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TYPES OF INTEGER HARMONIC NUMBERS (I)

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Abstract

In [11], J. Sándor presented in a table all 211 bi-unitary harmonic numbers up to 10^9. In the present paper we obtain several bi-unitary harmonic numbers which are higher than 10^9, using the Mersenne prime numbers. We also investigate bi-unitary harmonic numbers of some particular forms. Thus, we extend the study done by J. Sándor.

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1 Introduction

In number theory there are many results which involve the harmonic mean of divisors of a positive integer. In [8], O. Ore studies the positive integers n whose harmonic mean of its divisors is integer. These numbers were called *harmonic numbers*, by C. Pomerance in [10]. O. Ore linked the perfect numbers with the harmonic numbers, showing that every perfect number is harmonic. A list of the harmonic numbers less than $2 \cdot 10^9$ is given by G. L. Cohen in [1], finding a total of 130 harmonic numbers, and G. L. Cohen and R. M. Sorli in [2] have continued to this list up to 10^{10}.

The notion of harmonic numbers is extended to unitary harmonic numbers by K. Nageswara Rao in [7] and then to bi-unitary harmonic numbers by J. Sándor in [11].

Ch. Wall [12] shows that there are 23 unitary harmonic numbers n with $\omega(n) \leq 4$, where $\omega(n)$ represents the number of distinct prime factors of n, and T. Goto [3] proved that if n is a unitary perfect number with k distinct prime factors, then $n < \left(2^{2^k}\right)^k$. In [11] J. Sándor presented a table containing all the 211 bi-unitary harmonic numbers up to 10^9.

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In this paper we extend the J. Sándors's study, looking for other bi-unitary harmonic numbers, greater than 10^9. The paper is organised as follows. In section 2 we present some basic notions and well-known results about unitary and bi-unitary harmonic numbers. Starting with some Mersenne primes we detected new bi-unitary harmonic numbers in section 3.

2 Preliminaries

Let n be a positive integer and $1 = d_1 < d_2 < ... < d_s = n$ all its natural divisors. Let us consider the harmonic mean of divisors

$$H(n) = \frac{s}{\frac{1}{d_1} + \frac{1}{d_2} + ... + \frac{1}{d_s}}. \tag{1}$$

O. Ore studies in [8] numbers n with $H(n)$ an integer number.

We denote by $\sigma(n)$ and $\tau(n)$ the sum of divisors of n and the number of divisors of n, respectively, then $H(n)$ can be written as

$$H(n) = \frac{n\tau(n)}{\sigma(n)}, \tag{2}$$

Therefore, we remark that $H(n)$ is an integer if and only if $\sigma(n)|n\tau(n)$.

A number n satisfying the condition $\sigma(n)|n\tau(n)$ is called, [10], *harmonic number.*

We recall that a natural number n is perfect if $\sigma(n) = 2n$. It is proved, [8], that every perfect number is harmonic.

A divisor d of a positive integer n is called, [7], *unitary divisor* of n if $\left(d, \frac{n}{d}\right) = 1$. Let us denote by $\sigma^*(n)$, $\tau^*(n)$ the sum and the number of unitary divisors of n, respectively.

A positive integer n is called, [7], *unitary harmonic number* when $\sigma^*(n)|n\tau^*(n)$. This definition shows that a unitary perfect number n, so which satisfies $\sigma^*(n) = 2n$, is also a unitary harmonic number.

The notion of unitary divisor was extended to bi-unitary divisors. We recall that a divisor d of n is called *bi-unitary divisor* if the largest unitary common divisor of d and $\frac{n}{d}$ is 1. We denote by $\sigma^{**}(n)$ the sum of bi-unitary divisors of n.

In [12], Ch. Wall introduces the concept of bi-unitary perfect numbers, in the following way. A number n is called *bi-unitary perfect* number if $\sigma^{**}(n) = 2n$. It is proved that the only bi-unitary perfect numbers are $6, 60$ and 90.

We remark that the function $\sigma^{**}(n)$ is multiplicative and we have

$$\sigma^{**}(p^a) = \begin{cases} \sigma(p^a) = \frac{p^{a+1}-1}{p-1}, & \text{for } a \text{ odd} \\ \sigma(p^a) - p^{\frac{a}{2}} = \frac{p^{a+1}-1}{p-1} - p^{\frac{a}{2}}, & \text{for } a \text{ even} \end{cases} \tag{3}$$

We denote by $\tau^{**}(n)$ the number of bi-unitary divisors of n and it is easy to see that if $n = p_1^{a_1} p_2^{a_2} ... p_r^{a_r} > 1$, is the prime factorization of n, then

$$\tau^{**}(n) = \prod_{a_i = even} a_i \prod_{a_i = odd} (a_i + 1). \tag{4}$$

J. Sándor in [11] gives the following definition:

Definition 1. *A natural number n is called bi-unitary harmonic number if $\sigma^{**}(n)|n\tau^{**}(n)$.*

Remark 1. *In [13] there are all unitary harmonic numbers with the most 4 primes in their factorization, since in [11] there are all bi-unitary harmonic numbers smaller than 10^9. From these, we can remark that there are unitary harmonic numbers which are not bi-unitary harmonic numbers, for example $2^3 \cdot 3^3 \cdot 5 \cdot 7$, and there are bi-unitary harmonic numbers which are not unitary harmonic, for example $2^3 \cdot 3^3 \cdot 5^4 \cdot 7$.*

It is proved, [11], that a bi-unitary perfect number, is also bi-unitary harmonic number. How Wall shows that the numbers $6, 60$ and 90 are the bi-unitary perfect numbers, then these are bi-unitary harmonic numbers. Also in [11] it is showed that if n has the prime decomposition $n = p_1^{a_1} p_2^{a_2} ... p_r^{a_r} > 1$ with all exponents $\{a_i\}_{i=\overline{1,r}}$ odd numbers, then n is a bi-unitary harmonic number if and only if n is harmonic. It is also proved that bi-unitary harmonic numbers are not of the following forms: pq^4, p^3q^2 and p^3q^4, and the only number $5 \cdot 3^2$ is bi-unitary harmonic number in form pq^2, where p, q are primes.

From the table containing all the 211 bi-unitary harmonic numbers up to 10^9, presented by J. Sándor in [11], we can remark the following:

- There are only five odd numbers up to 10^9, which are bi-unitary harmonic, they are the following: 1, $45 = 3^2 \cdot 5$, $646425 = 3^2 \cdot 5^2 \cdot 13^2 \cdot 17$, $716625 = 3^2 \cdot 5^3 \cdot 7^2 \cdot 13$ and $29381625 = 3^2 \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 41$;
- There are only one perfect square up to 10^9, which are bi-unitary harmonic: $9922500 = 2^2 \cdot 3^4 \cdot 5^4 \cdot 7^2$;
- There are only five even powerful numbers (powerful numbers are 1 and every natural number in form $n = p_1^{a_1} p_2^{a_2} ... p_r^{a_r} > 1$, where $a_i \geq 2$, for all $i = \overline{1,r}$) up to 10^9, which are bi-unitary harmonic. They are the following: $3307500 = 2^2 \cdot 3^3 \cdot 5^4 \cdot 7^2$, $9922500 = 2^2 \cdot 3^4 \cdot 5^4 \cdot 7^2$, $23152500 = 2^2 \cdot 3^3 \cdot 5^4 \cdot 7^3$, $138915000 = 2^3 \cdot 3^4 \cdot 5^4 \cdot 7^3$ and $555660000 = 2^5 \cdot 3^4 \cdot 5^4 \cdot 7^3$.

3 Bi-unitary harmonic numbers obtained from Mersenne primes

In this section we search some bi-unitary harmonic numbers higher than 10^9. In the construction of the bi-unitary harmonic numbers higher than 10^9, we need sufficiently large primes, but we are looking for primes of a convenient form.

The ideea how to "build" these numbers starts from the following remark:

$$\sigma^{**}(2^{2k}) = (1 + 2 + 2^2 + ... + 2^{k-1} + 2^{k+1} + ... + 2^{2k}) = (2^k - 1)(2^{k+1} + 1), \quad (1)$$

where we used relation (3).

Since we are looking for a bi-unitaty number n which has a divisor 2^{2k}, we have $\sigma^{**}(n)|n\tau^{**}(n)$, hence we search bi-unitary harmonic numbers of the form

$$n = 2^{2k}(2^k - 1)(2^{k+1} - 1)p_1^{a_1}p_2^{a_2}...p_r^{a_r}. \tag{2}$$

Moreover, we choose $2^k - 1$ prime for an easy computation of $\sigma^{**}(2^k - 1) = 2^k$, $\tau^{**}(2^k - 1) = 2$.

Therefore, we use Mersenne primes. We recall that a number n is called *a Mersenne number*, [6], [9], if it is of the form $M_k = 2^k - 1$. It is well-known that the first eight Mersenne primes are the following:

$$\begin{array}{cccc} M_2 = 2^2 - 1, & M_3 = 2^3 - 1, & M_5 = 2^5 - 1, & M_7 = 2^7 - 1, \\ M_{13} = 2^{13} - 1, & M_{17} = 2^{17} - 1, & M_{19} = 2^{19} - 1, & M_{23} = 2^{31} - 1. \end{array}$$

Starting with each prime from above, we determine the unknowns $p_1, p_2, ..., p_r$, $a_1, a_2, ..., a_r$ from (2) such that relation from Definition 1 is satisfied. So, we have:

Theorem 1. *The following numbers:*

$$n_1 = 2^{14}(2^7 - 1)(2^8 + 1) \cdot 3^4 \cdot 11 \cdot 43, \tag{3}$$

$$n_2 = 2^{26}(2^{13} - 1) \cdot 3^3 \cdot 5^6 \cdot 19 \cdot 29 \cdot 31 \cdot 79 \cdot 113 \cdot 157 \cdot 313, \tag{4}$$

$$n_3 = 2^{34}(2^{17} - 1) \cdot 3^4 \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 19 \cdot 31 \cdot 37 \cdot 79 \cdot 109 \cdot 157 \cdot 313, \tag{5}$$

$$n_4 = 2^{38}(2^{19} - 1) \cdot 3^4 \cdot 5^3 \cdot 7^3 \cdot 13 \cdot 17 \cdot 29 \cdot 2203 \cdot 30841 \cdot 61681, \tag{6}$$

and

$$\begin{aligned} n_5 &= 2^{62}(2^{31} - 1) \cdot 3^5 \cdot 5^6 \cdot 7^5 \cdot 11 \cdot 13 \cdot 19 \cdot 43 \cdot 79 \cdot 107 \cdot 157 \cdot 313 \cdot & (7) \\ &\cdot 349 \cdot 641 \cdot 27919 \cdot 55837 \cdot 335021 \cdot 3350209 \cdot 6700417, & (8) \end{aligned}$$

are bi-unitary harmonic numbers, higher than 10^9.

Proof. Let n be the bi-unitary harmonic number sought of the form (2).

I. For $M_2 = 2^2 - 1 = 3$ we have $n = 2^4 \cdot 3^4 \cdot 7$ or $n = 2^4 \cdot 3^4 \cdot 5 \cdot 7$ or $n = 2^4 \cdot 3^4 \cdot 7 \cdot 11$.

II. For $M_3 = 2^3 - 1 = 7$ we find $n = 2^6 \cdot 3 \cdot 7 \cdot 17 = 22848$.

III. For $M_5 = 2^5 - 1 = 31$ we have $n = 2^{10} \cdot 7 \cdot 13 \cdot 31 = 2888704$.

The values of n written so far are found in the table given by J. Sándor in [11].

IV. For $M_7 = 2^7 - 1 = 127$ we take

$$n = 2^{14}(2^7 - 1)(2^8 + 1) \cdot v.$$

We have the Mersenne prime M_7 and the Fermat prime $2^8 + 1$ as divisors. Therefore, we calculate

$$\sigma^{**}(n) = (2^7 - 1)(2^8 + 1) \cdot 2^7 \cdot (2^8 + 2) \cdot w.$$

But $\sigma^{**}(n)|n\tau^{**}(n)$, which means that $2^8+2=2\cdot3\cdot43$ must be a divisor of the number $n\tau^{**}(n)$. So, a new form of n is given by

$$n=2^{14}(2^7-1)\cdot3\cdot43\cdot v^{'}.$$

But $\sigma^{**}(43)=44=2^2\cdot11$, implies $11|n$. Hence, we rewrite on n in the following way

$$n=2^{14}(2^7-1)(2^8+1)\cdot3\cdot11\cdot43\cdot v'',$$

which implies

$$\sigma^{**}(n)=(2^7-1)(2^8+1)\cdot2^7\cdot2\cdot3\cdot43\cdot2^2\cdot2^2\cdot3\cdot2^2\cdot11\cdot w''=2^{14}(2^7-1)(2^8+1)\cdot3^2\cdot11\cdot43\cdot w''.$$

We remark that divisor 3 appears once more, in addition, in the decomposition of $\sigma^{**}(n)$, but $\tau^{**}(n)=14\cdot2\cdot2\cdot2\cdot2=2^5\cdot7$ does not contain factor 3. Then the power of number 3 from decomposition of n must be increased because we need $\sigma^{**}(n)|n\tau^{**}(n)$.

If 3^2 is a unitary divisor of n, then $\sigma^{**}(3^2)=1+3^2=10$, which implies $5|n$, and the number 5 generates the appearance of another factor 3 in the decomposition of $\sigma^{**}(n)$. Therefore we take

$$n=2^{14}(2^7-1)(2^8+1)\cdot3^4\cdot11\cdot43,$$

for which we calculate

$$\begin{aligned}\sigma^{**}(n) &= (2^7-1)(2^8+1)\cdot2^7\cdot2\cdot3\cdot43\cdot2^4\cdot7\cdot2^2\cdot3\cdot2^2\cdot11=\\ &= 2^{16}(2^7-1)(2^8+1)\cdot3^2\cdot7\cdot11\cdot43.\end{aligned}$$

We also have $\tau^{**}(n)=14\cdot2\cdot2\cdot4\cdot2\cdot2=2^7\cdot7$, so it follows that

$$n\tau^{**}(n)=2^{21}(2^7-1)(2^8+1)\cdot3^4\cdot7\cdot11\cdot43,$$

which means that $\sigma^{**}(n)|n\tau^{**}(n)$. Consequently, the number

$$n_1=2^{14}(2^7-1)(2^8+1)\cdot3^4\cdot11\cdot43=20.488.159.346.688,$$

is a bi-unitary harmonic number higher than $2\cdot10^{13}$, and

$$H^{**}(n_1)=\frac{n_1\tau^{**}(n_1)}{\sigma^{**}(n_1)}=2^5\cdot3^2=288.$$

V. For $M_{13}=2^{13}-1=8191$ we have

$$n=2^{26}(2^{13}-1)(2^{14}+1)\cdot v,$$

where $2^{13}-1$ is a Mersenne prime number, and $2^{14}+1=5\cdot29\cdot113$. Therefore, we compute

$$\begin{aligned}\sigma^{**}(n) &= (2^{13}-1)(2^{14}+1)\cdot2^{13}\cdot2\cdot3\cdot2\cdot3\cdot5\cdot2\cdot3\cdot19\cdot w=\\ &= 2^{16}(2^{13}-1)\cdot3^3\cdot5^2\cdot19\cdot29\cdot113\cdot w,\end{aligned}$$

which means that 3, 5 and 19 must be the divisors of $n\tau^{**}(n)$, because the condition $\sigma^{**}(n)|n\tau^{**}(n)$ is imposed. So the new form of n is given by

$$n = 2^{26}(2^{13}-1)\cdot 3^a \cdot 5^b \cdot 19 \cdot 29 \cdot 113 \cdot v'.$$

After multiple checks, we choose the combination $3^3 \cdot 5^6$, so $\sigma^{**}(3^3\cdot 5^6) = 2^4 \cdot 5 \cdot 31 \cdot 313$. It follows that $31 \cdot 313|n$. According to all the above, rewrite number n in the following way:

$$n = 2^{26}(2^{13}-1)\cdot 3^3 \cdot 5^6 \cdot 19 \cdot 29 \cdot 31 \cdot 113 \cdot 313w'.$$

Since $\sigma^{**}(313) = 2\cdot 157$, implies $157|n$, and from the equality $\sigma^{**}(157) = 2\cdot 79$, we deduce $79|n$. Hence, we study if the number

$$n = 2^{26}(2^{13}-1)\cdot 3^3 \cdot 5^6 \cdot 19 \cdot 29 \cdot 31 \cdot 79 \cdot 113 \cdot 157 \cdot 313$$

is bi-unitary harmonic. We obtain

$$\begin{aligned}&\sigma^{**}(n) = (2^{13}-1)\cdot 5 \cdot 29 \cdot 113 \cdot 2^{13} \cdot 2^3 \cdot 5 \cdot 2 \cdot 31 \cdot 313 \cdot 2^2 \cdot 5 \cdot 2 \cdot 3 \cdot \\ &\cdot 5 \cdot 2^5 \cdot 2^4 \cdot 5 \cdot 2 \cdot 3 \cdot 19 \cdot 2 \cdot 79 \cdot 2 \cdot 157 = \\ &= 2^{32}(2^{13}-1)\cdot 3^2 \cdot 5^5 \cdot 19 \cdot 29 \cdot 31 \cdot 79 \cdot 113 \cdot 157 \cdot 313.\end{aligned}$$

In this case we have $\tau^{**}(n) = 26 \cdot 2 \cdot 4 \cdot 6 \cdot 2^7 = 2^{12} \cdot 3 \cdot 13$, which implies the following relation

$$H^{**}(n) = \frac{n\tau^{**}(n)}{\sigma^{**}(n)} = 2^6 \cdot 3^2 \cdot 5 \cdot 13$$

It follows that the number

$$\begin{aligned}&n_2 = 2^{26}(2^{13}-1)\cdot 3^3 \cdot 5^6 \cdot 19 \cdot 29 \cdot 31 \cdot 79 \cdot 113 \cdot 157 \cdot 313 = \\ &= 1.737.654.465.595.711.599.673.344.000.000,\end{aligned}$$

is a bi-unitary harmonic number. It is easy to see that this number is higher than 10^{30}.

VI. For $M_{17} = 2^{17} - 1 = 131071$, in the same way as above, we found the number

$$n = 2^{34}(2^{17}-1)\cdot 3^4 \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 19 \cdot 31 \cdot 37 \cdot 79 \cdot 109 \cdot 157 \cdot 313,$$

for which

$$\sigma^{**}(n) = 2^{41}(2^{17}-1)\cdot 3 \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 19 \cdot 31 \cdot 37 \cdot 79 \cdot 109 \cdot 157 \cdot 313,$$

and $\tau^{**}(n) = 2^{15} \cdot 3 \cdot 17$. It follows that

$$H^{**}(n) = \frac{n\tau^{**}(n)}{\sigma^{**}(n)} = 2^8 \cdot 3^4 \cdot 17,$$

which implies that the number

$$n_3 = 2^{34}(2^{17}-1)\cdot 3^4 \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 19 \cdot 31 \cdot 37 \cdot 79 \cdot 109 \cdot 157 \cdot 313,$$

is a bi-unitary harmonic number. It is easy to see that this number is higher than 10^{38}.

VII. Similar as above, starting with $M_{19} = 2^{19} - 1 = 524287$, we identify the number

$$n = 2^{38}(2^{19} - 1) \cdot 3^4 \cdot 5^3 \cdot 7^3 \cdot 13 \cdot 17 \cdot 29 \cdot 2203 \cdot 30841 \cdot 61681,$$

for which we calculate the sum of bi-unitary divisors of n

$$\begin{aligned} \sigma^{**}(n) &= (2^{19} - 1)(2^{20} + 1) \cdot 2^{19} \cdot 2^4 \cdot 7 \cdot 2^2 \cdot 3 \cdot 13 \cdot 2^4 \cdot 5^2 \cdot 2 \cdot \\ &\cdot 7 \cdot 2 \cdot 3^2 \cdot 2 \cdot 3 \cdot 5 \cdot 2^2 \cdot 19 \cdot 29 \cdot 2 \cdot 7 \cdot 2203 \cdot 2 \cdot 30841, \end{aligned}$$

so

$$\sigma^{**}(n) = 2^{36}(2^{19} - 1) \cdot 3^4 \cdot 5^3 \cdot 7^3 \cdot 13 \cdot 17 \cdot 19 \cdot 29 \cdot 2203 \cdot 30841 \cdot 61681,$$

where $2^{20} + 1 = 17 \cdot 61681$.

Because

$$\tau^{**}(n) = 38 \cdot 2 \cdot 4 \cdot 4 \cdot 4 \cdot 2^6 = 2^{14} \cdot 19,$$

it follows that

$$H^{**}(n) = \frac{n\tau^{**}(n)}{\sigma^{**}(n)} = 2^{16},$$

which means that the number

$$n_4 = 2^{38}(2^{19} - 1) \cdot 3^4 \cdot 5^3 \cdot 7^3 \cdot 13 \cdot 17 \cdot 19 \cdot 29 \cdot 2203 \cdot 30841 \cdot 61681,$$

is a bi-unitary harmonic number. It is easy to see that this number is higher than 10^{40}.

VIII. If we take $M_{31} = 2^{31} - 1 = 3221225471$, then we find the number

$$\begin{aligned} n &= 2^{62}(2^{31} - 1) \cdot 3^5 \cdot 5^6 \cdot 7^5 \cdot 11 \cdot 13 \cdot 19 \cdot 43 \cdot 79 \cdot 107 \cdot 157 \cdot 313 \cdot \\ &\cdot 349 \cdot 641 \cdot 27919 \cdot 55837 \cdot 335021 \cdot 3350209 \cdot 6700417, \end{aligned}$$

for which the sum of bi-unitary divisors of n is

$$\begin{aligned} \sigma^{**}(n) = (2^{31} - 1)(2^{32} + 1) \cdot 2^{31} \cdot 2^2 \cdot 7 \cdot 13 \cdot 2 \cdot 31 \cdot 313 \cdot 2^3 \cdot 3 \cdot 19 \cdot 43 \cdot \\ \cdot 2^3 \cdot 3 \cdot 2 \cdot 7 \cdot 2^2 \cdot 5 \cdot 2^2 \cdot 11 \cdot 2^4 \cdot 5 \cdot 2^2 \cdot 3^3 \cdot 2 \cdot 79 \cdot 2 \cdot 157 \cdot 2 \cdot 5^2 \cdot 7 \cdot 2 \cdot \\ \cdot 3 \cdot 107 \cdot 2^4 \cdot 5 \cdot 349 \cdot 2 \cdot 27919 \cdot 2 \cdot 3 \cdot 55837 \cdot 2 \cdot 5 \cdot 335021 \cdot 2 \cdot 3350209. \end{aligned}$$

It follows that

$$\begin{aligned} \sigma^{**}(n) = 2^{63}(2^{31} - 1) \cdot 3^7 \cdot 5^6 \cdot 7^3 \cdot 11 \cdot 13 \cdot 19 \cdot 31 \cdot 43 \cdot 79 \cdot 107 \cdot 157 \cdot \\ \cdot 313 \cdot 349 \cdot 641 \cdot 27919 \cdot 55837 \cdot 335021 \cdot 3350209 \cdot 6700417, \end{aligned}$$

where $2^{32} + 1 = 641 \cdot 6700417 = F_5$ is the fifth Fermat number, and the number of bi-unitary divisors of n is $\tau^{**}(n) = 2^{20} \cdot 3^3 \cdot 31$. So, we obtain

$$H^{**}(n) = \frac{n\tau^{**}(n)}{\sigma^{**}(n)} = 2^{19} \cdot 3 \cdot 7^2,$$

which implies that the number

$$n_5 = 2^{62}(2^{31}-1)\cdot 3^5\cdot 5^6\cdot 7^5\cdot 11\cdot 13\cdot 19\cdot 43\cdot 79\cdot 107\cdot 157\cdot 313\cdot$$
$$\cdot 349\cdot 641\cdot 27919\cdot 55837\cdot 335021\cdot 3350209\cdot 6700417,$$

is a bi-unitary harmonic number. It is easy to see that this number is higher than 10^{86}. □

Remark 2. *a) J. Sándor in [11] introduced the notion of bi-unitary k-perfect number given by the following: let $k \geq 2$ be an integer, then n is a bi-unitary k-perfect number if $\sigma^{**}(n) = kn$. From the above proof, we observe that for*

$$n = 2^{14}(2^7-1)(2^8+1)\cdot 3\cdot 11\cdot 43 = 758.820.716.544,$$

we have

$$\sigma^{**}(n) = 2^{14}(2^7-1)(2^8+1)\cdot 3^2\cdot 11\cdot 43,$$

so $\sigma^{**}(n) = 3n$. Consequently n is a bi-unitary 3-perfect number.

b) For $(m,n)=1$, we deduce that $H^{**}(m\cdot n) = H^{**}(m)H^{**}(n)$. This remark suggests how to search a positive integer m relatively prime with the number

$$n_1 = 2^{14}(2^7-1)(2^8+1)\cdot 3^4\cdot 11\cdot 43 = 20.488.159.346.688,$$

for which $H^{**}(m\cdot n_1)$ is integer.

Since $H^{**}(n) = 2^5\cdot 3^2$, we choose the natural number m as a prime number having the property that $\sigma^{**}(m) = m+1$ has in its factorization only primes 2 and 3. Also, we can choose m as the product of prime numbers with the previous property. Therefore, if the natural number m is prime, then we choose $m \in \{5, 7, 17, 23, 31, 47, 71\}$. Consequently, we obtain the following:

$$H^{**}(5\cdot n_1) = 2^5\cdot 3\cdot 5,\ H^{**}(7\cdot n_1) = 2^3\cdot 3^2\cdot 7,\ H^{**}(17\cdot n_1) = 2^5\cdot 17,$$

$$H^{**}(23\cdot n_1) = 2^3\cdot 3\cdot 23,\ H^{**}(31\cdot n_1) = 2\cdot 3^2\cdot 31,\ H^{**}(47\cdot n_1) = 2^2\cdot 3\cdot 47,$$

$$H^{**}(71\cdot n_1) = 2^3\cdot 71,$$

$$H^{**}(5\cdot 7\cdot n_1) = 2^3\cdot 3\cdot 5\cdot 7,\ H^{**}(5\cdot 23\cdot n_1) = 2^3\cdot 5\cdot 23,\ H^{**}(5\cdot 31\cdot n_1) = 2\cdot 3\cdot 5\cdot 31,$$

$$H^{**}(5\cdot 47\cdot n_1) = 2^2\cdot 5\cdot 47,\ H^{**}(7\cdot 17\cdot n_1) = 2^3\cdot 7\cdot 17,\ H^{**}(7\cdot 23\cdot n_1) = 2\cdot 3\cdot 7\cdot 23,$$

$$H^{**}(7\cdot 47\cdot n_1) = 3\cdot 7\cdot 47,\ H^{**}(7\cdot 71\cdot n_1) = 2\cdot 7\cdot 71,\ H^{**}(17\cdot 31\cdot n_1) = 2\cdot 17\cdot 31,$$

$$H^{**}(23 \cdot 47 \cdot n_1) = 23 \cdot 47.$$

Another combination can be made to the prime number 13*, because* $H^{**}(13) = \frac{13}{7}$*, so, we have the following relations:*

$$H^{**}(7 \cdot 13 \cdot n_1) = 2^3 \cdot 3^2 \cdot 13,\ H^{**}(5 \cdot 7 \cdot 13 \cdot n_1) = 2^3 \cdot 3 \cdot 5 \cdot 13,$$

$$H^{**}(7 \cdot 13 \cdot 17 \cdot n_1) = 2^3 \cdot 13 \cdot 17,\ H^{**}(7 \cdot 13 \cdot 23 \cdot n_1) = 2 \cdot 3 \cdot 13 \cdot 23,$$

$$H^{**}(7 \cdot 13 \cdot 47 \cdot n_1) = 3 \cdot 13 \cdot 47,\ H^{**}(7 \cdot 13 \cdot 71 \cdot n_1) = 2 \cdot 13 \cdot 71.$$

It follows that we obtained other bi-unitary harmonic numbers starting from n_1*:* $5 \cdot n_1$, $7 \cdot n_1$, $17 \cdot n_1$, $23 \cdot n_1$, $31 \cdot n_1$, $47 \cdot n_1$, $71 \cdot n_1$, $5 \cdot 7 \cdot n_1$, $5 \cdot 23 \cdot n_1$, $5 \cdot 31 \cdot n_1$, $5 \cdot 47 \cdot n_1$, $7 \cdot 17 \cdot n_1$, $7 \cdot 47 \cdot n_1$, $7 \cdot 71 \cdot n_1$, $17 \cdot 31 \cdot n_1$, $23 \cdot 47 \cdot n_1$, $7 \cdot 13 \cdot n_1$, $5 \cdot 7 \cdot 13 \cdot n_1$, $7 \cdot 13 \cdot 17 \cdot n_1$, $7 \cdot 13 \cdot 23 \cdot n_1$, $7 \cdot 13 \cdot 47 \cdot n_1$, $7 \cdot 13 \cdot 71 \cdot n_1$.

c) Starting from the number n_4 *with* $H^{**}(n_4) = 2^{16}$ *and the fifth Mersenne prime number* $M_{13} = 2^{13} - 1 = 8191$ *which has* $H^{**}(M_{13}) = \frac{2M_{13}}{2^{13}}$*, it follows that*

$$n_4 \cdot M_{13} = 2^{38}(2^{19} - 1)(2^{13} - 1) \cdot 3^4 \cdot 5^3 \cdot 7^3 \cdot 13 \cdot 17 \cdot 29 \cdot 2203 \cdot 30841 \cdot 61681,$$

is a bi-unitary harmonic number. It is easy to see the this number is higher than 10^{44} *and*

$$H^{**}(n_4 \cdot M_{13}) = 2^4(2^{13} - 1).$$

Using also the number n_4 *and the sixth Mersenne prime number* $M_{17} = 2^{17} - 1$, $H^{**}(M_{17}) = \frac{2M_{17}}{2^{17}}$ *we obtain that*

$$n_4 \cdot M_{17} = 2^{38}(2^{19} - 1)(2^{17} - 1) \cdot 3^4 \cdot 5^3 \cdot 7^3 \cdot 13 \cdot 17 \cdot 29 \cdot 2203 \cdot 30841 \cdot 61681,$$

is a bi-unitary harmonic number. It is easy to see the this number is higher than 10^{46} *and*

$$H^{**}(n_4 \cdot M_{17}) = M_{17} = 2^{17} - 1 = 131071.$$

d) For $M_{19} = 2^{19} - 1 = 524287$*, the seventh Mersenne prime number, we compute*

$$H^{**}(M_{19}) = \frac{2^{19} - 1}{2^{18}}.$$

But M_{19} *is relatively prime with every divisors of* n*, which means that*

$$H^{**}(n_5 \cdot M_{19} \cdot 293) = 2 \cdot 293 \cdot (2^{19} - 1).$$

Therefore we remark that the number

$$\begin{aligned} n_5 \cdot M_{19} \cdot 293 &= 2^{62}(2^{31} - 1)(2^{19} - 1) \cdot 3^5 \cdot 5^6 \cdot 7^5 \cdot 11 \cdot 13 \cdot 19 \cdot 43 \cdot 79 \cdot 107 \cdot \\ &\cdot 157 \cdot 293 \cdot 313 \cdot 349 \cdot 641 \cdot 27919 \cdot 55837 \cdot 335021 \cdot 3350209 \cdot 6700417, \end{aligned}$$

is a bi-unitary harmonic number. Obviously, this number is higher than 10^{94}.

Although we found bi-unitary harmonic numbers increasingly higher it is not known yet if they are infinite or not. This issue remains open.

Other types of integers harmonic numbers may be entered by the exponential and infinitary divisors [4].

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