Bulletin of the *Transilvania* University of Braşov • Vol 8(57), No. 2 - 2015
Series III: Mathematics, Informatics, Physics, 89-92

BROWNIAN PROBABILITIES UNDER SYMMETRIC REARRANGEMENT

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Abstract

We show that the probability that a Brownian motion lies in a given set at an arbitrarily fixed time is increased under the symmetric rearrangement of the set.

2000 *Mathematics Subject Classification:* 60J65, 46E30.
Key words: Brownian motion, symmetric rearrangement, inequality.

1 Introduction

The goal of this note is to prove a monotonicity property for Brownian probabilities under symmetric rearrangements. Specifically, we will consider the symmetric rearrangement of a set and a function, formally defined as follows.

For a Borel measurable set $A \subset \mathbb{R}^n$ of finite Lebesgue measure, the symmetric rearrangement of A (see e.g., [4]), denoted by A^*, is the open ball centered at the origin with the same Lebesgue measure as the set A, that is

$$A^* = \{x \in \mathbb{R}^n : \mathrm{Vol}\,(B\,(0, |x|)) < \mathrm{Vol}\,(A)\}\,, \tag{1}$$

where $|x| = \sqrt{x_1^2 + \ldots + x_n^2}$ denotes the Euclidean norm of $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $B\,(x, R) = \{y \in \mathbb{R}^n : |y - x| < R\}$ denotes the open ball centered at $x \in \mathbb{R}^n$ of radius $R > 0$, and $\mathrm{Vol}\,(A)$ denotes the volume (Lebesgue measure) of $A \subset \mathbb{R}^n$.

In particular note that if $A = B\,(0, R)$, its symmetric rearrangement is just $A^* = A = B\,(0, R)$.

The symmetric (nondecreasing) rearrangement of a given non-negative function $f : \mathbb{R}^n \to [0, \infty)$ which vanishes at infinity (i.e. $\mathrm{Vol}\,(\{x \in \mathbb{R}^n : f\,(x) > t\}) < \infty$ for all $t > 0$), is the function $f^* : \mathbb{R}^n \to [0, \infty)$ defined by

$$f^*\,(x) = \int_0^\infty 1_{\{y \in \mathbb{R}^n : f(y) > t\}^*}\,(x)\,dt, \qquad x \in \mathbb{R}^n. \tag{2}$$

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It can be shown (see [4], p. 81) that the symmetric rearrangement f^* is a non-decreasing, radially symmetric function, that is

$$f^*(x) \le f^*(y) \qquad \text{and} \qquad f^*(x) = f^*(|x|),$$

for any $x, y \in \mathbb{R}^n$ with $|x| > |y|$.

In the sequel we will need the following result showing that the symmetric rearrangement of a radially symmetric decreasing function is the same as the original function. More precisely we have the following.

Lemma 1. *Let $\varphi : [0,\infty) \to [0,\infty)$ be a strictly decreasing and continuous function with $\lim_{t\to\infty} \varphi(t) = 0$. If $f : \mathbb{R}^n \to [0,\infty)$ is given by $f(y) = \varphi(|y|)$, then $f^* = f$.*

Proof. Consider an arbitrarily fixed $t > 0$. If $t \ge \varphi(0)$, since φ is assumed strictly decreasing, we have $\{y \in \mathbb{R}^n : f(y) > t\} = \{y \in \mathbb{R}^n : \varphi(|y|) > t \ge \varphi(0)\} = \varnothing$, so by definition we have $\{y \in \mathbb{R}^n : f(y) > t\}^* = \varnothing = \{y \in \mathbb{R}^n : f(y) > t\}$ in this case.

If $t < \varphi(0)$, there exists $R > 0$ such that $t = \varphi(R)$. We have

$$\begin{aligned} \{y \in \mathbb{R}^n : f(y) > t\} &= \{y \in \mathbb{R}^n : \varphi(|y|) > t = \varphi(R)\} \\ &= \{y \in \mathbb{R}^n : |y| < R\} \\ &= B(0,R), \end{aligned}$$

and therefore $\{y \in \mathbb{R}^n : f(y) > t\}^* = B(0,R)^* = B(0,R) = \{y \in \mathbb{R}^n : f(y) > t\}$, since by the previous remark, open balls are invariant under symmetric rearrangement.

We showed that for any $t > 0$ we have

$$\{y \in \mathbb{R}^n : f(y) > t\}^* = \{y \in \mathbb{R}^n : f(y) > t\},$$

which combined with (2) and the layer cake representation of f (see for example [4], Theorem 1.13) gives

$$f^*(x) = \int_0^\infty 1_{\{y\in\mathbb{R}^n : f(y) > t\}^*}(x)\,dt = \int_0^\infty 1_{\{y\in\mathbb{R}^n : f(y) > t\}}(x)\,dt = f(x),$$

for any $x \in \mathbb{R}^n$, concluding the proof. □

One important property of the symmetric rearrangement of a function is that it increases the value of integrals, in the following sense.

Lemma 2 (Hardy-Littlewood inequality, [4], Theorem 3.4)**.** *If $f, g : \mathbb{R}^n \to [0,\infty)$ are measurable functions vanishing at infinity, then*

$$\int_{\mathbb{R}^n} f(x)\, g(x)\, dx \le \int_{\mathbb{R}^n} f^*(x)\, g^*(x)\, dx, \tag{3}$$

in the sense that when the left hand side is infinite, so is the right hand side.

Moreover, if g is a radially strictly decreasing function (i.e. $g(|x|) < g(|y|)$ for $x, y \in \mathbb{R}^n$ with $|x| > |y|$), then the above inequality holds iff $f = f^*$.

2 Main result

The main result is the following

Theorem 1. *If $(B_t)_{t\geq 0}$ is an $n-$dimensional Brownian motion starting at the origin, and $A \subset \mathbb{R}^n$ is a Borel measurable set with finite Lebesgue measure which contains the origin, then for any $t \geq 0$ we have*

$$P^0 (B_t \in A) \leq P^0 (B_t \in A^*). \tag{4}$$

Moreover, the equality holds in (4) if and only if $A = A^*$ a.e. with respect to the Lebesgue measure.

Proof. The transition density of the n-dimensional Brownian motion starting at $x \in \mathbb{R}^n$ (see e.g. [1]) is given by $p(t, x, y) = \frac{1}{(2\pi t)^{n/2}} e^{-\frac{|x-y|^2}{2t}}$.

Fixing an arbitrary $t > 0$, we have

$$P^0 (B_t \in A) = \int_A p(t, 0, y)\, dy = \int_{\mathbb{R}^n} 1_A(y)\, p(t, 0, y)\, dy.$$

Applying Lemma 2 with $f(y) = 1_A(y)$ and $g(y) = p(t, 0, y) = \frac{1}{(2\pi t)^{n/2}} e^{-\frac{|y|^2}{2t}}$, we obtain

$$P^0 (B_t \in A) \leq \int_{\mathbb{R}^n} 1_A^*(y)\, p^*(t, 0, y)\, dy.$$

It is not difficult to see that $1_A^*(y) = 1_{A^*}(y)$ for any $y \in \mathbb{R}^n$. Also, we have $p(t, 0, y) = \frac{1}{(2\pi t)^{n/2}} e^{-\frac{|y|^2}{2t}} = \varphi(|y|)$, where $\varphi(u) = \frac{1}{(2\pi t)^{n/2}} e^{-\frac{u^2}{2t}}$ is strictly decreasing and continuous on $[0, \infty)$, with $\lim_{u\to\infty} \varphi(u) = 0$. By Lemma 1 we have $p^*(t, 0, y) = p(t, 0, y)$, and therefore we obtain

$$\begin{aligned} P^0 (B_t \in A) &\leq \int_{\mathbb{R}^n} 1_A^*(y)\, p^*(t, 0, y)\, dy \qquad (5) \\ &= \int_{\mathbb{R}^n} 1_{A^*}(y)\, p(t, 0, y)\, dy \\ &= \int_{\mathbb{A}^*} p(t, 0, y)\, dy \\ &= P^0 (B_t \in A^*), \end{aligned}$$

concluding the first part of the proof.

For the second part, note that since $g(y) = p(t, 0, y)$ is radially (symmetric) strictly decreasing, the Hardy-Littlewood inequality also shows that the equality in (5) can hold if and only if $1_A^* = 1_A$, which is equivalent to $A^* = A$ a.e., concluding the proof. □

We conclude with some open questions. The result in Theorem 1 shows that the probability that Brownian motion hits a set of given volume (at a fixed, given time) is increased under the symmetric rearrangement of the set. We were led to consider such symmetric rearrangement inequalities for Brownian motion as a possible line of attack of the long-standing Gaussian correlation inequality, which can be stated equivalently in terms of Brownian motion as

$$P^0\left(B_t \in A \cap B\right) \geq P^0\left(B_t \in A\right) P^0\left(B_t \in B\right), \tag{6}$$

or

$$P^0\left(\left. B_t \in A \right| B_t \in B\right) \geq P^0\left(B_t \in A\right),$$

for arbitrary convex sets $A, B \subset \mathbb{R}^n$ symmetric with respect to the origin. See for example [3] or [5] for resolution of the conjecture in certain particular cases.

It seems natural to ask whether the result in Theorem 1 holds for other processes besides Brownian motion, for instance for conditional Brownian motion, or for the reflecting Brownian motion. We ask the following.

Question 1. In the notation of Theorem 1, is it true that the same inequality holds if one conditions on the event that the Brownian motion belongs to a certain measurable set $B \subset \mathbb{R}^n$ containing the origin? Explicitly, is it true that for any $t \geq 0$ we have

$$P^0\left(\left. B_t \in A \right| B_t \in B\right) \leq P^0\left(\left. B_t \in A^* \right| B_t \in B\right)?$$

Question 2. Consider an arbitrarily smooth domain $D \subset \mathbb{R}^n$ containing the origin, and let X_t be the reflecting Brownian motion on D starting at the origin (see for example [2]). Is it true that for any Borel measurable set $A \subset \mathbb{R}^n$ with finite Lebesgue measure which contains the origin, and any $t \geq 0$ we have

$$P^0\left(X_t \in A\right) \leq P^0\left(X_t \in A^*\right)?$$

Acknowledgement. The first and the last authors were supported by a grant of the Romanian National Authority for Scientific Research, CNCS - UEFISCDI, project number PNII-ID-PCCE-2011-2-0015.

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