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ESSENTIAL COMMUTATIVITY OF COMPOSITION AND DIFFERENTIATION OPERATOR

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Abstract

Let ϕ be an analytic self-map of the open unit disk in the complex plane. We study under which conditions on the symbol ϕ and the involved weights the composition operator C_ϕ and the differentiation operator D acting between weighted Bloch type spaces and weighted Banach spaces are essentially commutative.

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1 Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ the collection of all analytic functions on $\mathbb{D}$. For an analytic self-map ϕ of $\mathbb{D}$ the composition operator C_ϕ is defined by

$$C_\phi : H(\mathbb{D}) \to H(\mathbb{D}),\ f \mapsto f \circ \phi.$$

Composition operators have been studied widely in the literature. One of several reasons is that they link classical results in complex analysis with basic operator theoretical questions. For further exploration we refer the reader to the excellent monographs [8] by Cowen and MacCluer and [13] by Shapiro. On the other hand this kind of operator appears naturally in a variety of problems in several mathematical fields.

We are interested in the following setting: Let v and w be strictly positive continuous and bounded functions (*weights*) on $\mathbb{D}$. We study operators $DC_\phi - C_\phi D$ acting between *weighted Bloch type spaces*

$$B_v := \{f \in H(\mathbb{D}),\ \|f\|_{B_v} := \sup_{z\in\mathbb{D}} v(z)|f'(z)| < \infty\}$$

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endowed with the seminorm $\|.\|_{B_v}$ (provided we identify functions that differ by a constant, $\|.\|_{B_v}$ becomes a norm and B_v a Banach space) and *weighted Banach spaces of holomorphic functions*

$$H_w^\infty := \{f \in H(\mathbb{D}),\ \|f\|_w := \sup_{z\in\mathbb{D}} w(z)|f(z)| < \infty\}$$

endowed with the weighted sup-norm $\|.\|_w$.
Weighted Banach spaces of holomorphic functions arise quite naturally in various mathematical fields such as functional analysis, complex analysis, partial differential equations and convolution equations as well as distribution theory. For more information we refer the reader to [3] and [2].
Next, let us consider the classical differentiation operator

$$D : H(\mathbb{D}) \to H(\mathbb{D}),\ f \mapsto f'.$$

In [9] Harutyunyan and Lusky analyzed the diffferentiation operator $D : H_v^\infty \to H_w^\infty$. They gave a necessary as well as a sufficient condition on the involved weights when the operator is bounded resp. compact.
In this article we are interested in the question how these two classical operators are linked. More precisely we investigate the question when C_ϕ and D are *essentially commutative*, i.e. when $DC_\phi - C_\phi D : B_v^\infty \to H_w^\infty$ is a compact operator. Moreover we obtain conditions when this operator is bounded resp. an isometry.

2 Basics

2.1 Weights

A strictly positive continuous and bounded function $v : \mathbb{D} \to (0, \infty)$ is called a *weight*. Especially interesting are weights which satisfy $v(z) = v(|z|)$ for every $z \in \mathbb{D}$. We say that weights of this type are *radial*. Every radial weight which is non-increasing with respect to $|z|$ and such that $\lim_{|z|\to 1-} v(z) = 0$ is called a *typical* weight. In [11] Lusky studied weights satisfying the condition $(L1)$ (renamed after the author)

$$(L1) \quad \inf_{n\in\mathbb{N}} \frac{v(1-2^{-n-1})}{v(1-2^{-n})} > 0.$$

Examples of radial weights satisfying $(L1)$ are among others the standard weights $v(z) = (1-|z|^2)^p$, $p > 0$, for every $z \in \mathbb{D}$, and the logarithmic weights $v(z) = (1-\log(1-|z|^2))^q$, $q < 0$, for every $z \in \mathbb{D}$. In this work weights which satisfy $(L1)$ will play a great role. When dealing with weighted spaces often the so-called *associated weights* $\tilde{v}$ are required. For a weight v the associated weight $\tilde{v}$ is defined by

$$\tilde{v}(z) = \frac{1}{\sup\{|f(z)|;\ f \in H_v^\infty,\ \|f\|_v \leq 1\}},\ z \in \mathbb{D}.$$

The concept of associated weights was implicitly introduced by Anderson and Duncan in [1], while Bierstedt, Bonet and Taskinen studied them in [2]. They showed that the associated weight $\tilde{v}$ has the following properties:

1. $\tilde{v} \geq v > 0$.
2. $\tilde{v}$ is continuous.
3. For every $z \in \mathbb{D}$ there is $f_z \in H_v^\infty$, $\|f_z\|_v \leq 1$, such that $|f_z(z)| = \frac{1}{\tilde{v}(z)}$.

Since it is quite difficult to really compute the associated weights we are very interested in simple conditions on the weights which ensure that v and $\tilde{v}$ are equivalent weights, i.e. that there exists a constant $C > 0$ with

$$v(z) \leq \tilde{v}(z) \leq Cv(z) \text{ for every } z \in \mathbb{D}.$$

If v and $\tilde{v}$ are equivalent weights we say that v is an *essential* weight. By [4] we know that condition $(L1)$ implies the essentiality of v. Moreover, $v = \tilde{v}$ if $v(z) = |f(z)|$ for every $z \in \mathbb{D}$ and some $f \in H(\mathbb{D})$ whose Taylor series (at 0) has non-negative coefficients (see [2] Corollary 1.6).

2.2 Composition Operators

For general information on composition operators we refer the reader to the nicely written monographs [8] and [13]. In this section we collect the results we need on composition operators. In [4] Bonet, Domański, Lindström and Taskinen studied composition operators acting on weighted Banach spaces of holomorphic functions. Among other things they proved that in case w is an arbitrary weight and H_v^∞ is generated by a typical weight v the following assertions are true

1. $C_\phi : H_v^\infty \to H_w^\infty$ is a bounded operator if and only if $\sup_{z\in\mathbb{D}} \frac{w(z)}{\tilde{v}(\phi(z))} < \infty$.
2. $C_\phi : H_v^\infty \to H_w^\infty$ is a compact operator if and only if $\limsup_{|\phi(z)|\to 1} \frac{w(z)}{\tilde{v}(\phi(z))} = 0$.

Contreras and Hernandez-Díaz [7] and independently Montes [12] followed this branch of research by investigating weighted composition operators. We will not go into detail at this point. Moreover, Bonet, Lindström and Wolf studied differences of (weighted) composition operators, see [5] and [10]. The investigation of differences of (weighted) composition operators requires the *pseudohyperbolic metric* $\rho(z, w)$ for $z, w \in \mathbb{D}$, where $\rho(z, w)$ is defined as follows:

$$\rho(z, w) = |\varphi_z(w)| \text{ with the Möbius transformation } \varphi_z(w) = \frac{z - w}{1 - \overline{z}w}.$$

Furthermore, we will use the following well-known inequality

$$1 - |\varphi_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \overline{a}z|}.$$

In the sequel we need the following lemma which was given in [10].

Lemma 1 (Lindström, Wolf, [10]). *Let v be a radial weight satisfying $(L1)$ such that v is continuously differentiable w.r.t. $|z|$. There is a constant $M < \infty$ such that if $f \in H_v^\infty$, then*

$$|v(p)f(q) - v(p)f(p)| \leq M\|f\|_v \rho(p,q)$$

for all $p, q \in \mathbb{D}$.

Moreover we need the following proposition which can be easily deduced from Proposition 3.11 in [8].

Proposition 2. *Let v and w be arbitrary weights. Then the operator $DC_\phi - C_\phi D : B_v \to H_w^\infty$ is compact if and only if for every bounded sequence $(f_n)_n$ in B_v such that $f_n \to 0$ uniformly on the compact subsets of $\mathbb{D}$, then $[DC_\phi - C_\phi D](f_n) \to 0$ in H_w^∞.*

3 Essential Commutativity

We start this section by analyzing when the operator $DC_\phi - C_\phi D : B_v \to H_w^\infty$ is bounded. The methods are the same used by Bonet, Domański, Lindström and Taskinen in [4].

Proposition 3. *Let v and w be arbitrary weights. Then the operator $DC_\phi - C_\phi D : B_v \to H_w^\infty$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}} \frac{w(z}{\tilde{v}(\phi(z))}|\phi'(z) - 1| < \infty.$$

Proof. First we assume that $\sup_{z\in\mathbb{D}} \frac{w(z}{v(\phi(z))}|\phi'(z) - 1| < \infty$. Then, for every $f \in B_v$ we have

$$\begin{aligned}
\|(DC_\phi - C_\phi D)f\|_w &= \sup_{z\in\mathbb{D}} w(z)|\phi'(z)f'(\phi(z)) - f'(\phi(z))| \\
&= \sup_{z\in\mathbb{D}} \frac{w(z)}{\tilde{v}(\phi(z))}\tilde{v}(\phi(z))|f'(\phi(z))||\phi'(z) - 1| \\
&\leq \sup_{z\in\mathbb{D}} \frac{w(z)}{\tilde{v}(\phi(z))}|\phi'(z) - 1|\|f\|_{B_v},
\end{aligned}$$

since H_v^∞ and $H_{\tilde{v}}^\infty$ are isometrically isomorphic. Hence the operator $DC_\phi - C_\phi D : B_v \to H_w^\infty$ is bounded.
Conversely, we assume to the contrary that there is a sequence $(z_n)_n \subset \mathbb{D}$ with $|\phi(z_n)| \to 1$ such that

$$\frac{w(z_n)}{\tilde{v}(\phi(z_n))}|\phi'(z_n) - 1| \geq n \text{ for every } n \in \mathbb{N}.$$

For every $n \in \mathbb{N}$ there is a function $f_n \in B_v$ with $\|f_n\|_{B_v} \leq 1$ and $|f_n'(\phi(z_n))| = \frac{1}{\tilde{v}(\phi(z_n))}$. Then, by the boundedness of the operator $DC_\phi - C_\phi D : B_v \to H_w^\infty$ there is $c > 0$ such that

$$c \geq \|(DC_\phi - C_\phi D)f_n\|_w \geq w(z_n)|f'(\phi(z_n))\phi'(z_n) - f'(\phi(z_n))| \geq \frac{w(z_n)}{\tilde{v}(\phi(z_n))}|\phi'(z_n) - 1| \geq n$$

for every $n \in \mathbb{N}$, which is a contradiction. □

We continue with studying when D and C_ϕ are essentially commutative in the given context.

Proposition 4. *Let v and w be arbitrary weights. Then the operator $DC_\phi - C_\phi D : B_v \to H_w^\infty$ is compact if and only if*

$$\lim_{|\phi(z)|\to 1} \frac{w(z)}{\tilde{v}(\phi(z))}|\phi'(z) - 1| = 0.$$

Proof. We suppose that $\lim_{|\phi(z)|\to 1} \frac{w(z)}{v(\phi(z))}|\phi'(z)-1| = 0$. This means that for every $\varepsilon > 0$ there is $0 < r_0 < 1$ such that

$$\frac{w(z)}{v(\phi(z))}|\phi'(z) - 1| < \frac{\varepsilon}{2} \text{ for every } z \in \mathbb{D} \text{ with } |\phi(z)| > r_0.$$

Now let $(f_n)_n \subset B_v$ be an arbitrary sequence with the following properties

1. $\|f_n\|_{B_v} \leq 1$ for every $n \in \mathbb{N}$.
2. $f_n \to 0$ uniformly on the compact subsets of $\mathbb{D}$.

By Proposition 2 we have to show that $([DC_\phi - C_\phi D]f_n)_n$ converges to 0 in H_w^∞, that is,

$$\|[DC_\phi - DC_\phi]f_n\|_w = \sup_{z\in\mathbb{D}} w(z)|\phi'(z)f_n'(\phi(z)) - f_n'(\phi(z))| \to 0.$$

Since $f_n \to 0$ uniformly on the compact subsets of $\mathbb{D}$, for every $\varepsilon > 0$ and every $0 < r < 1$ there is $n_0 \in \mathbb{N}$ such that, if $|\phi(z)| \leq r$, then

$$|f_n'(\phi(z))| < \frac{\varepsilon}{2C},$$

where $C := \sup_{|\phi(z)|\leq r} w(z)|\phi'(z) - 1|$. Now, we obtain

$$\begin{aligned}
\|[DC_\phi - C_\phi D]f_n\|_w &= \sup_{z\in\mathbb{D}} w(z)|\phi'(z)f_n'(\phi(z)) - f_n'(\phi(z))| \\
&\leq \sup_{|\phi(z)|\leq r_0} w(z)|\phi'(z)f_n'(\phi(z)) - f_n'(\phi(z))| \\
&+ \sup_{|\phi(z)|> r_0} w(z)|\phi'(z)f_n'(\phi(z)) - f_n'(\phi(z))| \\
&\leq \sup_{|\phi(z)|\leq r_0} w(z)|\phi'(z) - 1||f_n'(\phi(z))| \\
&+ \sup_{|\phi(z)|> r_0} w(z)|\phi'(z) - 1|\|f_n\|_{B_v} \\
&\leq \frac{\varepsilon}{2} + C\frac{\varepsilon}{2C} = \varepsilon
\end{aligned}$$

for every $n \geq n_0$.
Conversely, we assume to the contrary that there are $c > 0$ and a sequence $(z_n)_n \subset \mathbb{D}$ with $|\phi(z_n)| \to 1$ such that

$$\frac{w(z_n)}{\tilde{v}(\phi(z_n))}|\phi'(z_n) - 1| \geq c$$

for every $n \in \mathbb{N}$. Since $|\phi(z_n)| \to 1$ there exist natural numbers $\alpha(n)$ with $\lim_{n\to\infty} \alpha(n) = \infty$ such that $|\phi(z_n)|^{\alpha(n)} \geq \frac{1}{2}$ for every $n \in \mathbb{N}$. Now, we consider functions g_n given by

$$g_n'(z) = z^{\alpha(n)} f_n(z) \text{ for every } z \in \mathbb{D} \text{ and every } n \in \mathbb{N},$$

where $\sup_{z\in\mathbb{D}} v(z)|f_n(z)| \leq 1$ and $|f_n(\phi(z_n))| = \frac{1}{\tilde{v}(\phi(z_n))}$ for every $n \in \mathbb{N}$. Hence obviously we have that $\|g_n\|_{B_v} \leq 1$ for every $n \in \mathbb{N}$. Moreover the sequence $(g_n)_n$ converges pointwise to 0 because of the factor $z^{\alpha(n)}$. Now, supposing that the operator $DC_\phi - C_\phi D : B_v \to H_w^\infty$ is compact, Proposition 2 implies that $\|(DC_\phi - C_\phi D)g_n\|_w \to 0$ as $n \to \infty$. On the other hand, we have for every $n \in \mathbb{N}$:

$$\begin{aligned} \|(DC_\phi - C_\phi D)g_n\|_w &\geq w(z_n)|g_n'(\phi(z_n))\phi'(z_n) - g_n'(\phi(z_n))| \\ &\geq \frac{w(z_n)}{\tilde{v}(\phi(z_n))}|\phi'(z_n) - 1||\phi(z_n)|^{\alpha(n)} \geq \frac{1}{2}c \end{aligned}$$

which is a contradiction.

Thus the claim follows. □

Corollary 5. *Let v and w be arbitrary weights. Then C_ϕ and D are essentially commutative if and only if* $\lim_{|\phi(z)|\to 1} \frac{w(z)}{\tilde{v}(\phi(z))}|\phi'(z) - 1| = 0$.

We close this section by investigating when $DC_\phi - C_\phi D : B_v \to H_w^\infty$ is an isometry. We use the methods given in [6].

Proposition 6. *Let w be an arbitrary weight.*

(a) Let v be a radial weight with (L1) such that v is continuously differentiable w.r.t. $|z|$. If $\sup_{z\in\mathbb{D}} \frac{w(z)}{v(\phi(z))}|\phi'(z) - 1| \leq 1$ *and*

$$(M_{\phi,D}) \qquad \forall a \in \mathbb{D}\ \exists\ (z_n)_n \subset \mathbb{D} \quad \text{such that}$$

$$\rho(\phi(z_n), a) \to 0 \text{ and } \frac{w(z_n)}{v(\phi(z_n))}|\phi'(z_n) - 1| \to 1,$$

then the operator $DC_\phi - C_\phi D : B_v \to H_w^\infty$ is an isometry.

(b) Let v be a radial weight such that $u(z) = \frac{v(z)}{(1-|z|^2)^p}$ is a weight on $\mathbb{D}$ for some $0 < p < \infty$ and $u = \tilde{u}$. If $DC_\phi - C_\phi D : B_v \to H_w^\infty$ is an isometry, then condition $(M_{\phi,D})$ is satisfied and $sup_{z\in\mathbb{D}}\frac{w(z)}{v(\phi(z))}|\phi'(z) - 1| \leq 1$.

Proof. We start with proving assertion (a). For every $f \in B_v$ we have that

$$\begin{aligned} \|(DC_\phi - C_\phi D)f\|_w &\leq \sup_{z\in\mathbb{D}} w(z)|\phi'(z)f'(\phi(z)) - f'(\phi(z))| \\ &\leq \sup_{z\in\mathbb{D}} \frac{w(z)}{v(\phi(z))}|\phi'(z) - 1|\|f\|_{B_v} \leq \|f\|_{B_v} \end{aligned}$$

In order to prove the reverse inequality let $f \in B_v$. Then $\|f\|_v = \lim_{m\to\infty} v(a_m)|f(a_m)|$ for some sequence $(a_m)_m \subset \mathbb{D}$. Now, let m be fixed. Hence by condition $(M_{\phi,D})$ there is a sequence $(z_n^m)_n \subset \mathbb{D}$ such that

$$\rho(\phi(z_n^m), a_m) \to 0 \text{ and } \frac{w(z_n^m)}{v(\phi(z_n^m))}|\phi'(z_n^m) - 1| \to 1$$

when $n \to \infty$. By Lemma 1, for all m and n

$$|v(a_m)f'(a_m) - v(\phi(z_n^m)f'(\phi(z_n^m))| \leq M\|f\|_{B_v}\rho(a_m, \phi(z_n^m)).$$

Hence

$$\begin{aligned} \|(DC_\phi - C_\phi D)f\|_w &= \sup_{z\in\mathbb{D}} \frac{w(z)}{v(\phi(z))}|\phi'(z) - 1|v(\phi(z))|f'(\phi(z))| \\ &\geq \limsup_{n\to\infty} \frac{w(z_n^m)}{v(\phi(z_n^m))}|\phi'(z_n^m) - 1| \times \\ &\qquad \times (v(a_m)|f'(a_m)| - M\|f\|_{B_v}\rho(\phi(z_n^m), a_m)) \\ &= v(a_m)|f(a_m)|. \end{aligned}$$

Since this is true for all m we arrive at

$$\|(DC_\phi - C_\phi D)f\|_w \geq \|f\|_{B_v}.$$

(b) Choose $p > 0$ such that $u(z) = \frac{v(z)}{(1-|z|^2)^p}$ is a weight on $\mathbb{D}$ such that $\tilde{u} = u$. Then by [6] we also have $v = \tilde{v}$. Now, by assumption $\|(DC_\phi - C_\phi D)f\|_w = \|f\|_{B_v}$ for every $f \in B_v$. Hence

$$\|DC_\phi - C_\phi D\| = \sup_{z\in\mathbb{D}} \frac{w(z)}{v(\phi(z))}|\phi'(z) - 1| \leq 1.$$

Let $a \in \mathbb{D}$. Then there exists $g_a \in B_u$, $\|g_a\|_{B_u} = 1$ such that $g_a'(a)\tilde{v}(a) = 1$. Put

$$f_a'(z) := g_a'(z)\left(\frac{1-|a|^2}{(1-\overline{a}z)^2}\right)^p \text{ for every } z \in \mathbb{D}.$$

It follows that $\|f_a\|_{B_v} = 1$ since $f_a'(a) = \frac{1}{\tilde{v}(a)}$. Thus we can pick a sequence $(z_n)_n \subset \mathbb{D}$ such that

$$w(z_n)|\phi'(z_n)f_a'(\phi(z_n)) - f_a'(\phi(z_n))| \to 1 \text{ when } n \to \infty.$$

Hence

$$\begin{aligned} 1 &\geq \frac{w(z_n)}{v(\phi(z_n))}|\phi'(z_n) - 1| \geq \frac{w(z_n)}{v(\phi(z_n))}v(\phi(z_n))|f'(\phi(z_n))|\phi'(z_n) - 1| \\ &= w(z_n)|f'(\phi(z_n))\phi'(z_n) - f'(\phi(z_n))|. \end{aligned}$$

So, we obtain $\lim_{n\to\infty} \frac{w(z_n)}{v(\phi(z_n))}|\phi'(z_n) - 1| = 1$.
Furthermore

$$\begin{aligned} 1 &\geq (1 - |\varphi_a(\phi(z_n))|^2)^p = \frac{(1-|a|^2)^p(1-|\phi(z_n)|^2)^p}{|1-\overline{\phi(z_n}a|^{2p}} \\ &= \frac{|f_a'(\phi(z_n))|v(\phi(z_n))(1-|\phi(z_n)|^2)^p}{|g_a'(\phi(z_n))|v(\phi(z_n))} \geq |f_a'(\phi(z_n))|v(\phi(z_n)) \end{aligned}$$

Sicne $|f_a'(\phi(z_n))|v(\phi(z_n)) \to 1$ when $n \to \infty$ we conclude, as $v = \tilde{v}$ that $\lim_{n\to\infty}(1 - |\varphi_a(\phi(z_n))|^2)^p = 1$ and $\rho(\phi(z_n), a) \to 0$ when $n \to \infty$. □

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