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NONEXISTENCE OF SUBNORMAL SOLUTIONS FOR A CLASS OF HIGHER ORDER COMPLEX DIFFERENTIAL EQUATIONS

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Abstract

In this article, we investigate the existence of subnormal solutions for a class of higher order complex differential equations. We generalize the result of N. Li and L. Z. Yang [14], L. P. Xiao [17] and also result of Z. X. Chen and K. H. Shon [4].

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1 Introduction

In this article, we use the standard notations of the Nevanlinna theory, see [11, 12, 18]. We denote the order of growth of a meromorphic function f by $\sigma(f)$. To express the rate of growth of meromorphic of infinite order, we recall the following definitions.

Definition 1 ([18]). *The hyper-order of growth of a meromorphic function f is defined by*

$$\sigma_2(f) = \overline{\lim_{r\to+\infty}} \frac{\log\log T(r,f)}{\log r},$$

where $T(r,f)$ is the Nevanlinna characteristic function of f.

In [7], Chiang and Gao gave the definition of the e-type order of a meromorphic function as follows.

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Definition 2 ([7])**.** *Let f be a meromorphic function. Define*

$$\sigma_e(f) = \varlimsup_{r\to+\infty} \frac{\log T(r,f)}{r}$$

to be the e-type order of f.

The following results are obvious.

1. If $0 < \sigma_e(f) < +\infty$, then $\sigma_2(f) = 1$.
2. If $\sigma_2(f) < 1$, then $\sigma_e(f) = 0$.
3. If $\sigma_2(f) = +\infty$, then $\sigma_e(f) = +\infty$.

Consider the second-order homogeneous linear periodic differential equation

$$f'' + P\left(e^z\right) f' + Q\left(e^z\right) f = 0, \tag{1}$$

where $P(w)$ and $Q(w)$ are not constants polynomials in $w = e^z$ $(z \in \mathbb{C})$. It's well known that every solution of equation (1) is entire.

Definition 3 ([8, 16])**.** *If $f \not\equiv 0$ is a solution of equation* (1), *and satisfies $\sigma_e(f) = 0$, then we say that f is a nontrivial subnormal solution of* (1). *For convenience, we also say that $f \equiv 0$ is a subnormal solution of* (1).

In [8, 16], subnormal solutions of (1) were investigated. In [16], H. Wittich has given the general forms of all subnormal solutions of (1) that are shown in the following theorem.

Theorem 1. *If $f \not\equiv 0$ is a subnormal solution of (1), then f must have the form*

$$f(z) = e^{cz}(a_0 + a_1 e^z + \cdots + a_m e^{mz}),$$

where $m \geq 0$ is an integer and $c, a_0, a_1, \ldots, a_m$ are constants with $a_0 a_m \neq 0$.

Based on the comparison of degrees of P and Q, Gundersen and Steinbart [8] refined Theorem 1 and obtained the exact forms of subnormal solutions of (1) as follows.

Theorem 2. *Under the assumption of Theorem 1, the following statements hold.*

(*i*) *If $\deg P > \deg Q$ and $Q \not\equiv 0$, then any subnormal solution $f \not\equiv 0$ of (1) must have the form*

$$f(z) = a_0 + a_1 e^{-z} + \cdots + a_m e^{-mz},$$

where $m \geq 1$ is an integer and $a_0, a_1, \ldots, a_m$ are constants with $a_0 a_m \neq 0$.

(*ii*) *If $Q \equiv 0$ and $\deg P \geq 1$, then any subnormal solution of (1) must be a constant.*

(*iii*) *If $\deg P < \deg Q$, then the only subnormal solution of (1) is $f \equiv 0$.*

For second order differential equations, Chen and Shon [4] studied the existence of subnormal solutions of the equation

$$f'' + \left[P_1\left(e^{z}\right) + P_2\left(e^{-z}\right)\right] f' + \left[Q_1\left(e^{z}\right) + Q_2\left(e^{-z}\right)\right] f = 0, \tag{2}$$

where $P_1(z), P_2(z), Q_1(z)$ and $Q_2(z)$ are polynomials in z, and obtained the following results.

Theorem 3. *Let $P_j(z), Q_j(z)$ $(j = 1, 2)$ be polynomials in z. If*

$$\deg Q_1 > \deg P_1 \ \textit{or}\ \deg Q_2 > \deg P_2,$$

then the equation (2) *has no nontrivial subnormal solution, and every solution of* (2) *satisfies $\sigma_2(f) = 1$.*

Theorem 4. *Let $P_j(z), Q_j(z)$ $(j = 1, 2)$ be polynomials in z. If*

$$\deg Q_1 < \deg P_1 \ \textit{and}\ \deg Q_2 < \deg P_2$$

and $Q_1 + Q_2 \not\equiv 0$, then the equation (2) *has no nontrivial subnormal solution, and every solution of* (2) *satisfies $\sigma_2(f) = 1$.*

Li-Yang [14] considered the case when $\deg Q_1 = \deg P_1$ and $\deg Q_2 = \deg P_2$ in the equation (2), and they proved it.

Theorem 5. *Let*

$$\begin{aligned} P_1(z) &= a_n z^n + \cdots + a_1 + a_0, \\ Q_1(z) &= b_n z^n + \cdots + b_1 + b_0, \\ P_2(z) &= c_m z^m + \cdots + c_1 + c_0, \\ Q_2(z) &= d_m z^m + \cdots + d_1 + d_0, \end{aligned}$$

where a_i, b_i $(i = 0, \ldots, n)$, c_j, d_j $(j = 0, \ldots, m)$ are constants, $a_n b_n c_m d_m \neq 0$. Suppose that $a_n d_m = b_n c_m$ and any one of the following three hypothesis hold:

1. *There exists i satisfying $(-\frac{b_n}{a_n})a_i + b_i \neq 0$, $0 < i < n$.*
2. *There exists j satisfying $(-\frac{b_n}{a_n})c_j + d_j \neq 0$, $0 < j < m$.*
3. $(-\frac{b_n}{a_n})^2 + (-\frac{b_n}{a_n})(a_0 + c_0) + b_0 + d_0 \neq 0$.

Then (2) *has no nontrivial subnormal solution, and every nontrivial solution f satisfies $\sigma_2(f) = 1$.*

In the same article [14], Li-Yang investigated the existence of subnormal solutions of the general form

$$f'' + \left[P_1\left(e^{\alpha z}\right) + P_2\left(e^{-\alpha z}\right)\right] f' + \left[Q_1\left(e^{\beta z}\right) + Q_2\left(e^{-\beta z}\right)\right] f = 0, \tag{3}$$

where $P_1(z), P_2(z), Q_1(z)$ and $Q_2(z)$ are polynomials in z. α, β are complex constants, and they proved the following results.

Theorem 6. *Let*

$$\begin{aligned} P_1(z) &= a_{1m_1}z^{m_1} + \cdots + a_{11} + a_{10}, \\ P_2(z) &= a_{2m_2}z^{m_2} + \cdots + a_{21} + a_{20}, \\ Q_1(z) &= b_{1n_1}z^{n_1} + \cdots + b_{11} + b_{10}, \\ Q_2(z) &= b_{2n_2}z^{n_2} + \cdots + b_{21} + b_{20}, \end{aligned}$$

where $m_k \geq 1$, $n_k \geq 1$ $(k = 1, 2)$ *are integers,* $a_{1i_1}(i_1 = 0, \ldots, m_1)$, $a_{2i_2}(i_2 = 0, \ldots, m_2)$, $b_{1j_1}(j_1 = 0, \ldots, n_1)$, $b_{2j_2}(j_2 = 0, \ldots, n_2)$, α *and* β *are complex constants,* $a_{1m_1}a_{2m_2}b_{1n_1}b_{2n_2} \neq 0$, $\alpha\beta \neq 0$. *Suppose* $m_1\alpha = c_1n_1\beta$ $(0 < c_1 < 1)$ *or* $m_2\alpha = c_2n_2\beta$ $(0 < c_2 < 1)$. *Then* (3) *has no nontrivial subnormal solution, and every nontrivial solution* f *satisfies* $\sigma_2(f) = 1$.

Theorem 7. *Let*

$$\begin{aligned} P_1(z) &= a_{1m_1}z^{m_1} + \cdots + a_{11} + a_{10}, \\ P_2(z) &= a_{2m_2}z^{m_2} + \cdots + a_{21} + a_{20}, \\ Q_1(z) &= b_{1n_1}z^{n_1} + \cdots + b_{11} + b_{10}, \\ Q_2(z) &= b_{2n_2}z^{n_2} + \cdots + b_{21} + b_{20}, \end{aligned}$$

where $m_k \geq 1$, $n_k \geq 1$ $(k = 1, 2)$ *are integers,* $a_{1i_1}(i_1 = 0, \ldots, m_1)$, $a_{2i_2}(i_2 = 0, \ldots, m_2)$, $b_{1j_1}(j_1 = 0, \ldots, n_1)$, $b_{2j_2}(j_2 = 0, \ldots, n_2)$, α *and* β *are complex constants,* $a_{1m_1}a_{2m_2}b_{1n_1}b_{2n_2} \neq 0$, $\alpha\beta \neq 0$. *Suppose* $m_1\alpha = c_1n_1\beta$ $(c_1 > 1)$ *and* $m_2\alpha = c_2n_2\beta$ $(c_2 > 1)$. *Then* (3) *has no nontrivial subnormal solution, and every nontrivial solution* f *satisfies* $\sigma_2(f) = 1$.

For higher order differential equations, Chen-Shon [5] and Liu-Yang [15] improved the Theorems 3, 4 to higher periodic differential equation

$$f^{(k)} + \left[P_{k-1}(e^z) + Q_{k-1}\left(e^{-z}\right)\right] f^{(k-1)} + \cdots + \left[P_0(e^z) + Q_0\left(e^{-z}\right)\right] f = 0 \quad (4)$$

and they proved the following results.

Theorem 8 ([15, 5])**.** *Let* $P_j(z), Q_j(z)$ $(j = 0, \ldots, k-1)$ *be polynomials in* z *with* $\deg P_j = m_j$, $\deg Q_j = n_j$. *If* P_0 *satisfies*

$$m_0 > \max\{m_j : 1 \leq j \leq k-1\} = m$$

or Q_0 *satisfies*

$$n_0 > \max\{n_j : 1 \leq j \leq k-1\} = n,$$

then (4) *has no nontrivial subnormal solution, and every solution of* (4) *is of hyper-order* $\sigma_2(f) = 1$.

Theorem 9 ([5])**.** *Let* $P_j(z), Q_j(z)$ $(j = 0, \ldots, k-1)$ *be polynomials in* z *with* $\deg P_j = m_j$, $\deg Q_j = n_j$, *and* $P_0 + Q_0 \not\equiv 0$. *If there exists* m_s, n_d $(s, d \in \{0, \ldots, k-1\})$ *satisfying both inequalities*

$$\begin{aligned} m_s &> \max\{m_j : j = 0, \ldots, s-1, s+1, \ldots, k-1\} = m, \\ n_d &> \max\{n_j : j = 0, \ldots, d-1, d+1, \ldots, k-1\} = n, \end{aligned}$$

then (4) *has no nontrivial subnormal solution, and every solution of* (4) *is of hyper-order* $\sigma_2(f) = 1$.

2 Main results

The main purpose of this article is to answer the following question.

Question. Can Theorems 6, 7 be generalized to higher order differential equation? We will prove the following results.

Theorem 10. *Let*

$$f^{(k)}+\left[P_{k-1}\left(e^{\alpha_{k-1}z}\right)+Q_{k-1}\left(e^{-\alpha_{k-1}z}\right)\right]f^{(k-1)}+\cdots+\left[P_0\left(e^{\alpha_0 z}\right)+Q_0\left(e^{-\alpha_0 z}\right)\right]f=0, \tag{5}$$

where

$$\begin{aligned} P_j(z) &= a_{jm_j}z^{m_j}+a_{j(m_j-1)}z^{m_j-1}+\cdots+a_{j0},\ j=0,\dots,k-1,\\ Q_j(z) &= b_{jn_j}z^{n_j}+b_{j(n_j-1)}z^{n_j-1}+\cdots+b_{j0},\ j=0,\dots,k-1 \end{aligned}$$

and $m_j\geq 1, n_j\geq 1$ $(j=0,\dots,k-1; k\geq 2)$ *are integers,* $a_{ju}\neq 0, b_{jv}\neq 0$ *and* $\alpha_j\neq 0$ $(j=0,\dots,k-1;\ u=0,\dots,m_j;\ v=0,\dots,n_j)$ *are complex constants. Suppose that*

$$\begin{cases} c_j m_0\alpha_0 = m_j\alpha_j,\ 0<c_j<1, \forall j=1,\dots,k-1\\ \text{or}\\ d_j n_0\alpha_0 = n_j\alpha_j,\ 0<d_j<1, \forall j=1,\dots,k-1, \end{cases}$$

then equation (5) *has no nontrivial subnormal solution, and every solution of* (5) *satisfies* $\sigma_2(f)=1$.

Theorem 11. *Let*

$$f^{(k)}+\left[P_{k-1}\left(e^{\alpha_{k-1}z}\right)+Q_{k-1}\left(e^{-\alpha_{k-1}z}\right)\right]f^{(k-1)}+\cdots+\left[P_0\left(e^{\alpha_0 z}\right)+Q_0\left(e^{-\alpha_0 z}\right)\right]f=0, \tag{6}$$

where

$$\begin{aligned} P_j(z) &= a_{jm_j}z^{m_j}+a_{j(m_j-1)}z^{m_j-1}+\cdots+a_{j0},\ j=0,\dots,k-1,\\ Q_j(z) &= b_{jn_j}z^{n_j}+b_{j(n_j-1)}z^{n_j-1}+\cdots+b_{j0},\ j=0,\dots,k-1 \end{aligned}$$

and $m_j\geq 1, n_j\geq 1$ $(j=0,\dots,k-1; k\geq 2)$ *are integers,* $a_{ju}\neq 0, b_{jv}\neq 0$ *and* $\alpha_j\neq 0$ $(j=0,\dots,k-1;\ u=0,\dots,m_j;\ v=0,\dots,n_j)$ *are complex constants such that* $P_0\left(e^{\alpha_0 z}\right)+Q_0\left(e^{-\alpha_0 z}\right)\not\equiv 0$. *If there exists* $s,t\in\{0,\dots,k-1\}$ *such that*

$$\begin{cases} m_s\alpha_s = c_j m_j\alpha_j,\ c_j>1,\ j=0,\dots,s-1,s+1,\dots,k-1,\\ \text{and}\\ n_t\alpha_t = d_j n_j\alpha_j,\ d_j>1,\ j=0,\dots,t-1,t+1,\dots,k-1, \end{cases}$$

then equation (6) *has no nontrivial subnormal solution, and every solution of* (6) *satisfies* $\sigma_2(f)=1$.

As a generalization of higher order equations of Theorem 1.5 and Theorem 1.6 in [17], we have the following results.

Theorem 12. *Let*

$$P_j\left(e^{\alpha_j z}\right) = a_{jm_j} e^{m_j \alpha_j z} + a_{j(m_j-1)} e^{(m_j-1)\alpha_j z} + \cdots + a_{j0},\ j = 0, \ldots, k-1,$$

where $m_j \geq 1$ $(j = 0, \ldots, k-1; k \geq 2)$ *are integers,* $a_{ju} \neq 0$ *and* $\alpha_j \neq 0$ $(j = 0, \ldots, k-1;\ u = 0, \ldots, m_j)$ *are complex constants. Suppose that* $c_j m_0 \alpha_0 = m_j \alpha_j$, $0 < c_j < 1$, $\forall j = 1, \ldots, k-1$. *Then equation*

$$f^{(k)} + P_{k-1}\left(e^{\alpha_{k-1} z}\right) f^{(k-1)} + \cdots + P_0\left(e^{\alpha_0 z}\right) f = 0 \tag{7}$$

has no nontrivial subnormal solution, and every solution satisfies $\sigma_2(f) = 1$.

Theorem 13. *Let*

$$P_j^*\left(e^{\alpha_j z}\right) = a_{jm_j} e^{m_j \alpha_j z} + a_{j(m_j-1)} e^{(m_j-1)\alpha_j z} + \cdots + a_{j1} e^{\alpha_j z},\quad j = 0, \ldots, k-1,$$

where $m_j \geq 1$ $(j = 1, \ldots, k-1; k \geq 2)$ *are integers,* $a_{ju} \neq 0$ *and* $\alpha_j \neq 0$ $(j = 0, \ldots, k-1;\ u = 0, \ldots, m_j)$ *are complex constants. Suppose that* $P_0\left(e^{\alpha_0 z}\right) \not\equiv 0$ *and there exists* $s \in \{1, \ldots, k-1\}$ *such that* $c_j m_s \alpha_s = m_j \alpha_j$, $0 < c_j < 1, \forall j = 0, \ldots, s-1, s+1, \ldots, k-1$. *Then the equation*

$$f^{(k)} + P_{k-1}^*\left(e^{\alpha_{k-1} z}\right) f^{(k-1)} + \cdots + P_0^*\left(e^{\alpha_0 z}\right) f = 0 \tag{8}$$

has no nontrivial subnormal solution, and every solution satisfies $\sigma_2(f) = 1$.

In [15], Liu-Yang gave an example that shows that in Theorem 8, if there exists $\deg P_i = \deg P_j$ and $\deg Q_i = \deg Q_j$ $(i \neq j)$, then equation (4) may have a nontrivial subnormal solution.

Example ([15, page 610])**.** A subnormal solution $f = e^{-z}$ satisfies the following equation

$$f^{(n)} + f^{(n-1)} + \cdots + f'' + \left(e^{2z} + e^{-2z}\right) f' + \left(e^{2z} + e^{-2z}\right) f = 0,$$

where n is an odd number.

Question. What can we say when $\deg P_0 = \deg P_1$ and $\deg Q_0 = \deg Q_1$ in equation (4)? We have the following result.

Theorem 14. *Let* $P_j(z), Q_j(z)$ $(j = 0, \ldots, k-1)$ *be polynomials in* z *with* $\deg P_0 = \deg P_1 = m, \deg Q_0 = \deg Q_1 = n$, $\deg P_j = m_j$, $\deg Q_j = n_j$ $(j = 2, \ldots, k-1)$, *let*

$$\begin{aligned} P_1(z) &= a_m z^m + a_{m-1} z^{m-1} + \cdots + a_0, \\ P_0(z) &= b_m z^m + b_{m-1} z^{m-1} + \cdots + b_0, \\ Q_1(z) &= c_n z^n + c_{n-1} z^{n-1} + \cdots + c_0, \\ Q_0(z) &= d_n z^n + d_{n-1} z^{n-1} + \cdots + d_0, \end{aligned}$$

where a_u, b_u, c_v, d_v $(u = 0, \ldots, m; v = 0, \ldots, n)$ *are complex constants,* $a_m b_m c_n d_n \neq 0$. *If* $a_m d_n = b_m c_n$, $m > \max\{m_j : j = 2, \ldots, k-1\}$, $n > \max\{n_j : j = 2, \ldots, k-1\}$ *and* $e^{-(b_m/a_m)z}$ *is not a solution of* (4), *then equation* (4) *has no nontrivial subnormal solution, and every solution* f *of* (4) *satisfies* $\sigma_2(f) = 1$.

Example. This example shows that Theorem 14, is not a particular case and it is different from Theorems 8, 9. Consider the differential equation

$$f''' + \left(e^{z} + e^{-z}\right) f'' + \left(e^{3z} - e^{-2z}\right) f' + \left(-2e^{3z} + 2e^{-2z}\right) f = 0.$$

By Theorems 8, 9, we can't say anything about the existence or nonexistence of nontrivial subnormal solutions, because neither hypotheses of Theorem 8 or of Theorem 9 are satisfied. But, we can see that all hypotheses of Theorem 14 are satisfied, then we guarantee that the above equation has no nontrivial subnormal solution. In fact, we have $k = 3$, $P_2(e^z) = e^z, Q_2(e^{-z}) = e^{-z}, P_1(e^z) = e^{3z}, Q_1(e^{-z}) = -e^{-2z}, P_0(e^z) = -2e^{3z}$ and $Q_0(e^{-z}) = 2e^{-2z}$, $m = 3$, $n = 2$, $m > 1 = \deg P_2$, $n > 1 = \deg Q_2$, $a_m = 1, b_m = -2, c_n = -1$ and $d_n = 2$, and we have $a_m d_n = b_m c_n$. It's clear that $e^{-(b_m/a_m)z} = e^{2z}$ is not a solution of the equation above.

Remark 1. *In Theorem 14, if the equation* (4) *accepts* $e^{-(b_m/a_m)z}$ *as a solution, then* (4) *has a subnormal solution. But, if* $e^{-(b_m/a_m)z}$ *doesn't satisfy* (4), *is there another subnormal solution may that satisfy* (4)? *The conditions of Theorem 14 guarantee that, if* (4) *doesn't accept* $e^{-(b_m/a_m)z}$ *as a subnormal solution, then* (4) *doesn't accept any other subnormal solution.*

Remark 2. *In Theorem 14, we can replace the condition " $e^{-(b_m/a_m)z}$ is not a solution of* (4) *" by many partial conditions. For example*

1. $P_j(0) + Q_j(0) = 0,\ (j = 0, \ldots, k-1)$.
2. $P_j(0) + Q_j(0) = 1,\ (j = 0, \ldots, k-1)$ *and* $a_m \neq b_m$.
3. $P_j(0) + Q_j(0) = 1,\ (j = 0, \ldots, k-1)$, $a_m = b_m$ *and* k *is an even number.*
4. $P_j(0) + Q_j(0) = 0, P_l(0) + Q_l(0) = 1\ (j = 0, \ldots, s;\ l = s+1, \ldots, k-1)$, $a_m = b_m$ *and* s, k *are both even or both odd. And so on.*

Remark 3. *In Theorem 5, the hypotheses (1)-(3) can be replaced by the condition "*$e^{-(b_n/a_n)z}$ *is not a solution of (2)".*

3 Some lemmas

Lemma 1 ([18, page 82]). *Let* $f_j(z)$ $(j = 1, \ldots, n)$ *be meromorphic functions, and* $g_j(z)$ $(j = 1, \ldots, n)$ *be entire functions satisfying*

1. $\sum_{j=0}^{n} f_j(z) e^{g_j(z)} \equiv 0$.
2. *when* $1 \leq j < k \leq n$, *then* $g_j(z) - g_k(z)$ *is not constant.*

3. when $1 \leq j \leq n$ *and* $1 \leq h < k \leq n$*, then*

$$T(r, f_j) = o\{T(r, e^{g_h - g_k})\}, \quad r \to \infty, r \notin E,$$

where $E \subset (1, +\infty)$ *is of finite linear measure or finite logarithmic measure.*

Then $f_j(z) \equiv 0 \quad (j = 1, \dots, n)$.

In [5], Chen-Shon proved [5, Lemma 2] that (4) has no polynomial solution under the hypotheses of Theorem 9. We will prove a similar result for the general case under one condition that factor f in (4) is not identically zero. Chen-Shon used the Lemma 1 in their proof, to get a contradiction in case that f is polynomial with $\deg(f) < s \leq d$. We use the same method but for all equations of the form (4), just with condition $P_0(e^z) + Q_0(e^{-z}) \not\equiv 0$. We will prove.

Lemma 2. *Let* $P_j(z), Q_j(z)$ $(j = 0, \dots, k-1)$ *be polynomials in* z *with* $\deg P_j = m_j$, $\deg Q_j = n_j$. *If* $P_0(e^z) + Q_0(e^{-z}) \not\equiv 0$, *then every solution of the equation*

$$f^{(k)} + \left[P_{k-1}(e^z) + Q_{k-1}\left(e^{-z}\right)\right] f^{(k-1)} + \cdots + \left[P_0(e^z) + Q_0\left(e^{-z}\right)\right] f = 0 \qquad (9)$$

is transcendental.

Proof. It's well known that every solution of the equation (9) is an entire function. $f \equiv 0$, is trivial solution. Since $P_0(e^z) + Q_0(e^{-z}) \not\equiv 0$, then f can't be a constant. Now, suppose that f is a nonconstant polynomial solution of (9). Let

$$P_j(e^z) + Q_j\left(e^{-z}\right) = \sum_{p=1}^{m_j} a_{jp} e^{pz} + c_j + \sum_{q=1}^{n_j} b_{jq} e^{-qz}, \qquad (10)$$

where a_{jp}, b_{jq} and c_j $(j = 0, \dots, k-1; p = 1, \dots, m_j$ and $q = 1, \dots, n_j)$ are complex constants. $m_j \geq 1, n_j \geq 1$ are integers and $a_{jm_j} b_{jn_j} \neq 0$, for all $j = 0, \dots, k-1$. Set $m = \max\{m_j : j = 0, \dots, k-1\}$ and $n = \max\{n_j : j = 0, \dots, k-1\}$. Then we can rewrite (10) as

$$P_j(e^z) + Q_j\left(e^{-z}\right) = \sum_{p=m_j+1}^{m} a_{jp} e^{pz} + \sum_{p=1}^{m_j} a_{jp} e^{pz} + c_j + \sum_{q=1}^{n_j} b_{jq} e^{-qz} + \sum_{q=n_j+1}^{n} b_{jq} e^{-qz}, \qquad (11)$$

where $a_{jp} = 0$, $(p = m_j + 1, \dots, m)$ and $b_{jq} = 0$, $(q = n_j + 1, \dots, n)$. By (9) and (11), we obtain

$$\sum_{p=1}^{m} A_p e^{pz} + C e^0 + \sum_{q=1}^{n} B_q e^{-qz} = 0, \qquad (12)$$

where

$$\begin{aligned} A_p &= \sum_{j=0}^{k-1} a_{jp} f^{(j)}, \ \ (p = 1, \dots, m), \\ B_q &= \sum_{j=0}^{k-1} b_{jq} f^{(j)}, \ \ (q = 1, \dots, n), \\ C &= f^{(k)} + \sum_{j=0}^{k-1} c_j f^{(j)}. \end{aligned} \tag{13}$$

Since f is polynomial, then A_p, B_q and C are also polynomial. And

$$\begin{aligned} T(r, A_p) &= o\{T(r, e^{(\alpha-\beta)z})\}, \ \ p = 1, \dots, m, \\ T(r, B_q) &= o\{T(r, e^{(\alpha-\beta)z})\}, \ q = 1, \dots, n, \\ T(r, C) &= o\{T(r, e^{(\alpha-\beta)z})\}, \end{aligned} \tag{14}$$

where $-n \le \beta < \alpha \le m$. By Lemma 1, (12) and (14), we obtain

$$A_p(z) \equiv 0 \ (p = 1, \dots, m), \ B_q(z) \equiv 0 \ (q = 1, \dots, n) \text{ and } C(z) \equiv 0. \tag{15}$$

Since $\deg f > \deg f' > \cdots > \deg f^{(k-1)} > \deg f^{(k)}$, then by (13) and (15), we see that

$$a_{0m} = \cdots = a_{01} = c_0 = b_{01} = \cdots = b_{0n} = 0.$$

Thus $P_0(e^z) + Q_0(e^{-z}) \equiv 0$, and this contradicts the assumption $P_0(e^z) + Q_0(e^{-z}) \not\equiv 0$. Therefore, every solution of (9) must be a transcendental entire function. □

Lemma 3 ([5, 1])**.** *Let $A_0, A_1, \dots, A_{k-1}$ be entire functions of finite order. If $f(z)$ is a solution of the equation*

$$f^{(k)} + A_{k-1} f^{(k-1)} + \cdots + A_1 f' + A_0 f = 0,$$

then $\sigma_2(f) \le \max\{\sigma(A_j) : j = 0, \dots, k-1\}$.

Lemma 4 ([9])**.** *Let f be a transcendental meromorphic function, and $\alpha > 1$ be a given constant. Then there exists a set $E \subset (1, \infty)$ with finite logarithmic measure and a constant $B > 0$ that depends only on α and $i, j (0 \le i < j)$, such that for all z satisfying $|z| = r \notin E \cup [0, 1]$*

$$\left| \frac{f^{(j)}(z)}{f^{(i)}(z)} \right| \le B \left[\frac{T(\alpha r, f)}{r} (\log^\alpha r) \log T(\alpha r, f) \right]^{j-i}.$$

Lemma 5 ([9])**.** *Let $f(z)$ be a transcendental meromorphic function with $\sigma(f) = \sigma < +\infty$. Let $H = \{(k_1, j_1), \dots, (k_q, j_q)\}$ be a finite set of distinct pairs of integers that satisfy $k_i > j_i \ge 0$, for $i = 1, \dots, q$. And let $\varepsilon > 0$ be a given constant.*

Then there exists a set $E \in [0, 2\pi)$ that has linear measure zero, such that if $\psi \in [0, 2\pi) \backslash E$, then there is a constant $R_0 = R_0(\psi) > 1$ such that for all z satisfying $\arg z = \psi$ and $|z| = r \geq R_0$ and for all $(k, j) \in H$, we have

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(\sigma - 1 + \varepsilon)(k - j)}.$$

Lemma 6 ([10, 13])**.** *Let $f(z)$ be an entire function and suppose that $|f^{(k)}(z)|$ is unbounded on some ray $\arg z = \theta$. Then, there exists an infinite sequence of points $z_n = r_n e^{i\theta}$ $(n = 1, 2, \dots)$, where $r_n \to +\infty$, such that $f^{(k)}(z_n) \to \infty$ and*

$$\left| \frac{f^{(j)}(z_n)}{f^{(k)}(z_n)} \right| \leq \frac{1}{(k - j)!} |z_n|^{k - j}(1 + o(1)), \quad (j = 0, \dots, k - 1).$$

Lemma 7 ([2])**.** *Let f be an entire function with $\sigma(f) = \sigma < +\infty$. Suppose there exists a set $E \cup [0, 2\pi)$ that has linear measure zero, such that for any ray $\arg z = \theta_0 \in [0, 2\pi) \backslash E$ and for sufficiently large r, we have*

$$\left| f(re^{i\theta_0}) \right| \leq M r^k,$$

where $M = M(\theta_0) > 0$ is a constant and $k > 0$ is a constant independent of θ_0, then f is a polynomial with $\deg f \leq k$.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function, $\mu_f(r)$ be the maximum term, i.e., $\mu_f(r) = \max\{|a_n| r^n; n = 0, 1, \cdots\}$, and let $\nu_f(r)$ be the central index of f, i.e., $\nu_f(r) = \max\{m; \mu_f(r) = |a_m| r^m\}$.

Lemma 8 ([6])**.** *Let f be an entire function of infinite order with $\sigma_2(f) = \alpha$ $(0 \leq \alpha < \infty)$ and a set $E \subset [1, +\infty)$ have finite logarithmic measure. Then there exists $\{z_k = r_k e^{i\theta_k}\}$ such that $|f(z_k)| = M(r_k, f)$, $\theta_k \in [0, 2\pi)$, $\lim_{k \to \infty} \theta_k = \theta_0 \in [0, 2\pi)$, $r_k \notin E$, $r_k \to \infty$, and such that*

1. *if $\sigma_2(f) = \alpha$ $(0 < \alpha < \infty)$, then for any given ε_1 $(0 < \varepsilon_1 < \alpha)$,*

$$\exp\{r_k^{\alpha - \varepsilon_1}\} < \nu_f(r_k) < \exp\{r_k^{\alpha + \varepsilon_1}\},$$

2. *if $\sigma(f) = \infty$ and $\sigma_2(f) = 0$, then for any given ε_2 $(0 < \varepsilon_2 < \frac{1}{2})$ and for any large $M > 0$, we have as r_k sufficiently large*

$$r_k^M < \nu_f(r_k) < \exp\{r_k^{\varepsilon_2}\}.$$

Lemma 9 ([12])**.** *Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ be a polynomial with $a_n \neq 0$. Then, for every $\varepsilon > 0$, there exists $r_0 > 0$ such that for all $r = |z| > r_0$ we have the inequalities*

$$(1 - \varepsilon)|a_n| r^n \leq |P(z)| \leq (1 + \varepsilon)|a_n| r^n.$$

Lemma 10 ([3]). *Consider $h(z)e^{az}$ where h is a nonzero entire function with $\sigma(h) = \alpha < 1$, $a = de^{i\varphi}$ $(d > 0, \varphi \in [0, 2\pi))$. Set $E_0 = \{\theta \in [0, 2\pi) : \cos(\varphi + \theta) = 0\}$. Then for any given ε $(0 < \varepsilon < 1 - \alpha)$, there is a set $E \subset [0, 2\pi)$ that has linear measure zero, if $z = re^{i\theta}$, $\theta \in [0, 2\pi)\backslash(E \cup E_0)$, we have as r sufficiently large*

1. *if* $\cos(\varphi + \theta) > 0$, *then*

$$\exp\{(1-\varepsilon)dr\cos(\varphi+\theta)\} \le |h(z)e^{az}| \le \exp\{(1+\varepsilon)dr\cos(\varphi+\theta)\},$$

2. *if* $\cos(\varphi + \theta) < 0$, *then*

$$\exp\{(1+\varepsilon)dr\cos(\varphi+\theta)\} \le |h(z)e^{az}| \le \exp\{(1-\varepsilon)dr\cos(\varphi+\theta)\}.$$

Let $P(z) = (a + ib)z^n + \cdots$ be a polynomial with degree $n \ge 1$, and $z = re^{i\theta}$. We denote $\delta(P, \theta) := a\cos(n\theta) - b\sin(n\theta)$.

Remark 4. *By definitions of P_j, Q_j $(j = 0, \dots, k-1)$ in Theorem 10 and Theorem 11, by Lemma 9 and Lemma 10, we can obtain that for all $z = re^{i\theta}$, $\theta \in [0, 2\pi)\backslash(E \cup E_0)$*

$$\left|P_j\left(e^{\alpha_j z}\right) + Q_j\left(e^{-\alpha_j z}\right)\right| = \begin{cases} \left|a_{jm_j}\right| e^{m_j\delta(\alpha_j z,\theta)r}(1+o(1)), & (\delta(\alpha_j z,\theta) > 0; r \to +\infty) \\ \left|b_{jn_j}\right| e^{-n_j\delta(\alpha_j z,\theta)r}(1+o(1)), & (\delta(\alpha_j z,\theta) < 0; r \to +\infty) \end{cases}$$

4 Proof of Theorem 10

Proof. **(1)** Suppose that f is a nontrivial solution of (5). Then f is an entire function. Since $P_0\left(e^{\alpha_0 z}\right) + Q_0\left(e^{-\alpha_0 z}\right) \not\equiv 0$, then every nonzero constant is not a solution of (5). Now, suppose that $f_0 = a_n z^n + \cdots + a_0$ $(n \ge 1;\ a_0, \dots, a_n$ are constants, $a_n \ne 0)$ is a polynomial solution of (5). Let $E_0 = \{\theta \in [0, 2\pi) : \delta(\alpha_0 z, \theta) = 0\}$, E_0 is a finite set. Take $z = re^{i\theta}, \theta \in [0, 2\pi)\backslash(E_0 \cup E)$ with E some set with linear measure zero. If $c_j m_0 \alpha_0 = m_j \alpha_j$, $(0 < c_j < 1, \forall j = 1, \dots, k-1)$, then we choose $\theta \in [0, 2\pi)\backslash(E_0 \cup E)$ such $\delta(\alpha_0 z, \theta) = |\alpha_0|\cos(\arg\alpha_0 + \theta) > 0$, then $\delta(\alpha_j z, \theta) = \dfrac{c_j}{m_j} m_0 \delta(\alpha_0 z, \theta) > 0, (\forall j = 1, \dots, k-1)$. By Lemma 9, Lemma 10 and (5) for a sufficiently large r, we have

$$\begin{aligned} |a_n|\left|a_{0m_0}\right| e^{m_0\delta(\alpha_0 z,\theta)r} r^n(1+o(1)) &= \left|P_0\left(e^{\alpha_0 z}\right) + Q_0\left(e^{-\alpha_0 z}\right)\right| |f_0| \\ &\le \left|f_0^{(k)}\right| + \sum_{j=1}^{k-1}\left|P_j\left(e^{\alpha_j z}\right) + Q_j\left(e^{-\alpha_j z}\right)\right|\left|f_0^{(j)}\right| \\ &\le M e^{cm_0\delta(\alpha_0 z,\theta)r} r^n(1+o(1)), \end{aligned}$$

where $0 < c = \max\{c_j : j = 1, \dots, k-1\} < 1$. This is a contradiction. Then (5) has no nonzero polynomial solution. If $d_j n_0 \alpha_0 = n_j \alpha_j$, $(0 < d_j < 1, \forall j = 1, \dots, k-1)$, then we choose $\theta \in [0, 2\pi)\backslash(E_0 \cup E)$, such that $\delta(\alpha_0 z, \theta) = |\alpha_0|\cos(\arg\alpha_0 + \theta) < 0$, then $\delta(\alpha_j z,\theta) = \dfrac{d_j}{n_j} n_0 \delta(\alpha_0 z,\theta) < 0, (\forall j = 1,\ldots,k-1)$. Using the similar method as in case $\delta(\alpha_0 z,\theta) > 0$, we obtain

$$|a_n|\,|b_{0n_0}|\,e^{-n_0\delta(\alpha_0 z,\theta)r} r^n (1+o(1)) \leq M e^{-dn_0\delta(\alpha_0 z,\theta)r} r^n (1+o(1)),$$

where $0 < d = \max\{d_j : j = 1,\ldots,k-1\} < 1$. This is a contradiction. So, (5) has no nonzero polynomial solution.
(2) By Lemma 4 we can see that there exists a set $E \subset (1,\infty)$ with finite logarithmic measure and there is a constant $B > 0$ such that for all z satisfying $|z| = r \notin E \cup [0,1]$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq B\,[T(2r,f)]^{j+1}\,,\ j = 1,\ldots,k. \tag{16}$$

Suppose that $f \not\equiv 0$ is a subnormal solution, then $\sigma_e(f) = 0$. Hence, for all $\varepsilon > 0$ and for sufficiently large r, we have

$$T(r,f) < e^{\varepsilon r}. \tag{17}$$

Substituting (17) into (16) with sufficiently large $|z| = r \notin E \cup [0,1]$, we obtain

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq B e^{2\varepsilon(j+1)r} \leq B e^{2\varepsilon(k+1)r},\ j = 1,\ldots,k. \tag{18}$$

(i) Suppose that $c_j m_0 \alpha_0 = m_j \alpha_j$, $(0 < c_j < 1, \forall j = 1,\ldots,k-1)$. Take $z = re^{i\theta}$ such that $r \notin E \cup [0,1]$ and $\delta(\alpha_0 z,\theta) = |\alpha_0|\cos(\arg\alpha_0 + \theta) > 0$, then $\delta(\alpha_j z,\theta) = \dfrac{c_j}{m_j} m_0 \delta(\alpha_0 z,\theta) > 0, (\forall j = 1,\ldots,k-1)$. Therefore

$$\left|P_0\left(e^{\alpha_0 z}\right) + Q_0\left(e^{-\alpha_0 z}\right)\right| = |a_{0m_0}|\,e^{m_0\delta(\alpha_0 z,\theta)r}(1+o(1)), \tag{19}$$

$$\begin{aligned}\left|P_j\left(e^{\alpha_j z}\right) + Q_j\left(e^{-\alpha_j z}\right)\right| &= \left|a_{jm_j}\right| e^{m_j\delta(\alpha_j z,\theta)r}(1+o(1)) \\ &= \left|a_{jm_j}\right| e^{c_j m_0\delta(\alpha_0 z,\theta)r}(1+o(1)) \\ &\leq D e^{cm_0\delta(\alpha_0 z,\theta)r}(1+o(1)),\end{aligned} \tag{20}$$

where $D = \max\limits_{1\leq j\leq k-1}\{\left|a_{jm_j}\right|\}$ and $0 < c = \max\limits_{1\leq j\leq k-1}\{|c_j|\} < 1$. Substituting (18), (19) and (20) into (5), we obtain

$$\begin{aligned}|a_{0m_0}|\,e^{m_0\delta(\alpha_0 z,\theta)r}(1+o(1)) &= \left|P_0\left(e^{\alpha_0 z}\right) + Q_0\left(e^{-\alpha_0 z}\right)\right| \\ &\leq \left|\frac{f^{(k)}}{f}\right| + \sum_{j=1}^{k-1}\left|P_j\left(e^{\alpha_j z}\right) + Q_j\left(e^{-\alpha_j z}\right)\right|\left|\frac{f^{(j)}}{f}\right| \\ &\leq Be^{2\varepsilon(k+1)r} + (k-1)DBe^{cm_0\delta(\alpha_0 z,\theta)r}e^{2\varepsilon(k+1)r}(1+o(1)).\end{aligned}$$

Hence,

$$|a_{0m_0}|\, e^{m_0\delta(\alpha_0 z,\theta)r}(1+o(1)) \le Me^{[cm_0\delta(\alpha_0 z,\theta)+2\varepsilon(k+1)]r}(1+o(1)) \tag{21}$$

for some constant $M > 0$. Since $0 < c < 1$, then we can see that (21) is a contradiction when

$$0 < \varepsilon < \frac{1-c}{2(k+1)} m_0\delta(\alpha_0 z,\theta).$$

Hence, equation (5) has no nontrivial subnormal solution.

(ii) Suppose that $d_j n_0\alpha_0 = n_j\alpha_j$, $(0 < d_j < 1, \forall j = 1,\dots,k-1)$. We choose $z = re^{i\theta}$, such that $r \notin E \cup [0,1]$ and $\delta(\alpha_0 z,\theta) = |\alpha_0|\cos(\arg\alpha_0+\theta) < 0$, then $\delta(\alpha_j z,\theta) = \dfrac{d_j}{n_j} n_0\delta(\alpha_0 z,\theta) < 0$, $(\forall j = 1,\dots,k-1)$. Using the similar method as in the proof of (i) above, we obtain

$$|b_{0n_0}|\, e^{-n_0\delta(\alpha_0 z,\theta)r}(1+o(1)) \le Me^{[-dn_0\delta(\alpha_0 z,\theta)+2\varepsilon(k+1)]r}(1+o(1)), \tag{22}$$

where $0 < d = \max\limits_{1\le j\le k-1}\{|d_j|\} < 1$, and for some constant $M > 0$. We see that (22) is a contradiction when

$$0 < \varepsilon < -\frac{1-d}{2(k+1)} n_0\delta(\alpha_0 z,\theta).$$

Hence, (5) has no nontrivial subnormal solution.

(3) By Lemma 3, every solution f of (5) satisfies $\sigma_2(f) \le 1$. Suppose that $\sigma_2(f) < 1$. Then $\sigma_e(f) = 0$, i.e., f is subnormal solution and this contradicts the conclusion above. So $\sigma_2(f) = 1$. □

5 Proof of Theorem 11

Proof. Suppose that $f \not\equiv 0$ is a solution of equation (6). Then f is an entire function. Since $P_0\left(e^{\alpha_0 z}\right) + Q_0\left(e^{-\alpha_0 z}\right) \not\equiv 0$, then f cannot be nonzero constant.

(1) We will prove that f is a transcendental function. We assume that f is a polynomial solution to (6), and we set

$$f(z) = a_n z^n + \cdots + a_0,$$

where $n \ge 1$, $a_0,\dots,a_n$ are constants with $a_n \ne 0$. Suppose that $s \le t$. Since $P_0\left(e^{\alpha_0 z}\right) + Q_0\left(e^{-\alpha_0 z}\right) \not\equiv 0$, then we can rewrite (6) as

$$f(z) = -\sum_{j=1}^{n} \frac{P_j(e^{\alpha_j z}) + Q_j(e^{-\alpha j z})}{P_0(e^{\alpha_0 z}) + Q_0(e^{-\alpha_0 z})} f^{(j)}(z) \tag{23}$$

which is a contradiction since the left side of equation (23) is a polynomial function but the right side is a transcendental function, and even in case

$$\frac{P_j(e^{\alpha_j z}) + Q_j(e^{-\alpha j z})}{P_0(e^{\alpha_0 z}) + Q_0(e^{-\alpha_0 z})} = K_j,\ \forall j = 1,\cdots,n,$$

where K_j, $\forall j = 1, \cdots, n$ are complex constants, we obtain $a_n = 0$, and this also contradicts the assumption $a_n \neq 0$. Hence, every solution of (6) is transcendental.

(2) Now, we will prove that every solution f of (6) satisfies $\sigma(f) = +\infty$. We assume that $\sigma(f) = \sigma < +\infty$. By Lemma 5, we know that for any given $\varepsilon > 0$ there exists a set $E \subset [0, 2\pi)$ that has linear measure zero, and for each $\psi \in [0, 2\pi)\backslash E$, there is a constant $R_0 = R_0(\psi) > 1$ such that for all z satisfying $\arg z = \psi$ and $|z| = r \geq R_0$, we have for $l \leq k - 1$

$$\left|\frac{f^{(j)}(z)}{f^{(l)}(z)}\right| \leq |z|^{(\sigma-1+\varepsilon)(j-l)}; \quad j = l+1, \ldots, k. \tag{24}$$

Let $H = \{\theta \in [0, 2\pi) : \delta(\alpha_s z, \theta) = 0\}$, H is a finite set. By the hypotheses of Theorem 11, we have $H = \{\theta \in [0, 2\pi) : \delta(\alpha_j z, \theta) = 0, (j = 0, \ldots, k-1)\}$. We take $z = re^{i\theta}$, such that $\theta \in [0, 2\pi)\backslash E \cup H$. Then $\delta(\alpha_s z, \theta) > 0$ or $\delta(\alpha_s z, \theta) < 0$. If $\delta(\alpha_s z, \theta) > 0$, then $\delta(\alpha_j z, \theta) > 0$ for all $j = 0, \ldots, s-1, s+1, \ldots, k-1$. We assert that $\left|f^{(s)}(z)\right|$ is bounded on the ray $\arg z = \theta$. If $\left|f^{(s)}(z)\right|$ is unbounded, then by Lemma 6, there exists an infinite sequence of points $z_u = r_u e^{i\theta}$ $(u = 1, 2, \ldots)$ where $r_u \to +\infty$ such that $f^{(s)}(z_u) \to \infty$ and

$$\left|\frac{f^{(j)}(z_u)}{f^{(s)}(z_u)}\right| \leq \frac{1}{(s-j)!}|z_u|^{s-j}(1+o(1)), \quad (j = 0, \ldots, s-1). \tag{25}$$

By (6) we obtain

$$\begin{aligned}
|a_{sm_s}|\, e^{m_s\delta(\alpha_s z_u,\theta)r_u}(1+o(1)) &= \left|P_s(e^{\alpha_s z_u}) + Q_s(e^{-\alpha_s z_u})\right| \\
&\leq \left|\frac{f^{(k)}(z_u)}{f^{(s)}(z_u)}\right| + \sum_{j=0, j\neq s}^{k-1}\left|P_j(e^{\alpha_j z_u}) + Q_j(e^{-\alpha_j z_u})\right|\left|\frac{f^{(j)}(z_u)}{f^{(s)}(z_u)}\right| \\
&\leq r_u^{(\sigma-1+\varepsilon)(k-s)} + \sum_{j>s}\left|a_{jm_j}\right| e^{m_j\delta(\alpha_j z_u,\theta)r_u} r_u^{(\sigma-1+\varepsilon)(j-s)} \\
&\quad + \sum_{j<s}\frac{1}{(s-j)!}\left|a_{jm_j}\right| e^{m_j\delta(\alpha_j z_u,\theta)r_u} r_u^{s-j}(1+o(1)) \\
&\leq M e^{Cm_s\delta(\alpha_s z_u,\theta)r_u} r_u^{\rho}(1+o(1)),
\end{aligned} \tag{26}$$

for some $M > 0$, where $\rho \geq \max\left\{\max_{s<j\leq k-1}\{(\sigma-1+\varepsilon)(j-s)\}; \max_{0\leq j<s}\{s-j\}\right\}$ $= \max\left\{\max_{s<j\leq k-1}\{(\sigma-1+\varepsilon)(j-s)\}; s\right\}$. Since $0 < C = \max_j\{\frac{1}{c_j}\} < 1$ and $\delta(\alpha_s z_u, \theta) > 0$, then (26) is a contradiction when $r_u \to +\infty$. Hence, $\left|f^{(s)}(z)\right|$ is bounded on the ray $\arg z = \theta$. Therefore, for sufficiently large r, we have

$$\left|f(re^{i\theta})\right| \leq C_1 r^s. \tag{27}$$

If $\delta(\alpha_s z, \theta) < 0$, then $\delta(\alpha_j z, \theta) < 0$ for all $j = 0, \ldots, s-1, s+1, \ldots, k-1$, in particular $\delta(\alpha_t z, \theta) < 0$, i.e., $-n_t\delta(\alpha_t z, \theta) > 0$. We assert that $\left|f^{(t)}(z)\right|$ is bounded on the ray $\arg z = \theta$. If $\left|f^{(t)}(z)\right|$ is unbounded, then by Lemma 6, there exists an infinite sequence of points $z_u = r_u e^{i\theta}$ $(u = 1, 2, \dots)$ where $r_u \to +\infty$ such that $f^{(t)}(z_u) \to \infty$ and

$$\left|\frac{f^{(j)}(z_u)}{f^{(t)}(z_u)}\right| \leq \frac{1}{(t-j)!}|z_u|^{t-j}(1+o(1)),\ (j = 0, \dots, t-1).$$

We obtain

$$|b_{tm_t}|\, e^{-n_t\delta(\alpha_t z_u,\theta)r_u}(1+o(1)) \leq Me^{-Dn_t\delta(\alpha_t z_u,\theta)r_u} r_u^{\rho}(1+o(1)) \tag{28}$$

for some $M > 0$, where $\rho \geq \max\left\{\max_{t<j\leq k-1}\{(\sigma-1+\varepsilon)(j-t)\}\,;\max_{0\leq j<t}\{t-j\}\right\}$ $= \max\left\{\max_{t<j\leq k-1}\{(\sigma-1+\varepsilon)(j-t)\}\,;t\right\}$. Since $0 < D = \max_j\{\frac{1}{d_j}\} < 1$ and $-n_t\delta(\alpha_t z, \theta) > 0$, then we see that (28) is a contradiction when $r_u \to +\infty$. Thus, for sufficiently large r, we have

$$\left|f(re^{i\theta})\right| \leq C_2 r^t. \tag{29}$$

Since the linear measure of $E \cup H$ is zero, by (27), (29) and Lemma 7, we conclude that f is polynomial, which contradicts the fact that f is transcendental. Therefore $\sigma(f) = +\infty$.

(3) Finally, we will prove that (6) has no nontrivial subnormal solution. Suppose that (6) has a subnormal solution f. So, $\sigma(f) = \infty$ and by Lemma 3, we see that $\sigma_2(f) \leq 1$. Set $\sigma_2(f) = \mu \leq 1$. By Lemma 4, there exists a set $E_1 \subset (1, \infty)$ having a finite logarithmic measure, and there is a constant $B > 0$ such that for all z satisfying $|z| = r \notin [0, 1] \cup E_1$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq B\left[T\left(2r, f\right)\right]^{j+1},\ \ j = 1, \dots, k. \tag{30}$$

From Wiman-Valiron theory, there is a set $E_2 \subset (1, \infty)$ having finite logarithmic measure, so we can choose z satisfying $|z| = r \notin E_2$ and $|f(z)| = M(r, f)$. Thus, we have

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z}\right)^j (1+o(1)),\ \ j = 1, \dots, k. \tag{31}$$

By Lemma 8, we can see that there exists a sequence $\{z_n = r_n e^{i\theta_n}\}$ such that $|f(z_n)| = M(r_n, f)$, $\theta_n \in [0, 2\pi)$, $\lim_{n\to\infty}\theta_n = \theta_0 \in [0, 2\pi)$, $r_n \notin [0, 1] \cup E_1 \cup E_2$, $r_n \to \infty$, and such that

1. if $\mu > 0$, then for any given ε_1 $(0 < \varepsilon_1 < \mu)$,

$$\exp\{r_n^{\mu-\varepsilon_1}\} < \nu_f(r_n) < \exp\{r_n^{\mu+\varepsilon_1}\}, \tag{32}$$

2. if $\mu = 0$, and since $\sigma(f) = \infty$, then for any given ε_2 $(0 < \varepsilon_2 < \frac{1}{2})$ and for any large $M > 0$, we have as r_n sufficiently large

$$r_n^M < \nu_f(r_n) < \exp\{r_n^{\varepsilon_2}\}. \tag{33}$$

From (32) and (33), we obtain that

$$\nu_f(r_n) > r_n, \ \ r_n \to \infty. \tag{34}$$

Since θ_0 may belong to $\{\theta \in [0, 2\pi) : \delta(\alpha_s z, \theta) > 0\}$, or $\{\theta \in [0, 2\pi) : \delta(\alpha_s z, \theta) < 0\}$, or $\{\theta \in [0, 2\pi) : \delta(\alpha_s z, \theta) = 0\}$, we divide the proof into three cases.

Case 1. $\theta_0 \in \{\theta \in [0, 2\pi) : \delta(\alpha_s z, \theta) > 0\}$. By $\theta_n \to \theta_0$, there exists $N > 0$ such that, as $n > N$, we have $\delta(\alpha_s z_n, \theta_n) > 0$. Since f is subnormal, then for any given $\varepsilon > 0$, we have

$$T(r, f) \le e^{\varepsilon r}. \tag{35}$$

By (30), (31) and (35), we obtain

$$\left(\frac{\nu_f(r_n)}{r_n}\right)^j (1+o(1)) = \left|\frac{f^{(j)}(z_n)}{f(z_n)}\right| \le B\left[T(2r_n, f)\right]^{k+1} \le Be^{2(k+1)\varepsilon r_n}, \ \ j = 1, \dots, k. \tag{36}$$

Because $\delta(\alpha_s z_n, \theta_n) > 0$, then $\delta(\alpha_j z_n, \theta_n) > 0$ $(j = 0, \dots, s-1, s+1, \dots, k-1)$, and we have

$$\left|P_s(e^{\alpha_s z_n}) + Q_s(e^{-\alpha_s z_n})\right| = |a_{sm_s}|\, e^{m_s \delta(\alpha_s z_n, \theta_n) r_n}(1 + o(1)) \tag{37}$$

and

$$\begin{aligned} \left|P_j(e^{\alpha_j z_n}) + Q_j(e^{-\alpha_j z_n})\right| &= \left|a_{jm_j}\right| e^{m_j \delta(\alpha_j z_n, \theta_n) r_n}(1 + o(1)) \\ &= \left|a_{jm_j}\right| e^{\frac{m_s}{c_j}\delta(\alpha_s z_n, \theta_n) r_n}(1 + o(1)) \\ &\le Me^{Cm_s \delta(\alpha_s z_n, \theta_n) r_n}(1 + o(1)), \ \ j \ne s, \end{aligned} \tag{38}$$

where $M = \max_j\{\left|a_{jm_j}\right|\}$ and $0 < C = \max_j\{\frac{1}{c_j}\} < 1$. We have by (6)

$$\begin{aligned} &\left|P_s(e^{\alpha_s z_n}) + Q_s(e^{-\alpha_s z_n})\right| \left|\frac{f^{(s)}(z_n)}{f(z_n)}\right| \\ &\le \left|\frac{f^{(k)}(z_n)}{f(z_n)}\right| + \sum_{j=0, j\ne s}^{k-1} \left|P_j(e^{\alpha_j z_n}) + Q_j(e^{-\alpha_j z_n})\right| \left|\frac{f^{(j)}(z_n)}{f(z_n)}\right|. \end{aligned}$$

By using Wiman-Valiron theory, we obtain

$$\begin{aligned} &\left|P_s(e^{\alpha_s z_n}) + Q_s(e^{-\alpha_s z_n})\right| \left(\frac{\nu_f(r_n)}{r_n}\right)^s (1 + o(1)) \\ &\le \left(\frac{\nu_f(r_n)}{r_n}\right)^k (1 + o(1)) + \sum_{j=0, j\ne s}^{k-1} \left|P_j(e^{\alpha_j z_n}) + Q_j(e^{-\alpha_j z_n})\right| \left(\frac{\nu_f(r_n)}{r_n}\right)^j (1 + o(1)). \end{aligned}$$

which implies

$$\left|P_s(e^{\alpha_s z_n}) + Q_s(e^{-\alpha_s z_n})\right|(1+o(1)) \le \left(\frac{\nu_f(r_n)}{r_n}\right)^{k-s}(1+o(1)) + \sum_{j=0, j\neq s}^{k-1} \left|P_j(e^{\alpha_j z_n}) + Q_j(e^{-\alpha_j z_n})\right| \left(\frac{\nu_f(r_n)}{r_n}\right)^{j-s}(1+o(1)).$$

By (34) we have

$$\frac{\nu_f(r_n)}{r_n} > 1, \quad r_n \to +\infty$$

then

$$\left|P_s(e^{\alpha_s z_n}) + Q_s(e^{-\alpha_s z_n})\right|(1+o(1)) \le \left(1 + \sum_{j=0, j\neq s}^{k-1} \left|P_j(e^{\alpha_j z_n}) + Q_j(e^{-\alpha_j z_n})\right|\right)\left(\frac{\nu_f(r_n)}{r_n}\right)^{k}(1+o(1))$$

and by (36), (37) and (38) we obtain

$$\begin{aligned} |a_{sm_s}|\, e^{m_s\delta(\alpha_s z_n,\theta_n)r_n}(1+o(1)) &= \left|P_s(e^{\alpha_s z_n}) + Q_s(e^{-\alpha_s z_n})\right|(1+o(1)) \\ &\le \left(1 + \sum_{j=0, j\neq s}^{k-1} \left|P_j(e^{\alpha_j z_n}) + Q_j(e^{-\alpha_j z_n})\right|\right)\left(\frac{\nu_f(r_n)}{r_n}\right)^{k}(1+o(1)) \\ &\le kMBe^{Cm_s\delta(\alpha_s z_n,\theta_n)r_n} e^{2(k+1)\varepsilon r_n}(1+o(1)). \end{aligned} \tag{39}$$

Since $0 < C < 1$ and $\delta(\alpha_s z_n, \theta_n) > 0$, then we can see that (39) is a contradiction when $r_n \to \infty$ and

$$0 < \varepsilon < \frac{1-C}{2(k+1)} m_s \delta(\alpha_s z_n, \theta_n).$$

Case 2. $\theta_0 \in \{\theta \in [0, 2\pi) : \delta(\alpha_s z, \theta) < 0\}$. By $\theta_n \to \theta_0$, there exists $N > 0$ such that, as $n > N$, we have $\delta(\alpha_s z_n, \theta_n) < 0$, then $\delta(\alpha_j z_n, \theta_n) > 0$ $(j = 0, \dots, s-1, s+1, \dots, k-1)$. In particular $\delta(\alpha_t z_n, \theta_n) < 0$, i.e., $-n_t\delta(\alpha_t z_n, \theta_n) > 0$. We have

$$\left|P_t(e^{\alpha_t z_n}) + Q_t(e^{-\alpha_t z_n})\right| = |b_{tn_t}|\, e^{-n_t\delta(\alpha_t z_n,\theta_n)r_n}(1+o(1)) \tag{40}$$

and

$$\begin{aligned} \left|P_j(e^{\alpha_j z_n}) + Q_j(e^{-\alpha_j z_n})\right| &= \left|b_{jn_j}\right| e^{n_j\delta(\alpha_j z_n,\theta_n)r_n}(1+o(1)) \\ &= \left|b_{jn_j}\right| e^{\frac{n_t}{d_j}\delta(\alpha_t z_n,\theta_n)r_n}(1+o(1)) \\ &\le Me^{Dn_t\delta(\alpha_t z_n,\theta_n)r_n}(1+o(1)), \quad j \neq t, \end{aligned} \tag{41}$$

where $M = \max_j\{\left|b_{jn_j}\right|\}$ and $0 < D = \max_j\{\frac{1}{d_j}\} < 1$. By the same way used to obtain (39) we deduce that, after (34), (36), (40), (41) and (6), we obtain

$$|b_{tn_t}|\, e^{-n_t\delta(\alpha_t z_n,\theta_n)r_n}(1+o(1)) \le kMBe^{-Dn_t\delta(\alpha_t z_n,\theta_n)r_n} e^{2(k+1)\varepsilon r_n}(1+o(1)). \tag{42}$$

Since $0 < D < 1$ and $-n_t\delta(\alpha_t z_n, \theta_n) > 0$, then we can see that (42) is a contradiction when $r_n \to \infty$ and

$$0 < \varepsilon < -\frac{1-D}{2(k+1)} n_t \delta(\alpha_t z_n, \theta_n).$$

Case 3. $\theta_0 \in H = \{\theta \in [0, 2\pi) : \delta(\alpha_s z, \theta) = 0\}$. By $\theta_n \to \theta_0$, for any given $\gamma > 0$, there exists $N > 0$ such that, as $n > N$, we have $\theta_n \in [\theta_0 - \gamma, \theta_0 + \gamma]$ and $z_n = r_n e^{i\theta_n} \in S(\theta_0) = \{z : \theta_0 - \gamma \leq \arg z \leq \theta_0 + \gamma\}$. By Lemma 4, there exists a set $E_3 \subset (1, \infty)$ having finite logarithmic measure, and there is a constant $B > 0$, such that for all z satisfying $|z| = r \notin [0,1] \cup E_3$, we have for $l \leq k-1$

$$\left| \frac{f^{(j)}(z)}{f^{(l)}(z)} \right| \leq B \left[T(2r, f)\right]^{j-l+1} \leq B \left[T(2r, f)\right]^{k+1}, \quad j = l+1, \dots, k. \tag{43}$$

Now, we consider the growth of $f(re^{i\theta})$ on the ray $\arg z = \theta \in [\theta_0 - \gamma, \theta_0) \cup (\theta_0, \theta_0 + \gamma]$. Denote $S_1(\theta_0) = [\theta_0 - \gamma, \theta_0)$ and $S_2(\theta_0) = (\theta_0, \theta_0 + \gamma]$. We can easily see that when $\theta_1 \in S_1(\theta_0)$ and $\theta_2 \in S_2(\theta_0)$ then $\delta(\alpha_s z, \theta_1)\delta(\alpha_s z, \theta_2) < 0$. Without loss of the generality, we suppose that $\delta(\alpha_s z, \theta) > 0$ for $\theta \in S_1(\theta_0)$ and $\delta(\alpha_s z, \theta) < 0$ for $\theta \in S_2(\theta_0)$. Since f is subnormal, then for any given $\varepsilon > 0$, we have

$$T(r, f) \leq e^{\varepsilon r}. \tag{44}$$

We assert that $\left|f^{(s)}(re^{i\theta})\right|$ is bounded on the ray $\arg z = \theta$. If $\left|f^{(s)}(z)\right|$ is unbounded, then by Lemma 6, there exists an infinite sequence of points $w_u = r_u e^{i\theta}$ $(u = 1, 2, \dots)$ where $r_u \to +\infty$ such that $f^{(s)}(w_u) \to \infty$ and

$$\left| \frac{f^{(j)}(w_u)}{f^{(s)}(w_u)} \right| \leq \frac{1}{(s-j)!} r_u^{s-j}(1+o(1)) \leq r_u^s(1+o(1)), \quad j = 0, \dots, s-1. \tag{45}$$

By (43) and (44), we obtain

$$\left| \frac{f^{(j)}(w_u)}{f^{(s)}(w_u)} \right| \leq B \left[T(2r_u, f)\right]^{j-s+1} \leq B \left[T(2r_u, f)\right]^{k+1} \leq e^{2(k+1)\varepsilon r_u}, \quad j = s+1, \dots, k. \tag{46}$$

By (6), (37), (38), (45) and (46), we deduce

$$|a_{sm_s}| e^{m_s \delta(\alpha_s z_n, \theta) r_u}(1+o(1)) \leq kMBe^{Cm_s\delta(\alpha_s w_u, \theta) r_u} e^{2(k+1)\varepsilon r_u} r_u^s (1+o(1)). \tag{47}$$

Since $0 < C < 1$ and $\delta(\alpha_s w_u, \theta) > 0$, then we can see that (47) is a contradiction when $r_u \to \infty$ and

$$0 < \varepsilon < \frac{1-C}{2(k+1)} m_s \delta(\alpha_s w_u, \theta).$$

Hence, for sufficiently large r, we have

$$\left|f(re^{i\theta})\right| \leq M_1 r^s \tag{48}$$

on the ray $\arg z = \theta \in [\theta_0 - \gamma, \theta_0)$. For $\theta \in S_2(\theta_0)$, we have $\delta(\alpha_s z, \theta) < 0$, $\delta(\alpha_t z, \theta) < 0$ and we assert that $\left|f^{(t)}(re^{i\theta})\right|$ is bounded on the ray $\arg z = \theta$. If $\left|f^{(t)}(z)\right|$ is unbounded, then by Lemma 6, there exists an infinite sequence of points $w_u = r_u e^{i\theta}$ $(u = 1, 2, \dots)$ where $r_u \to +\infty$ such that $f^{(t)}(w_u) \to \infty$ and

$$\left|\frac{f^{(j)}(w_u)}{f^{(t)}(w_u)}\right| \le \frac{1}{(t-j)!} r_u^{t-j}(1 + o(1)) \le r_u^t(1 + o(1)), \quad j = 0, \dots, t-1. \tag{49}$$

By (43) and (44), we obtain

$$\left|\frac{f^{(j)}(w_u)}{f^{(t)}(w_u)}\right| \le B\left[T\left(2r_u, f\right)\right]^{j-t+1} \le B\left[T\left(2r_u, f\right)\right]^{k+1} \le Be^{2(k+1)\varepsilon r_u},\ j = t+1, \dots, k. \tag{50}$$

By (6), (40), (41), (49) and (50), we deduce

$$\left|b_{tn_t}\right| e^{-n_t\delta(\alpha_t w_u, \theta) r_u}(1 + o(1)) \le kMBe^{-Dn_t\delta(\alpha_t w_u, \theta) r_u} e^{2(k+1)\varepsilon r_n} r_u^t(1 + o(1)). \tag{51}$$

Since $0 < D < 1$ and $-n_t\delta(\alpha_t z_n, \theta_n) > 0$, then we can see that (51) is a contradiction when $r_n \to \infty$ and

$$0 < \varepsilon < -\frac{1-D}{2(k+1)} n_t \delta(\alpha_t z_n, \theta_n).$$

Hence, for sufficiently large r

$$\left|f(re^{i\theta})\right| \le M_2 r^t \tag{52}$$

on the ray $\arg z = \theta \in (\theta_0, \theta_0 + \gamma]$. By (48) and (52), we have for sufficiently large r

$$\left|f(re^{i\theta})\right| \le Mr^k \tag{53}$$

on the ray $\arg z = \theta \ne \theta_0$, $z \in S(\theta_0)$. Since f has infinite order and $\{z_n = r_n e^{i\theta_n} \in S(\theta_0)\}$ satisfies $|f(z_n)| = M(r_n, f)$, we see that for any large $N > 0$, and as n sufficiently large, we have

$$\left|f(r_n e^{i\theta_n})\right| \ge \exp\{r_n^N\}. \tag{54}$$

Then, from (53) and (54), we get $Mr_n^k \ge \exp\{r_n^N\}$ that is a contradiction. Hence, (6) has no nontrivial subnormal solution.

(4) By Lemma 3, every solution f of (6) satisfies $\sigma_2(f) \le 1$. Suppose that $\sigma_2(f) < 1$, then $\sigma_e(f) = 0$, i.e., f is subnormal solution and this contradicts the conclusion above. So $\sigma_2(f) = 1$. □

6 Proof of Theorem 12

Proof. We consider $Q_j(z) \equiv 0$ $(j = 1, \dots, k-1)$ in (5). By a similar method of proof to Theorem 10, we conclude the result. □

7 Proof of Theorem 13

Proof. We consider $Q_j(z) \equiv 0$ $(j = 1, \dots, k-1)$ in (6). We use the same method as in the proof of Theorem 11. Just in the case when $\delta(\alpha_s z, \theta) < 0$, we use the fact that $\left|f^{(k)}(z)\right|$ is bounded on the ray $\arg z = \theta$. If $\left|f^{(k)}(z)\right|$ is unbounded, then by Lemma 6, there exists an infinite sequence of points $z_n = r_n e^{i\theta}$ $(n = 1, 2, \dots)$ where $r_n \to +\infty$ such that $f^{(k)}(z_n) \to \infty$ and

$$\left|\frac{f^{(j)}(z_n)}{f^{(k)}(z_n)}\right| \le r_n^k(1 + o(1)),\ (j = 0, \dots, k-1). \tag{55}$$

By the definition of $P_j^*(e^{\alpha_j z})$, and because $\delta(\alpha_s z, \theta) < 0$, i.e., $\delta(\alpha_j z, \theta) < 0, \forall j$, by $m_s\alpha_s = c_j m_j \alpha_j$. Then, we can write

$$\left|P_j^*(e^{\alpha_j z_n})\right| = |a_{j1}|\, e^{\delta(\alpha_j z_n, \theta) r_n}(1 + o(1)). \tag{56}$$

By (8), (55) and (56), we have

$$\begin{aligned} 1 &\le \sum_{j=0}^{k-1} \left|P_j^*(e^{\alpha_j z_n})\right| \left|\frac{f^{(j)}(z_n)}{f^{(k)}(z_n)}\right| \\ &\le \sum_{j=0}^{k-1} |a_{j1}|\, e^{\delta(\alpha_j z_n, \theta) r_n} r_n^k (1 + o(1)). \end{aligned} \tag{57}$$

Since $\delta(\alpha_j z, \theta) < 0, \forall j$, then (57) is a contradiction as $r_n \to \infty$. Thus, $\left|f^{(k)}(z)\right| \le M$, so $|f(z)| \le M r^k$. □

8 Proof of Theorem 14

Proof. Suppose that f is a nontrivial subnormal solution of (4). Let

$$h(z) = f(z) e^{(b_m/a_m)z}.$$

Then h is a nontrivial subnormal solution of the equation

$$h^{(k)} + \sum_{j=0}^{k-1} \left[R_j\left(e^z\right) + S_j\left(e^{-z}\right)\right] h^{(j)} = 0, \tag{58}$$

where

$$R_j\left(e^z\right) + S_j\left(e^{-z}\right) = C_k^j \left(-\frac{b_m}{a_m}\right)^{k-j} + \sum_{l=j}^{k-1} C_l^j \left(-\frac{b_m}{a_m}\right)^{l-j} \left[P_l\left(e^z\right) + Q_l\left(e^{-z}\right)\right].$$

Because $m > \max\{m_j : j = 2, \dots, k-1\}$ and $n > \max\{n_j : j = 2, \dots, k-1\}$, we have

$$\begin{aligned} \deg R_1 &= \deg P_1 = m, \\ \deg S_1 &= \deg Q_1 = n. \end{aligned}$$

From $a_m d_n = b_m c_n$, we see in the formula

$$\begin{aligned} R_0\left(e^z\right)+S_0\left(e^{-z}\right) &= \left(-\frac{b_m}{a_m}\right)^k+\sum_{l=2}^{k-1}\left(-\frac{b_m}{a_m}\right)^l\left[P_l\left(e^z\right)+Q_l\left(e^{-z}\right)\right] \\ &\quad +\left(-\frac{b_m}{a_m}\right)\left[P_1\left(e^z\right)+Q_1\left(e^{-z}\right)\right]+\left[P_0\left(e^z\right)+Q_0\left(e^{-z}\right)\right] \end{aligned}$$

that

$$\begin{aligned} \deg R_0 &< m, \\ \deg S_0 &< n. \end{aligned}$$

Then, we have

$$\begin{aligned} \deg R_1 &= m > \deg R_j : j = 0, 2, \ldots, k-1, \\ \deg S_1 &= n > \deg S_j : j = 0, 2, \ldots, k-1 \end{aligned}$$

and since $e^{-(b_m/a_m)z}$ is not a solution of (4), then

$$R_0\left(e^z\right)+S_0\left(e^{-z}\right)=\left(-\frac{b_m}{a_m}\right)^k+\sum_{l=0}^{k-1}\left(-\frac{b_m}{a_m}\right)^l\left[P_l\left(e^z\right)+Q_l\left(e^{-z}\right)\right]\not\equiv 0.$$

By applying Theorem 9 on equation (58), we obtain the conclusion. □

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