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PARAMETER ESTIMATION IN THE ARCH MODEL WITH WEIGHTED LIQUIDITY

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Abstract

We analyze a variant of the ARCH(1) model which captures the variation of the intra-day price. We study the asymptotic behavior of the least squares estimator for the parameters of the model.

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Key words: ARCH model, liquidity, intr-day price, least squares estimator, consistency, asymptotic normality

1 Introduction

Starting with the seminal paper by Engle [4], a vast literature on ARCH and related models has been developed. There are various extensions of this model, as the GARCH model introduced by Bollerslev in [2] or the EGARCH model of Nelson [10]. The purpose of this note is to study a new variant of the ARCH model that takes into account the fluctuation of the intra-day price and the liquidity existent in the market. This new model is motivated by our empirical studies, as noticed in [12]. Moreover, from the theoretical point of view, the new model that includes liquidity keeps the main properties of the standard ARCH model, that is, the estimators for the parameters of the model are consistent and asymptotically normal. We will discuss these theoretical aspects in Section 2.

In order to capture the fluctuation of the intra-day price in financial markets, we include in our model the price range for a financial asset in a given trading day, that is the difference between the maximum price (denoted h_t or the highest price at lag t or trading day t) and the minimum (lowest) price or l_t during the same trading day t. As a proxy for the liquidity we employ the number of shares traded during trading day t or the trading volume, denoted L_t. Therefore, our model weighs the impact of past shocks with their corresponding liquidity. In other

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words, if the past volatility was accompanied by high liquidity, then its impact on the future volatility will be larger. The model therefore does not allow the "false shocks" to have a high explanatory power, considering that the lack of liquidity creates a distorted picture of reality, and shocks that occur under such conditions must be corrected for their low liquidity. To better illustrate this, let us consider a certain asset A and let us imagine the hypothetical situation where during a certain trading day t only two transactions with asset A took place at large time intervals and with low trading volume, but with a large value for $|h_t - l_t|$, that is the two transactions took place at two significantly different prices. Consider also that in another trading session $t+i$, the asset A has also traded in the range of prices $[h_{t+i}, l_{t+i}]$ (where $h_{t+i} = h_t$ and $l_{t+i} = l_t$), but this time the transactions were numerous and the trading volume was high. Obviously, the trading day $t+i$ was more "turbulent" than the day t. But if in both cases only $|h_t - l_t|$ is considered as a measure of volatility without including the corresponding liquidity, then the shocks produced at time t and $t+i$ will have an identical impact on the future volatility, which would be clearly erroneous, as the shock from $t+i$ is much stronger than the shock from t.

The paper is structured as follows. In Section 2 we introduce and we analyze the

properties of the extended ARCH model: the existence of the stationary solution, the computation of the first moments and the behavior of the auto-correlation function. In Section 3 we study the least squares estimator for the parameters of the model.

2 The ARCH model with weighted liquidity

We will consider the following model: for every $t \in \mathbb{Z}$

$$X_t = \sigma_t \varepsilon_t \tag{1}$$

with

$$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \ell_1 L_{t-1}. \tag{2}$$

We assume that $(\varepsilon_t)_{t\in\mathbb{Z}}$ is a sequence of i.i.d. random variables such that $E\varepsilon_0 = 0$ and $E\varepsilon_0^2 = 1$. The sequence $(\varepsilon_t)_{t\in\mathbb{Z}}$ is referred to as the driving noise sequence. The parameters $\alpha_0, \alpha_1, \ell_1$ are strictly positive. The sequence $(L_t)_{t\in\mathbb{Z}}$ is also a sequence of i.i.d. positive random variables but in addition we will assume that it is independent of the sequence $(\varepsilon_t)_{t\in\mathbb{Z}}$. Actually L_t is interpreted as an indicator of the liquidity and it represents the volume of transactions at time t. It is again reasonable to assume that it is positive ($L_t = 0$ only when the market is closed).

The case $\ell_1 = 0$ corresponds to the classical ARCH(1) model introduced in [4].

2.1 Strictly stationarity and ergodicity of the model

From (1) and (2) we can immediately write, for every $t \in \mathbb{Z}$

$$\sigma_t^2 \quad = \quad \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 \sigma_{t-1}^2 + \ell_1 L_{t-1}. \tag{3}$$

We introduce the following notation:

$$y_t := \sigma_{t+1}^2, \quad A_t := \alpha_1 \varepsilon_t^2, \quad B_t = \alpha_0 + \ell_1 L_t, \qquad t \in \mathbb{Z}. \tag{4}$$

With the notation (4), relation (3) becomes

$$y_t = A_t y_{t-1} + B_t, \qquad t \in \mathbb{Z}. \tag{5}$$

Let us iterate the above relation (5). We get

$$\begin{aligned} y_t &= A_t y_{t-1} + B_t \\ &= A_t A_{t-1} y_{t-2} + A_t B_{t-1} + B_t \\ &= \ldots \\ &= \left(\prod_{i=0}^{k} A_{t-i} \right) y_{t-k-1} + \sum_{i=0}^{k} \left(\prod_{j=0}^{i-1} A_{t-j} \right) B_{t-i}. \end{aligned} \tag{6}$$

Let us check the existence of a stationary solution to problem (1)-(2).

Proposition 1. *Assume that L_0 has a density f that satisfies*

$$\int_{\mathbb{R}} (\log x)^2 f(x) dx < \infty. \tag{7}$$

Also suppose that $e^{E(\log A_1)} < 1$. Then the system (1)-(2) has a unique strictly stationary solution which can be written as $X_t = \sigma_t \varepsilon_t$ with

$$\sigma_t^2 = \sum_{i=0}^{\infty} \left(\prod_{j=0}^{i-1} A_{t-j} \right) B_{t-i}$$

and A, B given by (4). Moreover the strictly stationary solution is ergodic and X_t is independent of $(\varepsilon_j)_{j \geq t+1}$ for every $t \in \mathbb{Z}$.

Proof:

For every $k \geq 1$, let us denote by

$$h_t(k) := \sum_{i=0}^{k} \left(\prod_{j=0}^{i-1} A_{t-j} \right) B_{t-i}.$$

Since $h_t(k) > 0$ for every k, t, the limit

$$h_t := \lim_{k \to \infty} h_t(k)$$

exists in $[0,\infty]$. Moreover, $h_t(k)$ satisfies the recursive relation

$$h_t(k) = A_{t-1}h_{t-1}(k-1) + B_t, \qquad t \in \mathbb{Z}.$$

If h_t is finite for every $t \in \mathbb{Z}$, taking the limit as $k \to \infty$ in the above we get that h_t is a solution (clearly stationary) to (1)-(2). Notice that A and B are non-negative.

Therefore, we need to check that h_t is finite. To this end, it suffices to check that the series

$$\sum_{i=0}^{\infty}\left(\prod_{j=0}^{i-1} A_{t-j}\right) B_{t-i} = \alpha_0 \sum_{i=0}\left(\prod_{j=0}^{i-1} A_{t-j}\right) + \ell_1 \sum_{i=0}^{\infty}\left(\prod_{j=0}^{i-1} A_{t-j}\right) L_{t-i}$$

is convergent. The convergence of the first sum follows as in standard ARCH(1) model. To check the convergence of the series $\sum_{i=0}^{\infty}\left(\prod_{j=0}^{i-1} A_{t-j}\right) L_{t-i} := \sum_{i=0}^{\infty} u_i$ we will use the Cauchy criterion, that is, we prove that $\limsup_n u_n^{\frac{1}{n}} < 1$ almost surely. We have

$$u_n^{\frac{1}{n}} = e^{\frac{1}{n}\sum_{j=0}^{n-1}\log A_{t-j}} L_{t-n}^{\frac{1}{n}}$$

and by the law of large numbers $e^{\frac{1}{n}\sum_{j=0}^{n-1}\log A_{t-j}} \to_{n\to\infty} e^{E(\log A_1)} < 1$. The conclusion will then follow if we prove that $L_{t-n}^{\frac{1}{n}} \to_{n\to\infty} 1$ almost surely. For some $\gamma > 0$, we can write,

$$\begin{aligned}\sum_n P\left(|L_{t-n}^{\frac{1}{n}} - 1| \geq n^{-\gamma}\right) &\leq \sum_n n^{2\gamma} E\left(L_{t-n}^{\frac{1}{n}} - 1\right)^2 \\ &= \sum_n n^{2\gamma} E\left(L_{t-n}^{\frac{2}{n}} - 2L_{t-n}^{\frac{1}{n}} + 1\right)\end{aligned}$$

and

$$E\left(L_{t-n}^{\frac{2}{n}} - 2L_{t-n}^{\frac{1}{n}} + 1\right) = \int_{\mathbb{R}} (x^{\frac{2}{n}} - 2x^{\frac{1}{n}} + 1) f(x)dx \leq Cn^{-2}\int_{\mathbb{R}} (\log x)^2 f(x)dx$$

by analyzing the asymptotic behavior of the function $x^{\frac{2}{n}} - 2x^{\frac{1}{n}} + 1$. By condition (7),

$$\sum_n P\left(|L_{t-n}^{\frac{1}{n}} - 1| > n^{-\gamma}\right) \leq C\sum_n n^{2\gamma-2}$$

and the series converges for $\gamma > 0$ small enough. It follows that $L_{t-n}^{\frac{1}{n}}$ converges almost surely to 1 by Borel-Cantelli and the conclusion follows.

The ergodicity follows from Theorem 1.3 in [3]. □

Remark 1. *The condition (7) is satisfied if L_0 follows the normal or the Gamma distribution. But the class of examples is clearly bigger.*

The assumption $e^{E(\log \varepsilon_1)} < 0$ appears also in the standard ARCH(1) model (see e.g. [5]) and it is verified by a wider class of random variables.

Remark 2. *Supppose $P(B_0 = 0) < 1, P(A_0 = 0) = 0$, that $\prod_{i=0}^{n} A_{-i}$ converges almost surely to zero as $n \to \infty$ and that*

$$\int_1^\infty \frac{\log q}{T_A(\log q)} P_{|\alpha_0+\ell_1 L_0|}(dq) < \infty \tag{8}$$

where $P_{|\alpha_0+\ell_1 L_0|}$ denotes the distribution of the random variable $|\alpha_0 + \ell_1 L_0|$ and $T_A(y) := \int_0^y P\left(|A_0| < e^{-x}\right) dx$ for every $y \geq 0$. Then the series

$$\sum_{i=0}^{\infty} \left(\prod_{j=0}^{i-1} A_{t-j} \right) B_{t-i}$$

converges almost surely absolutely for every $t \in \mathbb{Z}$.This follows from Theorem 2.1 in [6] since $(A_t, B_t)_{t\in\mathbb{Z}}$ is a i.i.d. sequence of random variables in $\mathbb{R}^2$. See also Proposition 1 in [9]. Therefore, condition (8) could be used instead of (7).

As a consequence, we compute the mean of the squared volatility.

Lemma 1. *Assume that $\alpha_1 < 1$ and $E(L_0) < \infty$. Then for every $t \in \mathbb{Z}$, the squared volatility associated to the stationary solution is integrable and*

$$E(\sigma_t^2) = \frac{\alpha_0 + \ell_1 E(L_0)}{1 - \alpha_1}. \tag{9}$$

Proof: It suffices to take the expectation in (3) and then to iterate the relation. Denoting by $e_t = E(\sigma_t^2)$, we have

$$\begin{aligned} e_t &= \alpha_0 + \alpha_1 e_{t-1} + \ell_1 E(L_0) \\ &= \alpha_0 + \alpha_1(\alpha_0 + \alpha_1 e_{t-2} + \ell_1 E(L_0)) \\ &= \alpha_0(1 + \alpha_1 + \alpha_1^2) + l_1 E(L_0)(1 + \alpha_1 + \alpha_1^2) + \alpha_1^3 e_{t-3} \\ &= \dots \\ &= (\alpha_0 + \ell_1 E(L_0)) \sum_{k\geq 0} \alpha_1^k = \frac{\alpha_0 + \ell_1 E(L_0)}{1 - \alpha_1} \end{aligned}$$

where the equality at the beginning of the last line above is obtained by a trivial limit. □

Let us now calculate the fourth moment of the extended ARCH process.

Proposition 2. *For every $t \in \mathbb{Z}$ and for $\alpha_1 < \frac{1}{\sqrt{3}}$, we have*

$$E(\sigma_t^4) = \frac{A}{1 - 3\alpha_1^2} \tag{10}$$

with

$$A = \alpha_0^2 + 3\ell_1^2 + 2\frac{\alpha_0^2\alpha_1 + \alpha_0\ell_1 E(L_0)}{1 - \alpha_1} + 2\alpha_1\ell_1(\alpha_0 E(L_0) + \ell_1(E(L_0))^2)\frac{1}{1 - \alpha_1}. \tag{11}$$

Proof: First, fix $s,t \in \mathbb{Z}$ and notice that

$$\begin{aligned} E(\sigma_t^2 L_s) &= \sum_{i=0}^{\infty} \left(\prod_{j=0}^{i-1} E(A_{t-j}) \right) E(B_{t-i}L_s) \\ &= (\alpha_0 E(L_0) + l_1 (E(L_0))^2) \sum_{i=0}^{\infty} \alpha_1^i \\ &= (\alpha_0 E(L_0) + \ell_1 (E(L_0))^2) \frac{1}{1-\alpha_1} \end{aligned} \tag{12}$$

where we used

$$E(B_{t-i}L_s) = E(\alpha_0 + \ell_1 L_{t-i})L_s = \alpha_0 E(L_0) + \ell_1 (E(L_0))^2$$

since L_t is a i.i.d. sequence of random variables.

From relation (3), we get for every $t \in \mathbb{Z}$

$$\begin{aligned} E(\sigma_t^4) &= \alpha_0^2 + 3\alpha_1^2 E(\sigma_{t-1}^4) + 3\ell_1^2 + 2\alpha_0\alpha_1 E(\sigma_{t-1}^2) + 2\alpha_0\ell_1 + 2\alpha_1\ell_1 E(\sigma_{t-1}^2 L_{t-1}) \\ &= \alpha_0^2 + 3\alpha_1^2 E(\sigma_{t-1}^4) + 3\ell_1^2 + 2\frac{\alpha_0^2\alpha_1 + \alpha_0\ell_1 E(L_0)}{1-\alpha_1} + 2\alpha_1\ell_1 E(\sigma_{t-1}^2 L_{t-1}) \end{aligned}$$

and by relation (12),

$$E(\sigma_t^4) = 3E(\sigma_{t-1}^4) + A$$

with A given by (11). But iterating the above identity, we get (10). □

Remark 3. *The condition $\alpha_1 < \frac{1}{\sqrt{3}}$ also appears in the standard ARCH (1) model. Actually, from (10) one can recover the standard ARCH case by taking $l_1 = L_0 = 0$.*

Let us now compute the correlation function of the squared volatility and analyze its asymptotic behavior.

Proposition 3. *Assume $\alpha_1 < \frac{1}{\sqrt{3}}$. For every $t \in \mathbb{Z}$ and for every $k \geq 1$,*

$$E(y_t y_{t-k}) = a\alpha_1^k + b$$

where

$$a = E(\sigma_0^4) - \frac{1}{(1-\alpha_1)^2}\left[\alpha_0(\alpha_0 + l_1) + l_1\alpha_0 E(L_0) + \ell_1^2 (E(L_0))^2\right] \tag{13}$$

and

$$b = \alpha_0 l_1 (1 - E(L_0)). \tag{14}$$

Proof: Formula (6) reads, for every $k \geq 1$,

$$y_t = \left(\prod_{i=0}^{k-1} A_{t-i}\right) y_{t-k} + \sum_{i=0}^{k-1} \left(\prod_{j=0}^{i-1} A_{t-j}\right) B_{t-i}$$

and this implies

$$\begin{aligned}
E(y_t y_{t-k}) &= \left(\prod_{i=0}^{k-1} E(A_{t-i})\right) E(y_{t-k}^2) + \sum_{i=0}^{k-1} \left(\prod_{j=0}^{i-1} E(A_{t-j})\right) E(y_{t-k} B_{t-i}) \\
&= \alpha_1^k E(\sigma_0^4) + \sum_{i=0}^{k-1} \alpha_1^i E(y_{t-k} B_{t-i}) \\
&= \alpha_1^k E(\sigma_0^4) + \sum_{i=0}^{k-1} \alpha_1^i \left[\alpha_0 \frac{\alpha_0 + \ell_1}{1-\alpha_1} + \ell_1 E(y_{t-k} L_{t-i})\right]
\end{aligned}$$

and by (12) we can write

$$\begin{aligned}
E(y_t y_{t-k}) &= \alpha_1^k E(\sigma_0^4) + \sum_{i=0}^{k-1} \alpha_1^i \left[\alpha_0 \frac{\alpha_0 + \ell_1}{1-\alpha_1} + \ell_1 \frac{\alpha_0 E(L_0) + l_1 (E(L_0))^2}{1-\alpha_1}\right] \\
&= \alpha_1^k E(\sigma_0^4) + \left[\alpha_0 \frac{\alpha_0 + \ell_1}{1-\alpha_1} + \ell_1 \frac{\alpha_0 E(L_0) + l_1 (E(L_0))^2}{1-\alpha_1}\right] \frac{1-\alpha_1^k}{1-\alpha_1} \\
&= \alpha_1^k E(\sigma_0^4) + \frac{1-\alpha_1^k}{(1-\alpha_1)^2} \left[\alpha_0(\alpha_0 + l_1) + l_1 \alpha_0 E(L_0) + l_1^2 (E(L_0))^2\right]
\end{aligned}$$

and thus

$$\begin{aligned}
Cov(y_t, y_{t-k}) &= E(y_t y_{t-k}) - E(y_t) E(y_{t-k}) \\
&= \alpha_1^k E(\sigma_0^4) + \frac{1-\alpha_1^k}{(1-\alpha_1)^2} \left[\alpha_0(\alpha_0 + l_1) + l_1 \alpha_0 E(L_0) + l_1^2 (E(L_0))^2\right] \\
&\quad - \frac{(\alpha_0 + l_1 E(L_0))^2}{(1-\alpha_1)^2} \\
&= a \sigma_1^k + b
\end{aligned}$$

with

$$\begin{aligned}
a &= E(\sigma_0^4) - \frac{1}{(1-\alpha_1)^2} \left[\alpha_0(\alpha_0 + l_1) + l_1 \alpha_0 E(L_0) + l_1^2 (E(L_0))^2\right], \\
b &= \frac{1}{(1-\alpha_1)^2} \left[\alpha_0(\alpha_0 + l_1) + l_1 \alpha_0 E(L_0) + l_1^2 (E(L_0))^2 - (\alpha_0 + l_1 E(L_0)^2)\right] \\
&= \alpha_0 l_1 (1 - E(L_0)).
\end{aligned}$$

The conclusion of Proposition 3 is thus obtained. □

When the expectation of the noise satisfies $E(L_0) = 1$ (which is a natural assumption), we have the following behavior of the correlation function of the ARCH process.

Corollary 1. *Assume $E(L_0) = 1$. Then*

$$\alpha_1^{-k} Cov(y_t, y_{t-k}) \to_{k\to\infty} a$$

where a is defined by (13).

Proof: Is is an immediate consequence of Proposition 3 since $b = 0$ (b is given by (14)). □

Remark 4. *The ARCH process with liquidity has short memory when $E(L_0) = 1$ in the sense that*

$$\sum_{k\geq 0} |Cov(y_t, y_{t-k})| < \infty.$$

That is, it keeps the properties of the standard ARCH (1) model. If $E(L_0) \neq 0$, then clearly the above series is not convergent and the model has long memory.

3 Least squares estimator

We will use an idea from [1] in order to construct the Least Squares Estimator (LSE). Adding and subtracting X_t^2 to both sides of relation (2), we obtain

$$X_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + l_1 L_{t-1} + X_t^2 - \sigma_t^2$$

and since from (1), $X_t^2 = \sigma_t^2 \varepsilon_t^2$, we can write

$$\begin{aligned} X_t^2 &= \alpha_0 + \alpha_1 X_{t-1}^2 + l_1 L_{t-1} + \sigma_t^2(\varepsilon_t^2 - 1) \\ &= \alpha_0 + \alpha_1 X_{t-1}^2 + l_1 L_{t-1} + \eta_t \end{aligned}$$

where we denoted by

$$\eta_t = \sigma_t^2(\varepsilon_t^2 - 1). \tag{15}$$

The family $(\eta_t)_{t\in\mathbb{Z}}$ is a family of i.i.d. random variables, and for each t, the random variable $\varepsilon_t^2 - 1$ follows a chi-square distribution $\chi^2(1)$. Let us denote for every $t \in \mathbb{Z}$

$$Y_t = X_t^2, \qquad \alpha = (\alpha_0, \alpha_1, l_1), \qquad Z_t = (1, Y_{t-1}, L_{t-1}). \tag{16}$$

Then, with the notation (16), we have the vectorial expression

$$Y_t = Z_t^T \alpha + \eta_t \tag{17}$$

where Z_t^T denotes the transpose of the vector Z_t.

The purpose is to estimate the parameters α based on the observations $Z_1, ..., Z_N$. We will use a least squares method. The Least Squares Estimator is usually constructed by minimizing the quadratic error (here the error is interpreted as η_t)

$$\sum_{t=1}^{N} \eta_t^2 = \sum_{t=1}^{N} \left(\frac{Y_t - \alpha_0 - \alpha_1 Y_{t-1} - \ell_1 L_{t-1}}{\sigma_t^2} \right)^2.$$

By differentiating in the above relation with respect to $\alpha$and solving the equation $\frac{d}{d\alpha}\sum_{t=1}^{N}\eta_t^2 = 0$ we obtain the estimator

$$\widehat{\alpha} = \frac{\sum_{t=1}^{N} Z_t Y_t}{\sum_{t=1}^{N} Z_t Z_t^T} \tag{18}$$

and replacing in (18) Y_t by the right hand side of (17] we get

$$\widehat{\alpha} - \alpha = \frac{\sum_{t=1}^{N} Z_t \eta_t}{\sum_{t=1}^{N} Z_t Z_t^T}. \tag{19}$$

Proposition 4. *Assume that the sequence $(L_t)_{t\in\mathbb{Z}}$ in (2) is strongly mixing. Suppose (8) holds. Then the estimator $\widehat{\alpha}_N$ is strongly consistent, i.e. as $N \to \infty$*

$$\widehat{\alpha}_N \to \alpha \text{ almost surely.}$$

Moreover, $\widehat{\alpha}_N$ is asymptotically normal, that is, as $N \to \infty$,

$$N^{-\frac{1}{2}}(\widehat{\alpha}_N - \alpha) \to^d N(0, U_0 U_0^T)$$

where $\to^d$ stands for the convergence in distribution, $U_t := Z_t\eta_t$ for every $t \in \mathbb{Z}$ and $N(0, U_0U_0^T)$ is the Gaussian law with mean zero and variance $U_0U_0^T$.

Proof: Under (8), it follows from Proposition 1 that $(X_t)_{t\in\mathbb{Z}}$ is strictly stationary and ergodic. Since $(L_t)_{t\in\mathbb{Z}}$ in (2) is strongly mixing and for every $t \in \mathbb{Z}$, the random variable L_t is independent by X_t (from Proposition 1), we obtain that the vector $(X_t, L_t)_{t\in\mathbb{Z}}$ is also stationary and ergodic (see [7]). Then $Z_t\eta_t$ and $Z_tZ_t^T$ are also stationary and ergodic. It follows from the pointwise ergodic theorem for stationary sequences (see [11]) that almost surely

$$\frac{1}{N}\sum_{t=1}^{N} Z_t\eta_t \to_{N\to\infty} E(Z_0\eta_0) = 0$$

(from (15)) and

$$\frac{1}{N}\sum_{t=1}^{N} Z_tZ_t^T \to_{N\to\infty} E(Z_0Z_0^T) \neq 0.$$

This and (19) gives the strong consistency of the estimator. Let us regard the asymptotic distribution of the estimator (18). Denote $U_t = Z_t\eta_t$ for every $t \in \mathbb{Z}$. We first notice that $(U_t)_{t\in\mathbb{Z}}$ is a martingale difference sequence. Indeed, let consider $\mathcal{F}_t^U$ the sigma algebra generated by the random variables $U_j, j \leq t$ and $\mathcal{G}_t$ the sigma algebra generated by the random variables $((X_j)_{j\leq t}, (L_j)_{j\leq t+1})$. From Proposition 1 and our assumptions, ε_t is independent of $\mathcal{G}_{t-1}$ and thus $E\left(U_t|\mathcal{G}_{t-1}\right) = 0$ for every t. Since $\mathcal{F}_t^U \subset \mathcal{G}_t$ we obtain $E\left(U_t|\mathcal{F}_{t-1}^U\right) = 0$ for every t and therefore $(U_t)_{t\in\mathbb{Z}}$ is a martingale difference sequence. From the central limit theorem for stationary ergodic martingale differences (see e.g. [8]) we obtain that

$$\frac{1}{\sqrt{N}}\sum_{t=1}^{N} U_t \to N(0, E(U_0U_0^T))$$

and this concludes the proof. □

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