{1+x}\Big)^2 \leq \Big(\frac{t-x}{1+x}\Big)^2. \qquad (20)$$

Take $h = \sqrt{\frac{L(e_1-xe_0)^2,x)}{(1+x^2)(1+x)^2}}$. Then

$$\begin{aligned}&\rho(x)\left[L\Big(\frac{e_0}{\rho},x\Big)+\frac{1}{h^2}L_n\Big(\frac{(\psi^{-1}-\psi^{-1}(x)e_0)^2}{\rho},x\Big)\right]\omega_1(\Phi f,h)\\ = \quad &\left[\frac{L(e_0+e_2,x)}{1+x^2}+1\right]\omega_1\left(\Phi f,\sqrt{\frac{L((e_1-xe_0)^2,x)}{(1+x^2)(1+x)^2}}\right).\end{aligned}$$

In a similar way we can treat the third term for (10), multiplied by $\rho(x)$. Finally, in order to estimate the coefficient of the second term (10), multiplied by $\rho(x)$, by using the Cauchy-Schwarz inequality and inequality (20) we obtain

$$\begin{aligned}&\frac{\rho(x)}{h}\Big|L\Big(\frac{\psi^{-1}-\psi^{-1}(x)e_0}{\rho},x\Big)\Big| \leq \frac{\rho(x)}{h}L\Big(\Big|\frac{\psi^{-1}-\psi^{-1}(x)e_0}{\rho}\Big|,x\Big)\\ \leq \quad &\frac{\rho(x)}{h}\sqrt{L\Big(\frac{|\psi^{-1}-\psi^{-1}(x)e_0|^2}{\rho},x\Big)}\sqrt{L\Big(\frac{e_0}{\rho},x\Big)}\\ \leq \quad &\frac{\rho(x)}{h}\sqrt{\frac{L((e_1-xe_0)^2,x)}{(1+x)^2}}\sqrt{L(e_0+e_2,x)}\\ = \quad &\sqrt{\frac{L(e_0+e_2,x)}{1+x^2}}.\end{aligned}$$

$\square$

Remark 2. Similar estimates can be obtained from Theorem A and Theorem B for $I=[0,\infty)$, $g=f$, $y=x$ and $h=\sqrt{L((e_1-xe_0)^2,x)}$ if we multiply relations (16) and (17) by $\rho(x)$. The obtained relations are better in certain cases than relations (18) and (19), while in other cases the latter are better.

3 Applications to Szász-Mirakjan operators

The Szász-Mirakjan operators are defined as

$$S_n(f,x)=\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}e^{-nx}f\left(\frac{k}{n}\right),\quad x\geq 0$$

where $f:[0,\infty)\to\mathbb{R}$ is such that the series are convergent. Denote $s_{n,k}(x)=e^{-nx}\frac{(nx)^k}{k!}$. The following lemma is well known.

Lemma 1. *For $x\geq 0$ and $n\geq 1$ one has*

$$S_n(e_0,x)=1,$$
$$S_n(e_1,x)=x,$$
$$S_n(e_2,x)=x^2+\frac{x}{n}.$$

In order to apply the results from the previous sections to the Szász-Mirakjan operators for the weights $\rho=e_0$ and $\rho_0=\frac{e_0}{e_0+e_2}$ we must prove the following theorem.

Theorem 4. *For $\rho=e_0$ or $\rho=\frac{e_0}{e_0+e_2}$ we have*

$$S_n(C_{\rho,\infty}[0,\infty))\subset C_{\rho,\infty}[0,\infty),\ n\in\mathbb{N}. \tag{21}$$

Proof. Fix $n\in\mathbb{N}$. Let $f\in C_{\rho,\infty}[0,\infty)$ and denote $\ell=\lim_{x\to\infty}\rho(x)f(x)$. Let $\varepsilon>0$. We write

$$\rho(x)S_n(f,x)=\ell\rho(x)S_n\left(\frac{e_0}{\rho},x\right)+\rho(x)S_n\left(\frac{f\rho-\ell e_0}{\rho},x\right)=:T_n^1(x)+T_n^2(x). \tag{22}$$

If $\rho=e_0$, then $\rho(x)S_n(\frac{e_0}{\rho},x)=1$. If $\rho=\frac{e_0}{e_0+e_2}$, then from Lemma 1 we obtain $\rho(x)S_n(\frac{e_0}{\rho},x)=1+\frac{x}{n(x^2+1)}$. Therefore in both cases we have

$$\lim_{x\to\infty}\rho(x)S_n\left(\frac{e_0}{\rho},x\right)=1. \tag{23}$$

Then there is $x_1>0$, such that

$$|T_n^1(x)-\ell|<\varepsilon,\quad x\geq x_1. \tag{24}$$

Since $\ell=\lim_{x\to\infty}\rho(x)f(x)$, there is $\delta>0$, and $M>0$, such that

$$|\rho(x)f(x)-\ell|<\varepsilon,\quad \text{for } x\geq\delta,\quad \text{and}\quad |\rho(x)f(x)|\leq M,\quad \text{for } x\geq 0.$$

Denote $b_{n,k}(x) = (\rho\left(\frac{k}{n}\right) f\left(\frac{k}{n}\right) - \ell)/(\rho\left(\frac{k}{n}\right))s_{n,k}(x)$. Choose an integer $m > \delta n$. We have

$$\begin{aligned} |T_n^2(x)| &\leq \rho(x)\sum_{k=0}^{m}|b_{n,k}(x)| + \rho(x)\sum_{k=m+1}^{\infty}|b_{n,k}(x)| \\ &< 2M\rho(x)\sum_{k=0}^{m}\frac{1}{\rho\left(\frac{k}{n}\right)}s_{n,k}(x) + \varepsilon\rho(x)\sum_{k=m+1}^{\infty}\frac{1}{\rho\left(\frac{k}{n}\right)}s_{n,k}(x) \\ &= :V_n^1(x) + V_n^2(x). \end{aligned}$$

Then $V_n^2(x) \leq \varepsilon\rho(x)S_n\left(\frac{e_0}{\rho}, x\right)$. From relation (23) it follows that there is $x_2 > 0$, such that

$$V_n^2(x) < 2\varepsilon \quad \text{for} \quad x \geq x_1. \tag{25}$$

Finally let us estimate $V_n^1(x)$. If $\rho = e_0$ then we have $V_n^1(x) = 2M\sum_{k=0}^{m} s_{n,k}(x)$. If $\rho = \frac{e_0}{e_0+e_2}$, then using the equality $\left(\frac{k}{n}\right)^2 + 1 = \frac{k(k-1)}{n^2} + \frac{k}{n^2} + 1$ we obtain

$$\begin{aligned} V_n^1(x) &= 2M\rho(x)\left[x^2\sum_{k=2}^{m}s_{n,k-2}(x) + \frac{x}{n}\sum_{k=1}^{m}s_{n,k-1}(x) + \sum_{k=0}^{m}s_{n,k}(x)\right] \\ &\leq 8M\sum_{k=0}^{m}s_{n,k}(x). \end{aligned}$$

Then in both cases we obtain

$$V_n^1(x) \leq 8M\sum_{k=0}^{m}s_{n,k}(x).$$

Since $\lim_{y\to\infty}\sum_{k=0}^{m} e^{-y}\frac{y^k}{k!} = 0$, there is y_0, such that $\sum_{k=0}^{m} e^{-y}\frac{y^k}{k!} < \varepsilon$, for all $y \geq y_0$. Denote $x_3 = y_0/n$. Then

$$V_n^1(x) < 8M\varepsilon, \quad \text{for} \quad x \geq x_3. \tag{26}$$

From relations (22), (24), (25), (25), (26) we obtain

$$|\rho(x)S_n(f,x) - \ell| \leq \varepsilon(8M+3), \quad x \geq \max\{x_1, x_2, x_3\}.$$

Hence $\lim_{x\to\infty}\rho(x)S_n(f,x) = \ell$. Then $S_n(f) \in C_{\rho,\infty}[0,\infty)$. Since $f \in C_{\rho,\infty}[0,\infty)$ was chosen arbitrarily the theorem follows. $\square$

We move on to construct estimates of the degree of weighted approximation. First consider the weight $\rho(x) = 1$, $x \geq 0$.

Theorem 5. *For $\rho = e_0$, $f \in C_{\rho,\infty}[0,\infty)$, $x \in [0,\infty)$, $n \in \mathbb{N}$ it holds:*

$$|S_n(f,x) - f(x)| \leq 2\omega_1\left(f \circ \psi, \sqrt{\frac{x}{n(1+x)^2}}\right), \tag{27}$$

$$|S_n(f,x) - f(x)| \leq \omega_1\Big(f \circ \psi, \sqrt{\frac{x}{n(1+x)^2}}\Big) + \frac{3}{2}\omega_2\Big(f \circ \psi, \sqrt{\frac{x}{n(1+x)^2}}\Big), \tag{28}$$

$$\|S_n f - f\| \leq 2\omega_1\left(f \circ \psi, \frac{1}{2\sqrt{n}}\right), \tag{29}$$

$$\|S_n f - f\| \leq \omega_1\left(f \circ \psi, \frac{1}{2\sqrt{n}}\right) + \frac{3}{2}\omega_2\left(f \circ \psi, \frac{1}{2\sqrt{n}}\right). \tag{30}$$

Proof. By taking into account Lemma 1 we have $S_n((e_1 - xe_0)^2, x) = \frac{x}{n}$. We apply relations (14) and (15). □

Corollary 1. *If $f \in C_{e_0,\infty}[0,\infty)$ then*

$$\lim_{n\to\infty} \|S_n f - f\| = 0.$$

Proof. We take into account that $f \circ \psi \in C[0,1]$. □

Theorem 6. *For $\rho = e_0$, $f \in C_{\rho,\infty}[0,\infty)$, and $x \in [0,\infty)$ it holds:*

$$|S_n(f,x) - f(x)| \leq 2\omega_1\left(f, \sqrt{\frac{x}{n}}\right), \tag{31}$$

$$|S_n(f,x) - f(x)| \leq \frac{3}{2}\omega_2\left(f, \sqrt{\frac{x}{n}}\right). \tag{32}$$

Proof. We apply relations (16) and (17) and Lemma 1. □

Remark 3. From relations (31) and (32) we cannot obtain Corollary 1. We can only obtain the uniform convergence of $S_n f$ to f on the compacts subsets of $[0,\infty)$. Moreover, no other choice of the argument $h = h_n > 0$ in Theorem A or Theorem B, for $L = S_n$ leads estimates in sup-norm of the degree of approximation.

Remark 4. Corollary 1 can be obtained also in another mode. It is easy to show that if $f \in C_{e_0,\infty}[0,\infty)$, then $f \circ e_2 \in C_{e_0,\infty}[0,\infty)$. Also it is easy to show that all the functions from the space $C_{e_0,\infty}[0,\infty)$ are uniformly continuous. Then we can apply the following theorem of V. Totik [14].

Theorem C *If $f \in C[0,\infty)$ is so that $f \circ e_2$ is uniformly continuous on $[0,\infty)$, then*

$$\lim_{n\to\infty} S_n(f) = f \quad uniformly \ \ on \ \ [0,\infty)$$

.

Now consider the weight $\rho(x) = \frac{1}{1+x^2}$, $x \geq 0$.

Theorem 7. *Let* $\rho = \frac{e_0}{e_0+e_2}$ *and* $f \in C_{\rho,\infty}[0,\infty)$. *For* $x \in [0,\infty)$, $n \in \mathbb{N}$ *it holds:*

$$\begin{aligned}\rho(x)|S_n(f,x)-f(x)| &\le \frac{x}{n(1+x^2)}|\rho(x)f(x)| \\ &\quad + \left(2+\frac{x}{n(1+x^2)}\right)\omega_1\left(\Phi f, \sqrt{\frac{x}{n(1+x)^2(1+x^2)}}\right), \qquad (33)\\ \rho(x)|S_n(f,x)-f(x)| &\le \frac{x}{n(1+x^2)}|\rho(x)f(x)| \\ &\quad + \sqrt{1+\frac{x}{n(1+x^2)}}\omega_1\left(\Phi f, \sqrt{\frac{x}{n(1+x)^2(1+x^2)}}\right) \\ &\quad + \left(\frac{3}{2}+\frac{x}{n(1+x^2)}\right)\omega_2\left(\Phi f, \sqrt{\frac{x}{n(1+x)^2(1+x^2)}}\right), \qquad (34)\\ \|S_nf-f\|_\rho &\le \frac{1}{2n}\|f\|_\rho + \left(2+\frac{1}{2n}\right)\omega_1\left(\Phi f, \frac{0.423}{\sqrt{n}}\right). \qquad (35)\\ \|S_nf-f\|_\rho &\le \frac{1}{2n}\|f\|_\rho + \sqrt{2+\frac{1}{2n}}\omega_1\left(\Phi f, \frac{0.423}{\sqrt{n}}\right) \\ &\quad + \left(\frac{3}{2}+\frac{1}{2n}\right)\omega_2\left(\Phi f, \frac{0.423}{\sqrt{n}}\right). \qquad (36)\end{aligned}$$

Proof. Relations (33) and (34) follow from (18), (19) by taking into account Lemma 1.

Relations (35) and (36) follow from (33) and (34) and the following inequalities

$$\sqrt{\frac{x}{(1+x^2)(1+x)^2}} \le 0.423, \quad \frac{x}{1+x^2} \le \frac{1}{2}, \quad (x \ge 0).$$

□

Corollary 2. *If* $\rho = \frac{e_0}{e_0+e_2}$ *and* $f \in C_{\rho,\infty}[0,\infty)$ *then*

$$\lim_{n\to\infty} \|S_nf-f\|_\rho = 0. \qquad (37)$$

Proof. We apply (35) or (36) by taking into account that $\Phi f \in C[0,1]$. □

Theorem 8. *Let* $\rho = \frac{e_0}{e_0+e_2}$. *For any* $f \in C_{\rho,\infty}[0,\infty)$ *and* $x \in [0,\infty)$ *it holds:*

$$\begin{aligned}\rho(x)|S_n(f,x)-f(x)| &\le \frac{1+x}{1+x^2}\omega_1\Big(f,\frac{1}{\sqrt{n}}\Big), \qquad (38)\\ \rho(x)|S_n(f,x)-f(x)| &\le \frac{2+x}{2(1+x^2)}\omega_2\Big(f,\frac{1}{\sqrt{n}}\Big),. \qquad (39)\end{aligned}$$

Consequently, for any $n \in \mathbb{N}$ *we have*

$$\begin{aligned}\|S_nf-f\|_\rho &\le \frac{1+\sqrt{2}}{2}\omega_1\Big(f,\frac{1}{\sqrt{n}}\Big), \qquad (40)\\ \|S_nf-f\|_\rho &\le \frac{\sqrt{2}}{4}\omega_2\Big(f,\frac{1}{\sqrt{n}}\Big),. \qquad (41)\end{aligned}$$

Proof. Relations (38) and (39) follow from Theorem A and Theorem B, respectively, by taking $h = \frac{1}{\sqrt{n}}$. Then relations (40) and (41) follow from (38) and (39), respectively, since

$$\sup_{x\in[0,\infty)} \frac{1+x}{1+x^2} = \frac{1+\sqrt{2}}{2} \qquad \sup_{x\in[0,\infty)} \frac{2+x}{2(1+x^2)} = \frac{\sqrt{2}}{4}$$

□

Remark 5. In contrast with the case of the weight $\rho = e_0$ now we obtained estimates in norm of the degree of approximation. However from relations (40) and (41) we cannot deduce Corollary 2, because the condition $f \in C_{\rho,\infty}[0,\infty)$ does not imply that $\lim_{n\to\infty} \omega_j\left(f, \frac{1}{\sqrt{n}}\right) = 0$, for $j = 1, 2$. Indeed, consider the function $f \in \mathcal{F}[0,\infty)$ defined in the following mode. For any $k \in \mathbb{N} = \{1, 2, \ldots\}$, $f(k) = k$, $f\left(k \pm \frac{1}{2k}\right) = 0$ and f is linear on each of the intervals of the form $\left[k - \frac{1}{2k}, k\right]$, $\left[k, k + \frac{1}{2k}\right]$, $\left[k + \frac{1}{2k}, k + 1 - \frac{1}{2(k+1)}\right]$ and also $f(x) = 0$ for $x \in \left[0, \frac{1}{2}\right]$. Then $f \in C_{\rho,\infty}[0,\infty)$ and $\omega_j\left(f, \frac{1}{\sqrt{n}}\right) = \infty$, for $j = 1, 2$. No other choice of $h = h_n$ in (38) or (39) solves the problem, since all choices must be of the form $h_n = \frac{q}{\sqrt{n}}$, with q constant.

4 Applications to Baskakov operators

The Baskakov operators are defined by the relation

$$BA_n(f,x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{x^k}{(x+1)^{n+k}} f\left(\frac{k}{n}\right), \; x \geq 0, f : [0,\infty) \to \mathbb{R}, \; n \in \mathbb{N},$$

where f is taken such that the series are convergent. Denote $q_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(x+1)^{n+k}}$, $k \in \mathbb{N}$, $x \geq 0$.

Lemma 2. *For the test functions $e_j(x) = x^j, \; j = 0, 1, 2$ one has*

$$\begin{aligned} BA_n(e_0, x) &= 1 \\ BA_n(e_1, x) &= x \\ BA_n(e_2, x) &= x^2 + \frac{x(x+1)}{n} \end{aligned}$$

First we prove the following theorem.

Theorem 9. *For $\rho = e_0$ or $\rho = \frac{e_0}{e_0+e_2}$ we have*

$$BA_n(C_{\rho,\infty}[0,\infty)) \subset C_{\rho,\infty}[0,\infty), \; n \in \mathbb{N}. \tag{42}$$

Proof. Fix $n \in \mathbb{N}$. Let $f \in C_{\rho,\infty}[0,\infty)$ and denote $\ell = \lim_{x\to\infty} \rho(x)f(x)$. Let $\varepsilon > 0$. We write

$$\rho(x)BA_n(f,x) = \ell\rho(x)BA_n\left(\frac{e_0}{\rho},x\right) + \rho(x)BA_n\left(\frac{f\rho - \ell e_0}{\rho},x\right) =: \tilde{T}_n^1(x) + \tilde{T}_n^2(x). \tag{43}$$

If $\rho = e_0$, then $\rho(x)BA_n(\frac{e_0}{\rho},x) = 1$. If $\rho = \frac{e_0}{e_0+e_2}$, then from Lemma 1 we obtain $\rho(x)BA_n(\frac{e_0}{\rho},x) = 1 + \frac{x(1+x)}{n(x^2+1)}$. In this case we obtain

$$\lim_{x\to\infty} \rho(x)BA_n\left(\frac{e_0}{\rho},x\right) = \frac{n+1}{n}. \tag{44}$$

Denote

$$\tilde{\ell} = \begin{cases} \ell, & \rho = e_0 \\ \frac{n+1}{n}\ell, & \rho = \frac{e_0}{e_0+e_2}. \end{cases}$$

Then in both the cases there is $x_1 > 0$, such that

$$|\tilde{T}_n^1(x) - \tilde{\ell}| < \varepsilon, \quad x \geq x_1. \tag{45}$$

Since $\ell = \lim_{x\to\infty} \rho(x)f(x)$, there is $\delta > 0$, and $M > 0$, such that

$$|\rho(x)f(x) - \ell| < \varepsilon, \quad \text{for } x \geq \delta, \quad \text{and} \quad |\rho(x)f(x)| \leq M, \quad \text{for } x \geq 0.$$

Denote $c_{n,k}(x) = (\rho\left(\frac{k}{n}\right)f\left(\frac{k}{n}\right) - \ell)/(\rho\left(\frac{k}{n}\right))q_{n,k}(x)$. Choose an integer $m > \delta n$. We have

$$\begin{aligned} |\tilde{T}_n^2(x)| &\leq \rho(x)\sum_{k=0}^{m}|c_{n,k}(x)| + \rho(x)\sum_{k=m+1}^{\infty}|c_{n,k}(x)| \\ &< 2M\rho(x)\sum_{k=0}^{m}\frac{1}{\rho\left(\frac{k}{n}\right)}q_{n,k}(x) + \varepsilon\rho(x)\sum_{k=m+1}^{\infty}\frac{1}{\rho\left(\frac{k}{n}\right)}q_{n,k}(x) \\ &= :\tilde{V}_n^1(x) + \tilde{V}_n^2(x). \end{aligned}$$

Then $\tilde{V}_n^2(x) \leq \varepsilon\rho(x)BA_n\left(\frac{e_0}{\rho},x\right)$. From relation (44) it follows that there is $x_2 > 0$, such that

$$\tilde{V}_n^2(x) < 2\varepsilon \quad \text{for} \quad x \geq x_2. \tag{46}$$

It remains to estimate $\tilde{V}_n^1(x)$. If $\rho = e_0$ then we have $\tilde{V}_n^1(x) = 2M\sum_{k=0}^{m} q_{n,k}(x)$. If $\rho = \frac{e_0}{e_0+e_2}$, then we use the equality $\left(\frac{k}{n}\right)^2 + 1 = \frac{k(k-1)}{n^2} + \frac{k}{n^2} + 1$. We have

$$\begin{aligned} \frac{k}{n^2}q_{n,k}(x) &= \frac{n+k-1}{n^2}\cdot\frac{x}{1+x}q_{n,k-1}(x) \leq q_{n,k-1}(x),\ k \leq 1, \\ \frac{k(k-1)}{n^2}q_{n,k}(x) &= \frac{(n+k-1)(n+k-2)x^2}{n^2(1+x)^2}q_{n,k-2}(x) \leq q_{n,k-2}(x),\ k \leq 2. \end{aligned}$$

Then

$$\begin{aligned}\tilde{V}_n^1(x) &= 2M\rho(x)\left[\sum_{k=2}^{m} q_{n,k-2}(x) + \sum_{k=1}^{m} q_{n,k-1}(x) + \sum_{k=0}^{m} q_{n,k}(x)\right]\\ &\leq 8M\sum_{k=0}^{m} q_{n,k}(x).\end{aligned}$$

Then in both cases we have

$$\tilde{V}_n^1(x) \leq 8M\rho(x)\sum_{k=0}^{m} q_{n,k}(x).$$

But $\lim_{x\to\infty}\rho(x)\sum_{k=0}^{m} q_{n,k}(x) = 0$. Then there is $x_3 > 0$, such that

$$\tilde{V}_n^1(x) < \varepsilon, \quad \text{for} \quad x \geq x_3. \tag{47}$$

From relations (43), (45), (46), (46), (47) we obtain

$$|\rho(x)BA_n(f,x) - \tilde{\ell}| \leq 4\varepsilon \quad x \geq \max\{x_1, x_2, x_3\}.$$

Hence $\lim_{x\to\infty}\rho(x)BA_n(f,x) = \tilde{\ell}$. Then $BA_n(f) \in C_{\rho,\infty}[0,\infty))$. The theorem is proved. □

Now we can apply estimates given in the previous section. First consider the weight $\rho(x) = 1$, $x \geq 0$.

Theorem 10. *Let $\rho = e_0$ and $f \in C_{\rho,\infty}[0,\infty)$. For $x \geq 0$ and $n \in \mathbb{N}$ it holds:*

$$|BA_n(f,x) - f(x)| \leq 2\omega_1\left(f\circ\psi, \sqrt{\frac{x}{n(1+x)}}\right), \tag{48}$$

$$|BA_n(f,x) - f(x)| \leq \omega_1\left(f\circ\psi, \sqrt{\frac{x}{n(1+x)}}\right) + \frac{3}{2}\omega_2\left(f\circ\psi, \sqrt{\frac{x}{n(1+x)}}\right), \tag{49}$$

$$|BA_n(f,x) - f(x)| \leq 2\omega_1\left(f, \sqrt{\frac{x(1+x)}{n}}\right), \tag{50}$$

$$|BA_n(f,x) - f(x)| \leq \frac{3}{2}\omega_2\left(f, \sqrt{\frac{x(1+x)}{n}}\right). \tag{51}$$

Proof. We apply relations (14), (15), (16) and (17) for $L = BA_n$ by taking into account Lemma 2. □

From relations (48) and (48) we obtain:

Corollary 3. *Let $\rho = e_0$ and $f \in C_{\rho,\infty}[0,\infty)$. For $n \in \mathbb{N}$ it holds:*

$$\|BA_nf - f\| \leq 2\omega_1\left(f\circ\psi, \frac{1}{\sqrt{n}}\right), \tag{52}$$

$$\|BA_nf - f\| \leq \omega_1\left(f\circ\psi, \frac{1}{\sqrt{n}}\right) + \frac{3}{2}\omega_2\left(f\circ\psi, \frac{1}{\sqrt{n}}\right), \tag{53}$$

Consequently

$$\lim_{n\to\infty} \|BA_n f - f\| = 0. \tag{54}$$

Proof. Relations (52) and (53) follow immediately from relations (48) and (49). Relation (54) follows from (52) or (53) since $f\circ\psi \in C[0,1]$. □

Remark 6. From relations (50), (51) and more general, from Theorem A or Theorem B we cannot obtain an estimate in sup-norm of the degree of approximations.

Now we consider the weight $\rho(x) = \frac{1}{1+x^2}$, $x \geq 0$.

Theorem 11. *Let* $\rho = \frac{e_0}{e_0+e_2}$ *and* $f \in C_{\rho,\infty}[0,\infty)$. *For* $x\geq 0$ *and* $n\in\mathbb{N}$ *it holds:*

$$\begin{aligned}\rho(x)|BA_n(f,x)-f(x)| &\leq \frac{x(x+1)}{n(x^2+1)}\rho(x)|f(x)|\\ &+\left(2+\frac{x(x+1)}{n(x^2+1)}\right)\omega_1\left(\Phi f, \sqrt{\frac{x}{n(x^2+1)(x+1)}}\right),\end{aligned} \tag{55}$$

$$\begin{aligned}\rho(x)|BA_n(f,x)-f(x)| &\leq \frac{x(x+1)}{n(x^2+1)}\rho(x)|f(x)|\\ &+\sqrt{\frac{(n+1)x^2+x+n}{n(x^2+1)(x+1)^2}}\omega_1\left(\Phi f, \sqrt{\frac{x}{n(x^2+1)(x+1)}}\right)\\ &+\left(\frac{3}{2}+\frac{x(x+1)}{n(x^2+1)}\right)\omega_2\left(\Phi f, \sqrt{\frac{x}{n(x^2+1)(x+1)}}\right),\end{aligned} \tag{56}$$

$$\rho(x)|BA_n(f,x)-f(x)| \leq \frac{1+x+x^2}{1+x^2}\omega_1\Big(f,\frac{1}{\sqrt{n}}\Big),, \tag{57}$$

$$\rho(x)|BA_n(f,x)-f(x)| \leq \frac{2+x+x^2}{2(1+x^2)}\omega_2\Big(f,\frac{1}{\sqrt{n}}\Big),, \tag{58}$$

Proof. From Lemma 2 we obtain

$$\rho(x)BA_n\left(\frac{e_0}{\rho},x\right) = \frac{BA_n(e_0+e_2,x)}{1+x^2} = 1+\frac{x(1+x)}{n(1+x^2)}.$$

$$\sqrt{\frac{BA_n(e_0+e_2,x)}{(1+x^2)(1+x)^2}} = \sqrt{\frac{(n+1)x^2+x+n}{n(x^2+1)(x+1)^2}}$$

By replacing this in (18) and (19) we obtain relations (55) and (56).

Relations (57) and (58) can be obtained multiplying by $\rho(x)$ relations (7) and (8) from Theorem A and Theorem B for $I=[0,\infty)$, $L = BA_n$, $y=x$, $g=f$ and $h=\frac{1}{\sqrt{n}}$. For this note that

$$\rho(x)[BA_n(e_0,x)+h^{-2}BA_n((e_1-xe_0)^2,x)] = \frac{1}{1+x^2}+\frac{x(1+x)}{1+x^2} = \frac{1+x+x^2}{1+x^2},$$

$$\rho(x)\Big[BA_n(e_0,x)+\frac{1}{2}h^{-2}BA_n((e_1-xe_0)^2,x)\Big] = \frac{1}{1+x^2}+\frac{x(1+x)}{2(1+x^2)} = \frac{2+x+x^2}{2(1+x^2)}.$$

□

Corollary 4. *Let* $\rho = \frac{e_0}{e_0+e_2}$ *and* $f \in C_{\rho,\infty}[0,\infty)$. *For* $n \in \mathbb{N}$ *it holds:*

$$\|BA_n f - f\|_\rho \le \frac{1+\sqrt{2}}{2}\|f\|_\rho + \frac{1+\sqrt{2}}{2}\omega_1\left(\Phi f, \frac{0.529}{\sqrt{n}}\right), \tag{59}$$

$$\|BA_n f - f\|_\rho \le \frac{1+\sqrt{2}}{2}\|f\|_\rho + \sqrt{\frac{1+\sqrt{2}}{2}\omega_1\left(\Phi f, \frac{0.529}{\sqrt{n}}\right)} + \Big(\frac{3}{2} + \frac{1+\sqrt{2}}{2n}\Big)\omega_2\left(\Phi f, \frac{0.529}{\sqrt{n}}\right), \tag{60}$$

$$\|BA_n f - f\|_\rho \le \frac{3}{2}\omega_1\Big(f, \frac{1}{\sqrt{n}}\Big), \tag{61}$$

$$\|BA_n f - f\|_\rho \le 1.105\omega_2\Big(f, \frac{1}{\sqrt{n}}\Big). \tag{62}$$

Proof. We pass to supremum with regard to $x \ge 0$ in relations (55), (56), (57), (58). □

Corollary 5.

$$\lim_{n\to\infty} \|BA_n f - f\|_\rho = 0. \tag{63}$$

Proof. We can apply one of relations (55) or (56), since $\Phi f \in C[0,1]$. □

Remark 7. For obtaining relation (63) we cannot apply relations (57) or (58), since it is not sure that f is uniformly continuous. See the argument given in Remark 5.

Conclusion For both sequences of operators $(S_n)_n$ and $(BA_n)_n$ and for the both weights $\rho = e_0$ and $\rho = \frac{e_0}{e_0+e_2}$ we obtained quantitative results of the degree of approximation in norm $\|\cdot\|_\rho$ by applying the transformation. However, such estimates are impossible to be obtained directly from general estimates given in Theorem A and Theorem B. See the Remark 3 Remark 5, Remark 6 and Remark 7.

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