Bulletin of the *Transilvania* University of Braşov • Vol 9(58), No. 1 - 2016
Series III: Mathematics, Informatics, Physics, 39-52

SOME CHARACTERIZATIONS OF KENMOTSU MANIFOLDS ADMITTING A QUARTER-SYMMETRIC METRIC CONNECTION

Uday Chand DE[1], Dhananjoy MANDAL[2] and Krishanu MANDAL[3]

Abstract

In this paper we study certain curvature properties of Kenmotsu manifolds with respect to the quarter-symmetric metric connection. First we investigate Weyl projective symmetric Kenmotsu manifolds with respect to the quarter-symmetric metric connection. Next, we study Kenmotsu manifolds satisfying the curvature condition $\widetilde{P}\cdot\widetilde{S}=0$, where $\widetilde{P}$ and $\widetilde{S}$ are the projective curvature tensor and Ricci tensor respectively with respect to the quarter-symmetric metric connection. Further, we discuss about pseudoprojectively flat and ϕ-projectively semisymmetric Kenmotsu manifolds with respect to the quarter-symmetric metric connection. Finally, we give an example of a 5-dimensional Kenmotsu manifold admitting a quarter-symmetric metric connection for illustration.

2010 *Mathematics Subject Classification:* 53C05, 53D15.
Key words: Quarter-symmetric metric connection, Kenmotsu manifold, projective curvature tensor, Weyl projective symmetric, pseudoprojectively flat, ϕ-projectively semisymmetric.

1 Introduction

In a Riemannian manifold M a linear connection $\widetilde{\nabla}$ is called a quarter symmetric connection [8] if the torsion tensor T of the connection $\widetilde{\nabla}$

$$T(X,Y)=\widetilde{\nabla}_XY-\widetilde{\nabla}_YX-[X,Y] \tag{1}$$

satisfies

$$T(X,Y)=\eta(Y)\phi X-\eta(X)\phi Y, \tag{2}$$

[1]Department of Pure Mathematics, University of Calcutta, 35, Ballygunge Circular Road, Kol- 700019, West Bengal, India, e-mail: uc_de@yahoo.com
[2]Department of Pure Mathematics, University of Calcutta, 35, Ballygunge Circular Road, Kol- 700019, West Bengal, India, e-mail: dmandal.cu@gmail.com
[3]Department of Pure Mathematics, University of Calcutta, 35, Ballygunge Circular Road, Kol- 700019, West Bengal, India, e-mail: krishanu.mandal013@gmail.com

where η is a 1-form and ϕ is a $(1,1)$ tensor field. Moreover, a linear connection $\widetilde{\nabla}$ is said to be a metric connection of M if

$$(\widetilde{\nabla}_X g)(Y,U) = 0, \tag{3}$$

where $X, Y, U \in \chi(M)$, where $\chi(M)$ is the set of all differentiable vector fields on M. A linear connection $\widetilde{\nabla}$ satisfying both (2) and (3) is said to be a quarter-symmetric metric connection [8]. If we change ϕX by X, then the connection is known as semi-symmetric metric connection [29]. Thus the notion of quarter-symmetric connection generalizes the notion of the semi-symmetric connection. Semi-symmetric metric connections have been studied by several authors such as Barman [1], De [5], Özgür and Sular [16], Ozen et al [17, 18], Prvanovic [20], Prvanovic and Pušić [21], Smaranda and Andonie [24], Singh and Pandey [25] and many others.

Let M be an n-dimensional Riemannian manifold. If there exists a one-to-one correspondence between each coordinate neighborhood of M and a domain in Euclidean space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 3$, M is locally projectively flat if and only if the well-known projective curvature tensor P vanishes. Here P is defined by [26]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y], \tag{4}$$

for all $X, Y, Z \in \chi(M)$, where R is the curvature tensor and S is the Ricci tensor of type $(0,2)$. In fact, M is projectively flat if and only if it is of constant curvature. Thus the projective curvature tensor is the measure of the failure of a Riemannian manifold to be of constant curvature.

A Riemannian manifold (M,g) is called locally symmetric if its curvature tensor R is parallel (that is, $\nabla R = 0$). The notion of semisymmetric, a proper generalization of locally symmetric manifold, is defined by $R(X,Y)\cdot R = 0$, where $R(X,Y)$ acts on R as a derivation. A complete intrinsic classification of these manifolds was given by Szabo in [28]. A Riemannian manifold is said to be Weyl projective semisymmetric if the curvature tensor P satisfies $R(X,Y)\cdot P = 0$, where $R(X,Y)$ acts on P as a derivation.
We define endomorphisms $R(X,Y)$ and $X \wedge_A Y$ by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z,$$

$$(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y,$$

respectively, where $X, Y, Z \in \chi(M)$, $\chi(M)$, A is the symmetric $(0,2)$-tensor and ∇ is the Levi-Civita connection.

Quarter-symmetric metric connection in a Riemannian manifold have been studied by several authors such as Mandal and De [14], Rastogi [22, 23], Yano and Imai [30], Mukhopadhyay, Roy and Barua [15], Han et al [9], Biswas and De [3] and many others. Recently, Sular, Özgür and De [27] studied quarter-symmetric metric connection in a Kenmotsu manifold.

Motivated by these circumstances in this paper we study some curvature conditions in a Kenmotsu manifold admitting a quarter-symmetric metric connection. The paper is organized as follows: In section 2, we present a brief account of Kenmotsu manifolds. In section 3, we discuss the curvature tensor and the Ricci tensor of a Kenmotsu manifold with respect to the quarter-symmetric metric connection. In the next section we study Weyl projective symmetric Kenmotsu manifolds with respect to the quarter-symmetric metric connection and prove that the manifold is an Einstein manifold with respect to the Levi-Civita connecction. In section 5, we prove that a Kenmotsu manifold satisfies the curvature condition $\widetilde{P}\cdot\widetilde{S}=0$, where $\widetilde{P}$ and $\widetilde{S}$ are the projective curvature tensor and the Ricci tensor respectively with respect to the quarter-symmetric metric connection, if and only if the manifold is an Einstein manifold with respect to the quarter-symmetric metric connection. In the next two sections we study pseudoprojectively flat Kenmotsu manifolds and ϕ-projectively semisymmetric Kenmotsu manifolds with respect to the quarter-symmetric metric connection, respectively and both the cases the manifold is an Einstein manifold with respect to the Levi-Civita connection. Finally, we give an example of a 5-dimensional Kenmotsu manifold admitting a quarter-symmetric metric connection to verify some results.

2 Kenmotsu manifolds

Let M be an $n\ (=2m+1)$-dimensional almost contact metric manifold carries an almost contact metric structure (ϕ,ξ,η,g), where ϕ is a $(1,1)$-tensor field, ξ associated vector field, η a 1-form and g the Riemannian metric satisfying the following conditions [2]:

$$\phi^2X=-X+\eta(X)\xi,\ \eta(\xi)=1,\ \phi\xi=0,\ \eta(\phi X)=0, \tag{5}$$

$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y), \tag{6}$$

$$g(\phi X,Y)=-g(X,\phi Y),\ g(X,\xi)=\eta(X), \tag{7}$$

for all $X,Y\in\chi(M)$. If an almost contact metric manifold satisfies

$$(\nabla_X\phi)Y=g(\phi X,Y)\xi-\eta(Y)\phi X, \tag{8}$$

where ∇ denotes the Levi-Civita connection of g, then M is said to be a Kenmotsu manifold [12]. In a Kenmotsu manifold the following relations hold [12, 27, 11]:

$$\nabla_X\xi=X-\eta(X)\xi, \tag{9}$$

$$(\nabla_X\eta)Y=g(X,Y)-\eta(X)\eta(Y). \tag{10}$$

$$R(X,Y)\xi=\eta(X)Y-\eta(Y)X, \tag{11}$$

$$R(\xi,X)Y=\eta(Y)X-g(X,Y)\xi, \tag{12}$$

$$S(X,\xi)=-(n-1)\eta(X), \tag{13}$$

where R is the curvature tensor, S the Ricci tensor. From (9) we see that $div\xi = n-1$, for what a Kenmotsu manifold is not compact. It is well known [12] that a Kenmotsu manifold M^{2m+1} is locally a warped product $I \times_f N^{2m}$ where N^{2m} is a Kähler manifold, I is an open interval with coordinate t and the warping function f, defined by $f = ce^t$ for some positive constant c.
A Kenmotsu manifold M is said to be an η-Einstein manifold if the Ricci tensor S satisfies the following equation

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

where a and b are some scalars. For $b = 0$, the manifold M is an Einstein manifold.

Kenmotsu manifolds have been studied by several authors such as Calin [4], De and Pathak [7], Jun, De and Pathak [11], Pitis [19], Kirichenko [13], Hong et al [10] and many others.

3 Curvature tensor of a Kenmotsu manifold with respect to the quarter-symmetric metric connection

In a Kenmotsu manifold the quarter-symmetric metric connection $\widetilde{\nabla}$ and the Levi-Civita connection ∇ are related by [27]

$$\widetilde{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y, \tag{14}$$

for all vector fields X, Y on M.
Let $\widetilde{R}$ and R be the Riemannian curvature tensor with respect to the quarter-symmetric metric connection and Levi-Civita connection respectively of a Kenmotsu manifold. Then $\widetilde{R}$ and R are related by [27]

$$\begin{aligned} \widetilde{R}(X,Y)Z &= R(X,Y)Z + \eta(X)g(\phi Y, Z)\xi - \eta(Y)g(\phi X, Z)\xi \\ &\quad -\eta(X)\eta(Z)\phi Y + \eta(Y)\eta(Z)\phi X. \end{aligned} \tag{15}$$

Contracting (15) we have [27]

$$\widetilde{S}(Y,Z) = S(Y,Z) + g(\phi Y, Z), \tag{16}$$

where $\widetilde{S}$ and S are the Ricci tensor with respect to the quarter-symmetric metric connection and Levi-Civita connection, respectively. Moreover, for a Kenmotsu manifold with respect to the quarter-symmetric metric connection the following relations hold [27]:

$$\widetilde{R}(X,Y)\xi = \eta(X)Y - \eta(Y)X - \eta(X)\phi Y + \eta(Y)\phi X, \tag{17}$$

$$\widetilde{R}(X,\xi)Y = g(X,Y)\xi - \eta(Y)X - g(\phi X, Y)\xi + \eta(Y)\phi X, \tag{18}$$

$$\widetilde{R}(\xi,X)\xi = X - \eta(X)\xi - \phi X, \tag{19}$$

$$\widetilde{S}(X,\xi) = S(X,\xi) = -(n-1)\eta(X). \tag{20}$$

Further, it is noted that [27] the Ricci tensor $\widetilde{S}$ with respect to the quarter-symmetric metric connection is not symmetric.
Applying (15) and (16) in (4) gives

$$\begin{aligned}\widetilde{P}(X,Y)Z &= R(X,Y)Z+\eta(X)g(\phi Y,Z)\xi-\eta(Y)g(\phi X,Z)\xi \\ &\quad -\eta(X)\eta(Z)\phi Y+\eta(Y)\eta(Z)\phi X \\ &\quad -\frac{1}{n-1}[S(Y,Z)X+g(\phi Y,Z)X-S(X,Z)Y-g(\phi X,Z)Y]. \quad (21)\end{aligned}$$

Making use of (11)-(13) in (21), we obtain

$$\begin{aligned}\widetilde{P}(\xi,Y)Z &= g(\phi Y,Z)\xi-g(Y,Z)\xi-\eta(Z)\phi Y \\ &\quad -\frac{1}{n-1}[S(Y,Z)\xi+g(\phi Y,Z)\xi], \quad (22)\end{aligned}$$

$$\widetilde{P}(X,Y)\xi=\eta(Y)\phi X-\eta(X)\phi Y, \quad (23)$$

$$\widetilde{P}(\xi,Y)\xi=-\phi Y. \quad (24)$$

It should be note that

$$\widetilde{P}(X,Y)Z=-\widetilde{P}(Y,X)Z, \quad (25)$$

for all X,Y and $Z\in\chi(M)$.

4 Weyl projective symmetric Kenmotsu manifolds with respect to the quarter-symmetric metric connection

In this section we study Weyl projective symmetric Kenmotsu manifolds with respect to the quarter-symmetric metric connection $\widetilde{\nabla}$. At first we prove the following:

Theorem 1. *Let M be an $n(=2m+1)$-dimensional Kenmotsu manifold. If M is Weyl projective symmetric Kenmotsu manifolds with respect to the quarter-symmetric metric connection, then M is an Einstein manifold with respect to the Levi-Civita connection.*

Proof. Assume that M is an $n(=2m+1)$-dimensional Weyl projective symmetric Kenmotsu manifolds with respect to the quarter-symmetric metric connection. Therefore we have $(\widetilde{R}(X,Y)\cdot\widetilde{P})(U,V)=0$ for all X,Y,U and $V\in\chi(M)$. This is equivalent to

$$\begin{aligned}&\widetilde{R}(X,Y)\widetilde{P}(U,V)W-\widetilde{P}(\widetilde{R}(X,Y)U,V)W \\ &-\widetilde{P}(U,\widetilde{R}(X,Y)V)W-\widetilde{P}(U,V)\widetilde{R}(X,Y)W=0, \quad (26)\end{aligned}$$

where $X,Y,U,V,W\in\chi(M)$.
Substituting $X=U=\xi$ in the above equation gives

$$\begin{aligned}&\widetilde{R}(\xi,Y)\widetilde{P}(\xi,V)W-\widetilde{P}(\widetilde{R}(\xi,Y)\xi,V)W \\ &-\widetilde{P}(\xi,\widetilde{R}(\xi,Y)V)W-\widetilde{P}(\xi,V)\widetilde{R}(\xi,Y)W=0. \quad (27)\end{aligned}$$

Making use of (18) and (19) in (27) we have

$$\begin{aligned}
&\eta(\widetilde{P}(\xi,V)W)Y - g(Y,\widetilde{P}(\xi,V)W)\xi - \eta(\widetilde{P}(\xi,V)W)\phi Y \\
&+g(\phi Y,\widetilde{P}(\xi,V)W)\xi - \widetilde{P}(Y,V)W + \eta(Y)\widetilde{P}(\xi,V)W \\
&+\widetilde{P}(\phi Y,V)W - \eta(V)\widetilde{P}(\xi,Y)W + \eta(V)\widetilde{P}(\xi,\phi Y)W \\
&-\eta(W)\widetilde{P}(\xi,V)Y + g(Y,W)\widetilde{P}(\xi,V)\xi + \eta(W)\widetilde{P}(\xi,V)\phi Y \\
&-g(\phi Y,W)\widetilde{P}(\xi,V)\xi = 0. \qquad (28)
\end{aligned}$$

Using (21), (22) and (24) in (28) and then taking inner product with arbitrary vector field Z, we obtain

$$\begin{aligned}
&g(\phi V,W)g(Y,Z) - g(V,W)g(Y,Z) - g(\phi V,W)\eta(Y)\eta(Z) \\
&+g(\phi V,Y)\eta(W)\eta(Z) - g(\phi V,W)g(\phi Y,Z) + g(V,W)g(\phi Y,Z) \\
&-g(R(Y,V)W,Z) - g(\phi Y,W)\eta(V)\eta(Z) + g(V,Y)\eta(W)\eta(Z) \\
&+g(\phi V,Z)\eta(W)\eta(Y) + g(R(\phi Y,V)W,Z) + g(Y,W)\eta(V)\eta(Z) \\
&+g(\phi Y,W)g(\phi V,Z) - g(Y,W)g(\phi V,Z) \\
&+\frac{1}{n-1}\{S(\phi Y,W)g(V,Z) - S(Y,W)g(V,Z) - g(\phi Y,W)g(V,Z) \\
&-g(Y,W)g(V,Z) + g(V,Z)\eta(Y)\eta(W) + S(Y,W)\eta(V)\eta(Z) \\
&+g(\phi Y,W)\eta(V)\eta(Z) - S(\phi Y,W)\eta(V)\eta(Z) + g(Y,W)\eta(V)\eta(Z) \\
&+S(V,Y)\eta(W)\eta(Z) + g(\phi V,Y)\eta(W)\eta(Z) - S(V,\phi Y)\eta(W)\eta(Z) \\
&-g(V,Y)\eta(W)\eta(Z)\} = 0. \qquad (29)
\end{aligned}$$

Substituting $V = W = e_i$ in (29), where $\{e_i\}(1 \le i \le n)$ is an orthonormal basis of the tangent space at any point of the manifold M^n, we have

$$\begin{aligned}
&-ng(Y,Z) + ng(\phi Y,Z) - S(Y,Z) + S(\phi Y,Z) - g(\phi Y,Z) \\
&-g(Y,Z) + \eta(Y)\eta(Z) + \frac{1}{n-1}\{S(\phi Y,Z) + \eta(Y)\eta(Z) \\
&-g(Y,Z) - S(Y,Z) - g(\phi Y,Z)\} = 0. \qquad (30)
\end{aligned}$$

Replacing Y by ϕY in (30) yields

$$\begin{aligned}
&-ng(\phi Y,Z) - ng(Y,Z) - S(\phi Y,Z) - S(Y,Z) - \eta(Y)\eta(Z) \\
&+g(Y,Z) - g(\phi Y,Z) + \frac{1}{n-1}\{g(Y,Z) - S(\phi Y,Z) - \eta(Y)\eta(Z) \\
&-S(Y,Z) - g(\phi Y,Z)\} = 0. \qquad (31)
\end{aligned}$$

Adding (30) and (31), it follows that

$$S(Y,Z) + g(\phi Y,Z) = -(n-1)g(Y,Z). \qquad (32)$$

Interchanging Y and Z in (32) gives

$$S(Z,Y) + g(\phi Z,Y) = -(n-1)g(Z,Y). \qquad (33)$$

Adding (32) and (33) and then applying (7) we get

$$S(Y,Z) = -(n-1)g(Y,Z),$$

which shows that the manifold is an Einstein manifold with respect to the Levi-Civita connection. Thus our theorem is proved. □

5 Kenmotsu manifolds satisfying the curvature condition $\widetilde{P}\cdot\widetilde{S}=0$

In this section we consider a Kenmotsu manifold satisfying the curvature condition

$$(\widetilde{P}(X,Y)\cdot\widetilde{S})(U,V) = 0,$$

which is equivalent to

$$\widetilde{S}(\widetilde{P}(X,Y)U,V) + \widetilde{S}(U,(\widetilde{P}(X,Y)V) = 0. \tag{34}$$

Substituting $X = U = \xi$ in the above equation we have

$$\widetilde{S}(\widetilde{P}(\xi,Y)\xi,V) + \widetilde{S}(\xi,(\widetilde{P}(\xi,Y)V) = 0. \tag{35}$$

Using (24) and (20) in (35) we obtain

$$\widetilde{S}(\phi Y,V) + (n-1)\eta(\widetilde{P}(\xi,Y)V) = 0. \tag{36}$$

Making use of (16) and (22) in (36) it follows that

$$\begin{aligned} &S(\phi Y,V) + \eta(Y)\eta(V) - ng(Y,V) \\ &+(n-1)g(\phi Y,V) - S(Y,V) - g(\phi Y,V) = 0. \end{aligned} \tag{37}$$

Putting $Y = \phi Y$ in the above equation yields

$$\begin{aligned} &-S(Y,V) - ng(\phi Y,V) - (n-1)g(Y,V) \\ &-S(\phi Y,V) + g(Y,V) - \eta(Y)\eta(V) = 0. \end{aligned} \tag{38}$$

Adding (37) and (38) we get

$$S(Y,V) + g(\phi Y,V) + (n-1)g(Y,V) = 0. \tag{39}$$

Applying (16) in (39) gives

$$\widetilde{S}(Y,V) = -(n-1)g(Y,V), \tag{40}$$

from which it follows that the manifold is an Einstein manifold with respect to the quarter-symmetric metric connection.

Conversely, if the manifold is an Einstein manifold of the form (40), then it is obvious that $\widetilde{S}(\widetilde{P}(X,Y)U,V) + \widetilde{S}(U,(\widetilde{P}(X,Y)V) = 0$, for any $X,Y,U,V \in \chi(M)$, that is, $\widetilde{P}\cdot\widetilde{S} = 0$. By the above discussions we have the following:

Theorem 2. *An $n(=2m+1)$-dimensional Kenmotsu manifold satisfies the curvature condition $\widetilde{P}\cdot\widetilde{S}=0$ if and only if the manifold is an Einstein manifold with respect to the quarter-symmetric metric connection.*

Again interchanging Y and V in (39) we obtain

$$S(V,Y)+g(\phi V,Y)+(n-1)g(V,Y)=0. \tag{41}$$

Adding (39) and (41) and also using (7) we have

$$S(Y,V)=-(n-1)g(Y,V),$$

that is, the manifold is an Einstein manifold with respect to the Levi-Civita connection. Hence, we can state the following:

Corollary 1. *If an $n(=2m+1)$-dimensional Kenmotsu manifold satisfies the curvature condition $\widetilde{P}\cdot\widetilde{S}=0$, then the manifold is an Einstein manifold with respect to the Levi-Civita connection.*

6 Pseudoprojectively flat Kenmotsu manifolds with respect to the quarter-symmetric metric connection

This section is devoted to study pseudoprojectively flat Kenmotsu manifolds with respect to the quarter-symmetric metric connection.

A Kenmotsu manifold is said to be pseudoprojectively flat [6] if the following condition holds

$$g(P(\phi X,Y)Z,\phi W)=0, \tag{42}$$

for all X,Y,Z and $W\in\chi(M)$.
Therefore we have

$$g(\widetilde{P}(\phi X,Y)Z,\phi W)=0. \tag{43}$$

Making use of (21) and (43) we obtain

$$\begin{aligned} g(R(\phi X,Y)Z,\phi W) &= \frac{1}{n-1}[S(Y,Z)g(\phi X,\phi W)+g(\phi Y,Z)g(\phi X,\phi W)\\ &\quad -S(\phi X,Z)g(Y,\phi W)+g(X,Z)g(Y,\phi W)\\ &\quad -\eta(X)\eta(Z)g(Y,\phi W)]+\eta(Y)\eta(Z)g(X,\phi W). \end{aligned} \tag{44}$$

Replacing X by ϕX and W by ϕW in (44) implies

$$\begin{aligned} g(R(\phi^2X,Y)Z,\phi^2W) &= \frac{1}{n-1}[S(Y,Z)g(\phi^2X,\phi^2W)+g(\phi Y,Z)g(\phi^2X,\phi^2W)\\ &\quad -S(\phi^2X,Z)g(Y,\phi^2W)+g(\phi X,Z)g(Y,\phi^2W)]\\ &\quad +\eta(Y)\eta(Z)g(\phi X,\phi^2W). \end{aligned} \tag{45}$$

Making use of (5) we get

$$\begin{aligned} &g(R(\phi^2X,Y)Z,\phi^2W)\\ &= g(R(X,Y)Z,W) - \eta(W)g(R(X,Y)Z,\xi)\\ &-\eta(X)g(R(\xi,Y)Z,W) + \eta(X)\eta(W)g(R(\xi,Y)Z,\xi). \end{aligned} \tag{46}$$

Applying (5) and the above equation in (45) gives

$$\begin{aligned} &g(R(X,Y)Z,W) - \eta(W)g(R(X,Y)Z,\xi)\\ &-\eta(X)g(R(\xi,Y)Z,W) + \eta(X)\eta(W)g(R(\xi,Y)Z,\xi)\\ &= \frac{1}{n-1}[S(Y,Z)g(X,W) - S(Y,Z)\eta(X)\eta(W) + g(\phi Y,Z)g(X,W)\\ &-g(\phi Y,Z)\eta(X)\eta(W) - S(X,Z)g(Y,W) + S(X,Z)\eta(Y)\eta(W)\\ &-(n-1)g(Y,W)\eta(X)\eta(Z) + (n-1)\eta(X)\eta(Y)\eta(Z)\eta(W)\\ &-g(\phi X,Z)g(Y,W) + g(\phi X,Z)\eta(Y)\eta(W)] + g(X,\phi Z)\eta(Y)\eta(Z). \end{aligned} \tag{47}$$

Putting $X = W = e_i$ in (47), where $\{e_i\}(1 \leq i \leq n)$ is an orthonormal basis of the tangent space at any point of the manifold M^n, we get

$$S(Y,Z) - g(R(\xi,Y)Z,\xi) = \frac{n-2}{n-1}[S(Y,Z) + g(\phi Y,Z)] - \eta(Y)\eta(Z). \tag{48}$$

Using (12) and (48) we obtain

$$S(Y,Z) = (n-2)g(\phi Y,Z) - (n-1)g(Y,Z). \tag{49}$$

Interchanging Y and Z in (49) yields

$$S(Z,Y) = (n-2)g(\phi Z,Y) - (n-1)g(Z,Y). \tag{50}$$

Adding (49) and (50), we have $S(Y,Z) = -(n-1)g(Y,Z)$, for all $Y, Z \in \chi(M)$. Thus we see that the manifold is an Einstein manifold with respect to the Levi-Civita connection. This leads to the following:

Theorem 3. *An $n(=2m+1)$-dimensional pseudoprojectively flat Kenmotsu manifold with respect to the quarter-symmetric metric connection is an Einstein manifold with respect to the Levi-Civita connection.*

7 ϕ-projectively semisymmetric Kenmotsu manifolds with respect to the quarter-symmetric metric connection

A Kenmotsu manifold is said to be ϕ-projectively semisymmetric if $P(X,Y)\cdot\phi = 0$ holds on M, for any $X, Y \in \chi(M)$. In this section we consider M be an $n(=2m+1)$-dimensional ϕ-projectively semisymmetric Kenmotsu manifold with respect to the quarter-symmetric metric connection. Therefore $\widetilde{P}(X,Y)\cdot\phi = 0$ implies

$$(\widetilde{P}(X,Y)\cdot\phi)Z = \widetilde{P}(X,Y)\phi Z - \phi\widetilde{P}(X,Y)Z = 0, \tag{51}$$

for any X, Y and $Z \in \chi(M)$.
Substituting $X = \xi$ in (51) we have

$$(\widetilde{P}(\xi,Y)\cdot\phi)Z = \widetilde{P}(\xi,Y)\phi Z - \phi\widetilde{P}(\xi,Y)Z = 0. \tag{52}$$

Applying (22) in (52) we obtain

$$\begin{aligned} & g(Y,Z)\xi - g(Y,\phi Z)\xi - \frac{1}{n-1}S(Y,\phi Z)\xi \\ & -\frac{1}{n-1}g(Y,Z)\xi + \frac{1}{n-1}\eta(Y)\eta(Z)\xi - \eta(Z)Y = 0. \end{aligned} \tag{53}$$

Taking inner product of (53) with ξ yields

$$(n-2)g(Y,Z) - (n-1)g(Y,\phi Z) - S(Y,\phi Z) - (n-2)\eta(Y)\eta(Z) = 0. \tag{54}$$

Setting $Z = \phi Z$ in (54) gives

$$S(Y,Z) + (n-2)g(Y,\phi Z) + (n-1)g(Y,Z) = 0. \tag{55}$$

Interchanging Y and Z in (55) we obtain

$$S(Z,Y) + (n-2)g(Z,\phi Y) + (n-1)g(Z,Y) = 0. \tag{56}$$

Adding (55) and (56), we have $S(Y,Z) = -(n-1)g(Y,Z)$, which implies that the manifold is an Einstein manifold with respect to the Levi-Civita connection. Therefore we can state the following:

Theorem 4. *An $n(= 2m+1)$-dimensional ϕ-projectively semisymmetric Kenmotsu manifold with respect to the quarter-symmetric metric connection is an Einstein manifold with respect to the Levi-Civita connection.*

8 Example of a 5-dimensional Kenmotsu manifold admitting a quarter-symmetric metric connection

We consider the 5-dimensional manifold $M = \{(x,y,z,u,v) \in \mathbb{R}^5\}$, where (x,y,z,u,v) are the standard coordinates in $\mathbb{R}^5$.
We choose the vector fields

$$e_1 = e^{-v}\frac{\partial}{\partial x},\ \ e_2 = e^{-v}\frac{\partial}{\partial y},\ \ e_3 = e^{-v}\frac{\partial}{\partial z},\ \ e_4 = e^{-v}\frac{\partial}{\partial u},\ \ e_5 = \frac{\partial}{\partial v},$$

which are linearly independent at each point of M.
Let g be the Riemannian metric defined by

$$g(e_i,e_j) = 0,\ \ i \neq j,\ \ i,j = 1,2,3,4,5$$

and

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = g(e_4, e_4) = g(e_5, e_5) = 1.$$

Let η be the 1-form defined by

$$\eta(Z) = g(Z, e_5),$$

for any $Z \in \chi(M)$.
Let ϕ be the $(1, 1)$-tensor field defined by

$$\phi e_1 = e_3,\ \phi e_2 = e_4,\ \phi e_3 = -e_1,\ \phi e_4 = -e_2,\ \phi e_5 = 0.$$

Using the linearity of ϕ and g, we have

$$\eta(e_5) = 1,$$

$$\phi^2(Z) = -Z + \eta(Z)e_5$$

and

$$g(\phi Z, \phi U) = g(Z, U) - \eta(Z)\eta(U),$$

for any $U, Z \in \chi(M)$. Thus, for $e_5 = \xi$, $M(\phi, \xi, \eta, g)$ defines an almost contact metric manifold. The 1-form η is closed.

We have

$$\Omega(\frac{\partial}{\partial x}, \frac{\partial}{\partial z}) = g(\frac{\partial}{\partial x}, \phi\frac{\partial}{\partial z}) = g(\frac{\partial}{\partial x}, -\frac{\partial}{\partial x}) = -e^{2v}.$$

Hence, we obtain $\Omega = -e^{2v} dx \wedge dz$. Thus, $d\Omega = -2e^{2v} dv \wedge dx \wedge dz = 2\eta \wedge \Omega$. Therefore, $M(\phi, \xi, \eta, g)$ is an almost Kenmotsu manifold. It can be seen that $M(\phi, \xi, \eta, g)$ is normal. So, it is a Kenmotsu manifold.

Then we have

$$[e_1, e_2] = [e_1, e_3] = [e_1, e_4] = [e_2, e_3] = 0, [e_1, e_5] = e_1,$$

$$[e_4, e_5] = e_4, [e_2, e_4] = [e_3, e_4] = 0, [e_2, e_5] = e_2, [e_3, e_5] = e_3.$$

The Levi-Civita connection ∇ of the metric tensor g is given by Koszul's formula which is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) \\ &\quad -g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned}$$

Taking $e_5 = \xi$ and using the above formula we obtain the following:

$$\nabla_{e_1} e_1 = -e_5,\quad \nabla_{e_1} e_2 = 0,\quad \nabla_{e_1} e_3 = 0,\quad \nabla_{e_1} e_4 = 0,\quad \nabla_{e_1} e_5 = e_1,$$

$$\nabla_{e_2} e_1 = 0,\quad \nabla_{e_2} e_2 = -e_5,\quad \nabla_{e_2} e_3 = 0,\quad \nabla_{e_2} e_4 = 0,\quad \nabla_{e_2} e_5 = e_2,$$

$$\nabla_{e_3} e_1 = 0,\quad \nabla_{e_3} e_2 = 0,\quad \nabla_{e_3} e_3 = -e_5,\quad \nabla_{e_3} e_4 = 0,\quad \nabla_{e_3} e_5 = e_3,$$

$$\nabla_{e_4}e_1 = 0,\quad \nabla_{e_4}e_2 = 0,\quad \nabla_{e_4}e_3 = 0,\quad \nabla_{e_4}e_4 = -e_5,\quad \nabla_{e_4}e_5 = e_4,$$

$$\nabla_{e_5}e_1 = 0,\quad \nabla_{e_5}e_2 = 0,\quad \nabla_{e_5}e_3 = 0,\quad \nabla_{e_5}e_4 = 0,\quad \nabla_{e_5}e_5 = 0.$$

Further we obtain the following:

$$\widetilde{\nabla}_{e_1}e_1 = -e_5,\quad \widetilde{\nabla}_{e_1}e_2 = 0,\quad \widetilde{\nabla}_{e_1}e_3 = 0,\quad \widetilde{\nabla}_{e_1}e_4 = 0,\quad \widetilde{\nabla}_{e_1}e_5 = e_1,$$

$$\widetilde{\nabla}_{e_2}e_1 = 0,\quad \widetilde{\nabla}_{e_2}e_2 = -e_5,\quad \widetilde{\nabla}_{e_2}e_3 = 0,\quad \widetilde{\nabla}_{e_2}e_4 = 0,\quad \widetilde{\nabla}_{e_2}e_5 = e_2,$$

$$\widetilde{\nabla}_{e_3}e_1 = 0,\quad \widetilde{\nabla}_{e_3}e_2 = 0,\quad \widetilde{\nabla}_{e_3}e_3 = -e_5,\quad \widetilde{\nabla}_{e_3}e_4 = 0,\quad \widetilde{\nabla}_{e_3}e_5 = e_3,$$

$$\widetilde{\nabla}_{e_4}e_1 = 0,\quad \widetilde{\nabla}_{e_4}e_2 = 0,\quad \widetilde{\nabla}_{e_4}e_3 = 0,\quad \widetilde{\nabla}_{e_4}e_4 = -e_5,\quad \widetilde{\nabla}_{e_4}e_5 = e_4,$$

$$\widetilde{\nabla}_{e_5}e_1 = -e_3,\quad \widetilde{\nabla}_{e_5}e_2 = -e_4,\quad \widetilde{\nabla}_{e_5}e_3 = e_1,\quad \widetilde{\nabla}_{e_5}e_4 = e_2,\quad \widetilde{\nabla}_{e_5}e_5 = 0.$$

By the above results, we can easily obtain the non-vanishing components of the curvature tensors as follows:

$$R(e_1,e_2)e_2 = R(e_1,e_3)e_3 = R(e_1,e_4)e_4 = R(e_1,e_5)e_5 = -e_1,$$

$$R(e_1,e_2)e_1 = e_2,\ R(e_1,e_3)e_1 = R(e_5,e_3)e_5 = R(e_2,e_3)e_2 = e_3,$$

$$R(e_2,e_3)e_3 = R(e_2,e_4)e_4 = R(e_2,e_5)e_5 = -e_2, R(e_3,e_4)e_4 = -e_3,$$

$$R(e_2,e_5)e_2 = R(e_1,e_5)e_1 = R(e_4,e_5)e_4 = R(e_3,e_5)e_3 = e_5,$$

$$R(e_1,e_4)e_1 = R(e_2,e_4)e_2 = R(e_3,e_4)e_3 = R(e_5,e_4)e_5 = e_4$$

and

$$\widetilde{R}(e_1,e_2)e_2 = \widetilde{R}(e_1,e_3)e_3 = \widetilde{R}(e_1,e_4)e_4 = -e_1,$$

$$\widetilde{R}(e_1,e_2)e_1 = e_2,\ \widetilde{R}(e_1,e_3)e_1 = \widetilde{R}(e_2,e_3)e_2 = e_3,$$

$$\widetilde{R}(e_2,e_3)e_3 = \widetilde{R}(e_2,e_4)e_4 = -e_2, \widetilde{R}(e_2,e_5)e_5 = e_4 - e_2,$$

$$\widetilde{R}(e_3,e_4)e_4 = -e_3,\ \widetilde{R}(e_2,e_5)e_2 = \widetilde{R}(e_1,e_5)e_1 = \widetilde{R}(e_4,e_5)e_4 = e_5,$$

$$\widetilde{R}(e_3,e_5)e_3 = e_5, \widetilde{R}(e_1,e_4)e_1 = \widetilde{R}(e_2,e_4)e_2 = \widetilde{R}(e_3,e_4)e_3 = e_4,$$

$$\widetilde{R}(e_1,e_5)e_5 = e_3 - e_1, \widetilde{R}(e_3,e_5)e_5 = -e_1 - e_3, \widetilde{R}(e_4,e_5)e_5 = -e_2 - e_4.$$

Making use of the above results we obtain the Ricci tensors as follows:

$$S(e_1,e_1) = S(e_2,e_2) = S(e_3,e_3) = S(e_4,e_4) = S(e_5,e_5) = -4$$

and

$$\widetilde{S}(e_1,e_1) = \widetilde{S}(e_2,e_2) = \widetilde{S}(e_3,e_3) = \widetilde{S}(e_4,e_4) = \widetilde{S}(e_5,e_5) = -4.$$

It can be easily verified that the manifold is an Einstein manifold with respect to the quarter-symmetric metric connection. Therefore Theorem 2 is verified.

References

[1] Barman, A., *On para-Sasakian manifolds admitting semi-symmetric metric connection,* Publ. De L'Institut Math., **109** (2014), 239-247.

[2] Blair, D.E., *Contact manifolds in Reimannian geometry,* Lecture Notes in Math., **509**, Springer-verlag., 1976.

[3] Biswas, S.C. and De, U.C., *Quarter-symmetric metric connection in SP-Sasakian manifold,* Common. Fac. Sci. Univ. Ank. Al., **46**(1997), 49-56.

[4] Calin, C., *Kenmotsu manifolds with η-parallel Ricci tensor,* Bull. Soc. Math. Banja Luka, **10** (2003), 10-15.

[5] De, U.C., *On a type of semi-symmetric metric connection on a Riemannian manifold,* Anal. Stiint. Ale Univ. "Al.I.Cuza" Din Iasi Math., **XXXVIII** (1991), 105-108.

[6] De, U.C. and De, A.,*On some curvature properties of K-contact manifolds,* Extracta Math., **27** (2012), 125-134.

[7] De, U.C. and Pathak, G., *On* 3*-dimensional Kenmotsu manifolds,* Indian J. Pure App. Math., **35** (2004), 159-165.

[8] Golab, S., *On semi-symmetric and quarter-symmetric linear connections,* Tensor(N.S.) **29**(1975), 249-254.

[9] Han, Y., Yun, H.T., Zhao, P., *Some invariants of quarter-symmetric metric connections under the projective transformation,* Filomat, **27** (2013), 679-691.

[10] Hong, S., Özgür, C. and Tripathi, M.M., *On some special classes of Kenmotsu manifolds,* Kuwait J. Sci. Eng. **33** (2006), 19-32.

[11] Jun, J.-B., De, U.C. and Pathak, G., *On Kenmotsu manifolds,* J. Korean Math. Soc., **42** (2005), 435-445.

[12] Kenmotsu, K., *A class of almost contact Riemannian manifolds,* Tohoku Math. J. **24** (1972), 93-103.

[13] Kirichenko, V.F., *On the geometry of Kenmotsu manifolds,* Doklady Akademii Nauk, **380** (2001), 585-587.

[14] Mandal, K. and De, U.C., *Quarter-symmetric metric connection in a P-Sasakian manifold,* Anal. Univ. de Vest Timi., Seria Mat. Inf., **53** (2015), 137-150.

[15] Mukhopadhyay, S., Roy, A.K. and Barua, B., *Some properties of quarter-symmetric metric connection on a Riemannian manifold,* Soochow J. Math., **17** (1991), 205-211.

[16] Özgür, C. and Sular, S., *Warped product manifolds with semi-symmetric metric connections,* Taiwanese J. Math., **15** (2011), 1701-1719.

[17] Ozen, Z.F., Demirbag, S.A., Uysal, S.A. and Yilmaz, H.B., *Some vector fields on a Riemannian manifold with semi-symmetric metric connection,* Bull. Iranian Math. Soc., **38** (2012) 479-490.

[18] Ozen, Z.F., Uysal, S.A. and Demirbag, S.A., *On sectional curvature of a Riemannian manifold with semi-symmetric metric connection,* Ann. Polon. Math., **101**(2011), 131-138.

[19] Pitiş, G., *Geometry of Kenmotsu manifolds,* University Transllivania of Brasov, Romania, 2007.

[20] Prvanovic, M., *On pseudo metric semi-symmetric connections,* Publ. De L' Institut Math., (N.S.), **18** (1975), 157-164.

[21] Prvanovic, M. and Pušić, N., *On manifolds admitting some semi-symmetric metric connection,* Indian J. Math., **37** (1995), 37-47.

[22] Rastogi, S.C., *On quarter-symmetric metric connection,* C. R. Acad. Sci. Bulgar, **31** (1978), 811-814.

[23] Rastogi, S.C., *On quarter-symmetric metric connection,* Tensor, N.S. **44** (1987),133-141.

[24] Smaranda, D. and Andonie, O.C., *On semi-symmetric connections,* Ann. Fac. Sci. Univ. Nat. Zaire(Kinshasa), Sec. Math.-Phys., **2**(1976), 265-270.

[25] Singh, R.N. and Pandey, M.K., *On a type of semi-symmetric metric connection on a Riemannian manifold,* Bull. Calcutta Math. Soc., **16** (2008) 179–184.

[26] Soośs, G. *Über die geodätischen Abbildungen von Riemannaschen Räumen auf projektiv symmetrische Riemannsche Räume,* Acta. Math. Acad. Sci. Hungar., **9** (1958), 359-361.

[27] Sular, S., Özgür, C. and De, U.C., *Quarter-symmetric metric connection in a Kenmotsu manifold,* SUT J. Math., **44** (2008), 297-306.

[28] Szabó, Z.I., *Structure theorems on Riemannian spaces satisfying* $R(X,Y) \cdot R = 0$ *I the local version,* J. Diff. Geom. **17** (1982), 531-582.

[29] Yano, K., *On semi-symmetric metric connection,* Rev. Roumaine Math. Pures Appl. **15** (1970), 1579-1586.

[30] Yano, K. and Imai, T., *Quarter-symmetric metric connections and their curvature tensors,* Tensor (N.S.) **38** (1982), 13-18.