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# ON A WEIGHTED $K$-FUNCTIONAL

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**Abstract**

In this paper we give some results related to a $K$-functional defined for some seminormed subspaces of the Banach spaces of continuous functions.

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## 1 Introduction

Let $(X,d)$ be a compact metric space and let $C(X)$ denote the Banach space of all real-valued continuous functions on $X$ equipped with the supremum norm $\|f\| = \|f\|_X = \sup\limits_{x\in X} |f(x)|$, $f \in C(X)$.

For $f \in C(X)$ and $t > 0$, the modulus of continuity is defined by [6], [7]

$$\omega_d(f,t) = \sup \left\{ |f(x) - f(y)| \, | x, y \in X, \, d(x,y) \le t \right\}. \tag{1}$$

The Lipschitz space $Lip_d(X)$ (the space of all real-valued Lipschitz functions on $X$) is a seminormed subspace dense in $C(X)$ with the seminorm

$$|f|_{Lip_d(X)} = \sup_{d(x,y)>0} \frac{|f(x) - f(y)|}{d(x,y)}, \, f \in Lip_d(X). \tag{2}$$

For a seminormed subspace $(Y, |\cdot|_Y)$ dense in $C(X)$, the Peetre's $K$-functional is defined by

$$K\left(f,t;C(X),Y\right) = \inf_{g\in Y} \left\{ \|f-g\|_X + t\,|g|_Y \right\}, \, f \in C(X), \, t > 0. \tag{3}$$

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The Brudnyĭ's representation [13], [10], [5] is well known:

$$K\left(f, \frac{t}{2}; C(X), Lip_d(X)\right) = \frac{1}{2}\overline{\omega}_d(f,t),\ 0 < t \leq \delta(X), \tag{4}$$

where $\overline{\omega}_d(f,\cdot)$ is the least concave majorant of $\omega_d(f,\cdot)$ given by

$$\overline{\omega}_d(f,t) = \begin{cases} \sup\limits_{\substack{0\leq x\leq t\leq y\leq \delta(X)\\ x\neq y}} \dfrac{(t-x)\omega_d(f,y)+(y-t)\omega_d(f,x)}{y-x} & ,0 \leq t \leq \delta(X), \\ \omega_d(f,\delta(X)) & ,t > \delta(X), \end{cases}$$

$\delta(X)$ being the diameter of the compact space $X$.

For $X = [a,b] \subset \mathbb{R}$ endowed with the euclidian distance $d(x,y) = |x-y|$ we denote simply by $\omega$ the modulus of continuity and by $Lip[a,b]$ the Lipschitz space. In this case the following equality

$$K\left(f,t; C[a,b], Lip[a,b]\right) = K\left(f,t; C[a,b], C^1[a,b]\right), \tag{5}$$

holds [13], [5], [2], [12], where $C^1[a,b]$ is the space of all continuously differentiable functions on $[a,b]$ with the seminorm $|g|_{C^1[a,b]} = \|g'\|_{[a,b]}$, $g \in C^1[a,b]$.

In [3] (see also [4]), is given a generalized Shisha-Mond type inequality in terms of the $\varphi$-pseudomodulus of continuity

$$\omega_\varphi(f,t) = \sup\left\{|f(x)-f(y)|\,|x,y\in Q,\ |\varphi(x)-\varphi(y)| \leq t\right\},\ f\in C(Q),\ t>0,$$

where $Q$ is a connected compact Hausdorff space and $\varphi \in C(Q)$ fixed, $C(Q)$ being the space of real-valued continuous functions on $Q$.

In the following we consider $\varphi : [a,b] \longrightarrow [c,d]$ a strictly increasing and continuous one-to-one map. In this case the weighted modulus $\omega_\varphi$ is a particular case of the modulus $\omega_d$ with the distance $d(x,y) = |\varphi(x)-\varphi(y)|$ and the following equality

$$\omega_\varphi(f,t) = \omega\left(f\circ\varphi^{-1},t\right),\ f\in C[a,b],\ t>0, \tag{6}$$

holds [9]. For the $K$-functional $K\left(f,t;C[a,b],Lip_d[a,b]\right)$, $f \in C[a,b]$, $t > 0$ , in Section 2 we establish an equality of type (5) by considering a generalized derivatives. In Section 3 general estimates of the degree of approximation by positive linear operator in terms of this K-functional are established. The constants appearing in these estimates are optimal in the sense of [11].

## 2 The weighted $K$-functional

Let $\varphi : [a,b] \longrightarrow [c,d]$ be a strictly increasing and continuous one-to-one map. We denote by $Lip_\varphi[a,b]$ the Lipschitz space $Lip_d[a,b]$ for $d(x,y) = |\varphi(x)-\varphi(y)|$ ie

$$Lip_\varphi[a,b] = \left\{ f\in C[a,b]\,|\,|f|_{Lip_\varphi[a,b]} = \sup_{x\neq y}\frac{|f(x)-f(y)|}{|\varphi(x)-\varphi(y)|} < \infty \right\}.$$

**Lemma 1.** *We have*

$$f \in Lip_\varphi[a,b] \Longleftrightarrow f \circ \varphi^{-1} \in Lip[c,d];$$

*furthermore* $|f|_{Lip_\varphi[a,b]} = \left|f \circ \varphi^{-1}\right|_{Lip[c,d]}$.

*Proof.* We suppose that $f \in Lip_\varphi[a,b]$. For $u$, $v \in [c,d]$, $u \neq v$, we have

$$\frac{\left|\left(f \circ \varphi^{-1}\right)(u) - \left(f \circ \varphi^{-1}\right)(v)\right|}{|u-v|} = \frac{\left|f\left(\varphi^{-1}(u)\right) - f\left(\varphi^{-1}(v)\right)\right|}{\left|\varphi\left(\varphi^{-1}(u)\right) - \varphi\left(\varphi^{-1}(v)\right)\right|} \leq |f|_{Lip_\varphi[a,b]}.$$

Thus $f \circ \varphi^{-1} \in Lip[c,d]$ and $\left|f \circ \varphi^{-1}\right|_{Lip[c,d]} \leq |f|_{Lip_\varphi[a,b]}$.
Conversely, if $f \circ \varphi^{-1} \in Lip[c,d]$, then for $x$, $y \in [a,b]$, $x \neq y$, we have

$$\frac{|f(x) - f(y)|}{|\varphi(x) - \varphi(y)|} = \frac{\left|\left(f \circ \varphi^{-1}\right)(\varphi(x)) - \left(f \circ \varphi^{-1}\right)(\varphi(y))\right|}{|\varphi(x) - \varphi(y)|} \leq \left|f \circ \varphi^{-1}\right|_{Lip[c,d]}.$$

Therefore $f \in Lip_\varphi[a,b]$ and $|f|_{Lip_\varphi[a,b]} \leq \left|f \circ \varphi^{-1}\right|_{Lip[c,d]}$. □

**Proposition 1.** *The following equality*

$$K\left(f,t;C[a,b],Lip_\varphi[a,b]\right) = K\left(f \circ \varphi^{-1},t;C[c,d],Lip[c,d]\right) \tag{7}$$

*holds.*

*Proof.* Let $g \in Lip[c,d]$. Then $g \circ \varphi \in Lip_\varphi[a,b]$ and

$$\begin{aligned} K\left(f,t;C[a,b],Lip_\varphi[a,b]\right) &\leq \|f - g\circ\varphi\|_{[a,b]} + t\,|g\circ\varphi|_{Lip_\varphi[a,b]} \\ &= \left\|f\circ\varphi^{-1} - g\right\|_{[c,d]} + t\,|g|_{Lip[c,d]} \end{aligned}$$

Since $g$ is arbitrary it involves

$$K\left(f,t;C[a,b],Lip_\varphi[a,b]\right) \leq K\left(f \circ \varphi^{-1},t;C[c,d],Lip[c,d]\right).$$

For the reverse inequality, for $g \in Lip_\varphi[a,b]$, we have $g \circ \varphi^{-1} \in Lip[c,d]$ and

$$\begin{aligned} K\left(f\circ\varphi^{-1},t;C[c,d],Lip[c,d]\right) &\leq \left\|f\circ\varphi^{-1} - g\circ\varphi^{-1}\right\|_{[c,d]} + t\left|g\circ\varphi^{-1}\right|_{Lip[c,d]} \\ &= \|f-g\|_{[a,b]} + t\,|g|_{Lip_\varphi[a,b]} \end{aligned}$$

Since $g$ is arbitrary it involves

$$K\left(f \circ \varphi^{-1},t;C[c,d],Lip[c,d]\right) \leq K\left(f,t;C[a,b],Lip_\varphi[a,b]\right).$$

□

By following the paper [1] we consider the following generalization of differentiation:

**Definition 1.** *The function $f : [a,b] \longrightarrow \mathbb{R}$ is $\varphi$-differentiable in $x \in (a,b)$ if*

$$f'_\varphi(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{\varphi(x+h)-\varphi(x)} \tag{8}$$

*exists and it is finite.*

*We say a function $f$ is $\varphi$-differentiable on $[a,b]$ if $f$ is $\varphi$-differentiable at each $x \in (a,b)$ and exists (in $\mathbb{R}$)*

$$f'_\varphi(a) = \lim_{h\downarrow 0} \frac{f(a+h)-f(a)}{\varphi(a+h)-\varphi(a)}, \tag{9}$$

$$f'_\varphi(b) = \lim_{h\downarrow 0} \frac{f(b)-f(b-h)}{\varphi(b)-\varphi(b-h)}; \tag{10}$$

*in this case we denote by $f'_\varphi : [a,b] \longrightarrow \mathbb{R}$ the $\varphi$-derivative of $f$.*

We denote by

$$C^1_\varphi[a,b] = \left\{ f : [a,b] \longrightarrow \mathbb{R} | f \text{ is } \varphi\text{-differentiable on } [a,b] \text{ and } f'_\varphi \in C[a,b] \right\}$$

**Lemma 2.** *We have*

$$f \in C^1_\varphi[a,b] \Longleftrightarrow f\circ\varphi^{-1} \in C^1[c,d];$$

*furthermore* $\left\| f'_\varphi \right\|_{[a,b]} = \left\| \left( f\circ\varphi^{-1}\right)' \right\|_{[c,d]}$.

*Proof.* If $f \in C^1_\varphi[a,b]$, then for $y \in [c,d]$ we have

$$\begin{aligned} \lim_{h\to 0} \frac{\left(f\circ\varphi^{-1}\right)(y+h) - \left(f\circ\varphi^{-1}\right)(y)}{h} &= \lim_{h\to 0} \frac{f\left(\varphi^{-1}(y+h)\right) - f\left(\varphi^{-1}(y)\right)}{\varphi\left(\varphi^{-1}(y+h)\right) - \varphi\left(\varphi^{-1}(y)\right)} \\ &= f'_\varphi\left(\varphi^{-1}(y)\right) \end{aligned}$$

Thus $(f\circ\varphi^{-1})' = f'_\varphi \circ \varphi^{-1}$.
Conversely, if $f\circ\varphi^{-1} \in C^1[c,d]$, then for $x \in [a,b]$ we have

$$\begin{aligned} \lim_{h\to 0} \frac{f(x+h)-f(x)}{\varphi(x+h)-\varphi(x)} &= \lim_{h\to 0} \frac{\left(f\circ\varphi^{-1}\right)(\varphi(x+h)) - \left(f\circ\varphi^{-1}\right)(\varphi(x))}{\varphi(x+h)-\varphi(x)} \\ &= \left(f\circ\varphi^{-1}\right)'(\varphi(x)) \end{aligned}$$

Thus $f'_\varphi = \left(f\circ\varphi^{-1}\right)' \circ \varphi$. □

**Proposition 2.** *The following equality*

$$K\left(f,t;C[a,b],C^1_\varphi[a,b]\right) = K\left(f\circ\varphi^{-1},t;C[c,d],C^1[c,d]\right) \tag{11}$$

*holds.*

*Proof.* Let $g \in C^1[c,d]$. Then $g \circ \varphi \in C^1_\varphi[a,b]$ and

$$\begin{aligned} K\left(f,t;C[a,b],C^1_\varphi[a,b]\right) &\leq \|f-g\circ\varphi\|_{[a,b]} + t\left\|(g\circ\varphi)'_\varphi\right\|_{[a,b]} \\ &= \left\|f\circ\varphi^{-1}-g\right\|_{[c,d]} + t\left\|g'\right\|_{[c,d]}. \end{aligned}$$

Since $g$ is arbitrary this implies

$$K\left(f,t;C[a,b],C^1_\varphi[a,b]\right) \leq K\left(f\circ\varphi^{-1},t;C[c,d],C^1[c,d]\right).$$

For the reverse inequality, for $g \in C^1_\varphi[a,b]$, we have $g\circ\varphi^{-1} \in C^1[c,d]$ and

$$\begin{aligned} K\left(f\circ\varphi^{-1},t;C[c,d],C^1[c,d]\right) &\leq \left\|f\circ\varphi^{-1}-g\circ\varphi^{-1}\right\|_{[c,d]} + t\left\|\left(g\circ\varphi^{-1}\right)'\right\|_{[c,d]} \\ &= \|f-g\|_{[a,b]} + t\left\|g'_\varphi\right\|_{[a,b]}. \end{aligned}$$

Since $g$ is arbitrary this implies

$$K\left(f\circ\varphi^{-1},t;C[c,d],C^1[c,d]\right) \leq K\left(f,t;C[a,b],C^1_\varphi[a,b]\right).$$

□

From (7) and (11) we obtain

**Proposition 3.** *For any $f \in C[a,b]$ and any $t > 0$ we have*

$$K\left(f,t;C[a,b],C^1_\varphi[a,b]\right) = K\left(f,t;C[a,b],Lip_\varphi[a,b]\right). \tag{12}$$

## 3 Estimates

We agree to denote by $K_\varphi(f,t) = K\left(f,t;C[a,b],C^1_\varphi[a,b]\right)$, $f \in C[a,b]$, $t > 0$. We use the notation $e_k$ for the function $e_k(x) = x^k$, $k = 0,1$.

**Theorem 1.** *Let $L : \mathbf{C}[a,b] \longrightarrow \mathbf{C}[a,b]$ a positive linear operator and $f \in \mathbf{C}[a,b]$. Then $(\forall)x \in [a,b], (\forall)t > 0$ we have*

$$\begin{aligned} |L(f,x)-f(x)| &\leq |f(x)|\cdot|L(e_0,x)-1| \\ &+\max\left\{2L(e_0,x),\, \frac{L(|\varphi-\varphi(x)e_0|,x)}{t}\right\}K_\varphi(f,t). \end{aligned} \tag{13}$$

*Conversely, if $(\exists)A$, $B$, $C \geq 0$ such that*

$$\begin{aligned} |L(f,x)-f(x)| &\leq A\cdot|f(x)|\cdot|L(e_0,x)-1| \\ &+\max\left\{B\cdot L(e_0,x),\, C\cdot\frac{L(|\varphi-\varphi(x)e_0|,x)}{t}\right\}K_\varphi(f,t) \end{aligned} \tag{14}$$

*holds for all positive linear operator, any $f \in \mathbf{C}[a,b]$, any $x \in [a,b]$ and any $t > 0$, then $A \geq 1$, $C \geq 1$ and $B \geq 2$.*

*Proof.* Let $g \in C^1_\varphi[a,b]$. Then $g \circ \varphi^{-1} \in C^1[c,d]$ and

$$\begin{aligned} |g(y)-g(x)| &= \left|\left(g\circ\varphi^{-1}\right)(\varphi(y)) - \left(g\circ\varphi^{-1}\right)(\varphi(x))\right| \\ &= \left|\left(g\circ\varphi^{-1}\right)'(\varphi(\xi))\right| \cdot |\varphi(y)-\varphi(x)| \\ &\leq \left\|\left(g\circ\varphi^{-1}\right)'\right\| \cdot |\varphi(y)-\varphi(x)| \\ &= \left\|g'_\varphi\right\| \cdot |\varphi(y)-\varphi(x)|, \end{aligned}$$

where $\xi$ is between $x$ and $y$.

We have

$$\begin{aligned} &|L(f,x)-f(x)| \leq |f(x)| \cdot |L(e_0,x)-1| + L\left(|f-f(x)e_0|, x\right) \\ &\leq |f(x)| \cdot |L(e_0,x)-1| + L\left(2\|f-g\| e_0 + \left\|g'_\varphi\right\| \cdot |\varphi - \varphi(x)e_0|, x\right) \\ &\leq |f(x)| \cdot |L(e_0,x)-1| \\ &+ \max\left\{2L(e_0,x), \frac{L\left(|\varphi-\varphi(x)e_0|, x\right)}{t}\right\} \cdot \left\{\|f-g\| + t\left\|g'_\varphi\right\|\right\}. \end{aligned}$$

Since $g$ is arbitrary this implies (13).

If we choose $L(h,x)=0$ and $f=e_0$ and replace in (14) we obtain $A \geq 1$.

If we choose $[a,b]=[0,1]$, $L(h,x)=h(1)$, $f=\varphi$, $x=0$ and replace in (14) we obtain $\varphi(1)-\varphi(0) \leq \max\left\{Bt, C\left(\varphi(1)-\varphi(0)\right)\right\} \leq Bt + C\left(\varphi(1)-\varphi(0)\right)$, $(\forall)t>0$ (we use $K_\varphi(f,t) = K(f\circ\varphi^{-1},t) = K(e_1,t) \leq t$). Passing to the limit $t \to 0$ we obtain $C \geq 1$.

To show that $B \geq 2$ we choose $[a,b]=[0,1]$, $L(h,x) = \left(h\circ\psi^{-1}\right)(1)$, $f = 2\psi - e_0$, $x=0$, where $\psi = \frac{\varphi-\varphi(0)e_0}{\varphi(1)-\varphi(0)}$. We have $L(f,0)=1$, $f(0)=2\psi(0)-1=-1$, $L(e_0,0)=1$, $L\left(|\varphi-\varphi(0)|, 0\right) = L\left(\left(\varphi(1)-\varphi(0)\right)\psi, 0\right) = \varphi(1)-\varphi(0)$ and

$$K_\varphi(f,t) \leq \|f\| = \|2\psi - e_0\| = 1.$$

Replace in (14) and we obtain $2 \leq B + \dfrac{C(\varphi(1)-\varphi(0))}{t}$, $(\forall)t>0$. Passing to the limit $t \to \infty$ we obtain $B \geq 2$. □

**Theorem 2.** *Let* $L : \mathbf{C}[a,b] \longrightarrow \mathbf{C}[a,b]$ *a positive linear operator and* $f \in C^1_\varphi[a,b]$. *Then* $(\forall)x \in [a,b]$, $(\forall)t > 0$ *we have*

$$\begin{aligned} &|L(f,x)-f(x)| \leq |f(x)| \cdot |L(e_0,x)-1| \\ &\quad + \left|f'_\varphi(x)\right| \cdot |L(\varphi-\varphi(x)e_0, x)| \\ &\quad + \max\left\{2L\left(|\varphi-\varphi(x)e_0|, x\right), \frac{L\left(\left(\varphi-\varphi(x)e_0\right)^2, x\right)}{2t}\right\} \cdot K_\varphi\left(f'_\varphi, t\right) \end{aligned} \tag{15}$$

*Conversely, if* $(\exists)A_0$, $A_1$, $B$, $C \geq 0$ *such that*

$$|L(f,x)-f(x)| \leq A_0 \cdot |f(x)| \cdot |L(e_0,x)-1|$$

$$+A_1 \cdot \left|f'_\varphi(x)\right| \cdot \left|L\left(\varphi - \varphi(x)e_0, x\right)\right| \tag{16}$$
$$+\max\left\{B \cdot L\left(\left|\varphi - \varphi(x)e_0\right|, x\right), C \cdot \frac{L\left(\left(\varphi - \varphi(x)e_0\right)^2, x\right)}{t}\right\} \cdot K_\varphi\left(f'_\varphi, t\right)$$

*holds for all positive linear operator* $L : \mathbf{C}[a,b] \longrightarrow \mathbf{C}[a,b]$, *any* $f \in C^1_\varphi[a,b]$, *any* $x \in [a,b]$ *and any* $t > 0$ *then* $A_0$, $A_1 \geq 1$, $B \geq 2$ *and* $C \geq \frac{1}{2}$.

*Proof.* Let $g \in C[a,b]$ such that $g \circ \varphi^{-1} \in C^2[c,d]$. Then $g'_\varphi \in C^1_\varphi[a,b]$ and $\left(g'_\varphi\right)'_\varphi = \left(g \circ \varphi^{-1}\right)'' \circ \varphi$. Following the method from [8] we obtain

$$\begin{aligned}
&\left|f(y) - f(x) - f'_\varphi(x)\left(\varphi(y) - \varphi(x)\right)\right| \\
&\leq \left|(f-g)(y) - (f-g)(x) - (f-g)'_\varphi(x)\left(\varphi(y) - \varphi(x)\right)\right| \\
&+ \left|g(y) - g(x) - g'_\varphi(x)\left(\varphi(y) - \varphi(x)\right)\right| \\
&= \left|(f-g)'_\varphi(\eta) - (f-g)'_\varphi(x)\right| \cdot \left|\varphi(y) - \varphi(x)\right| \\
&+ \left|\left(g \circ \varphi^{-1}\right)(\varphi(y)) - \left(g \circ \varphi^{-1}\right)(\varphi(x)) - \left(g \circ \varphi^{-1}\right)'(\varphi(x))\left(\varphi(y) - \varphi(x)\right)\right| \\
&\leq 2\left\|f'_\varphi - g'_\varphi\right\| \cdot \left|\varphi(y) - \varphi(x)\right| + \frac{\left|\left(g \circ \varphi^{-1}\right)''(\varphi(\mu))\right|}{2}\left(\varphi(y) - \varphi(x)\right)^2 \\
&= 2\left\|f'_\varphi - g'_\varphi\right\| \cdot \left|\varphi(y) - \varphi(x)\right| + \frac{\left|\left(g'_\varphi\right)'_\varphi(\mu)\right|}{2}\left(\varphi(y) - \varphi(x)\right)^2 \\
&\leq 2\left\|f'_\varphi - g'_\varphi\right\| \left|\varphi(y) - \varphi(x)\right| + \frac{\left\|\left(g'_\varphi\right)'_\varphi\right\|}{2}\left(\varphi(y) - \varphi(x)\right)^2
\end{aligned}$$

where $\eta$ and $\mu$ are between $x$ and $y$. Then

$$\begin{aligned}
&L\left(\left|f(y) - f(x) - f'_\varphi(x)\left(\varphi(y) - \varphi(x)\right)\right|, x\right) \\
&\leq 2\left\|f'_\varphi - g'_\varphi\right\| L\left(\left|\varphi - \varphi(x)e_0\right|, x\right) + \frac{\left\|\left(g'_\varphi\right)'_\varphi\right\|}{2} L\left(\left(\varphi - \varphi(x)e_0\right)^2, x\right) \\
&\leq \max\left\{2L\left(\left|\varphi - \varphi(x)e_0\right|, x\right), \frac{L\left(\left(\varphi - \varphi(x)e_0\right)^2, x\right)}{2t}\right\} \times \\
&\times \left\{\left\|f'_\varphi - g'_\varphi\right\| + t\left\|\left(g'_\varphi\right)'_\varphi\right\|\right\}
\end{aligned}$$

from where

$$\begin{aligned}
&L\left(\left|f(y) - f(x) - f'_\varphi(x)\left(\varphi(y) - \varphi(x)\right)\right|, x\right) \\
&\leq \max\left\{2L\left(\left|\varphi - \varphi(x)e_0\right|, x\right), \frac{L\left(\left(\varphi - \varphi(x)e_0\right)^2, x\right)}{2t}\right\} \cdot K_\varphi\left(f'_\varphi, t\right).
\end{aligned}$$

So

$$
\begin{aligned}
&|L(f,x)-f(x)| \le |f(x)|\cdot|L(e_0,x)-1| + \left|f'_\varphi(x)\right|\cdot|L((\varphi-\varphi(x)e_0),x)|\\
&+L\left(\left|f-f(x)e_0-f'_\varphi(x)(\varphi-\varphi(x)e_0)\right|,x\right)\\
&\le |f(x)|\cdot|L(e_0,x)-1| + \left|f'_\varphi(x)\right|\cdot|L((\varphi-\varphi(x)e_0),x)|\\
&\quad+\max\left\{2L(|\varphi-\varphi(x)e_0|,x),\frac{L\left((\varphi-\varphi(x)e_0)^2,x\right)}{2t}\right\}\cdot K_\varphi\left(f'_\varphi,t\right).
\end{aligned}
$$

If we choose $L(h,x)=0$ and $f=e_0$ and replace in (16) we obtain $A_0 \ge 1$.

If we choose $[a,b]=[0,1]$, $L(h,x)=h(1)$, $f=\varphi$, $x=0$ and replace in (16) we obtain $A_1 \ge 1$.

To show that $C \ge \frac{1}{2}$ we choose $[a,b]=[0,1]$, $L(h,x)=h(1)$, $f=\psi^2$ and $x=0$, where $\psi=\varphi-\varphi(0)e_0$. We have $L(f,0)=(\varphi(1)-\varphi(0))^2$, $f(0)=\psi(0)^2=0$, $f'_\varphi=f'_\psi=(f\circ\psi^{-1})'\circ\psi=2\psi$ and

$$
K_\varphi\left(f'_\varphi,t\right)=K_\psi\left(f'_\psi,t\right)=K\left(f'_\psi\circ\psi^{-1},t\right)=K(2e_1,t)\le 2t.
$$

From (16) we obtain $(\varphi(1)-\varphi(0))^2 \le 2Bt(\varphi(1)-\varphi(0))+2C(\varphi(1)-\varphi(0))^2\ (\forall)t>0$. Passing to the limit $t\to 0$ we obtain $C\ge\frac{1}{2}$.

To show that $B\ge 2$ we choose $[a,b]=[0,1]$, $L(h,x)=h(1)$, $f(x)=2\psi(x)^{\alpha+1}$ with $\alpha>0$ and $x=0$, where $\psi=\frac{\varphi-\varphi(0)e_0}{\varphi(1)-\varphi(0)}$. We have $L(f,0)=2\psi(1)^{\alpha+1}=2$, $f(0)=2\psi(0)^{\alpha+1}=0$, $f'_\varphi=(f\circ\varphi^{-1})'\circ\varphi=\frac{2(\alpha+1)\psi^\alpha}{\varphi(1)-\varphi(0)}$ and

$$
K_\varphi\left(f'_\varphi,t\right)\le\left\|\frac{2(\alpha+1)\psi^\alpha}{\varphi(1)-\varphi(0)}-\frac{(\alpha+1)e_0}{\varphi(1)-\varphi(0)}\right\|=\frac{\alpha+1}{\varphi(1)-\varphi(0)}.
$$

From (16) we obtain

$$
2\le\left[B(\varphi(1)-\varphi(0))+\frac{C}{t}(\varphi(1)-\varphi(0))^2\right]\frac{\alpha+1}{\varphi(1)-\varphi(0)},\ (\forall)t>0.
$$

Passing to the limit $t\to\infty$ and $\alpha\to 0$ we obtain $B\ge 2$. □

**Remark 1.** *In particular, for $\varphi=e_1$ we obtain the results given in [14].*

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## References


[1] Amat, S., Busquier, S., Escudero, A., Trillo, J. C., *Lagrange interpolation for continuous piecewise smooth functions*, Journal of Computational and Applied Mathematics 221(2008), 47–51

[2] Anastassiou, G.A., Gal, S.G., *Approximation Theory. Moduli of Continuity and Global Smoothness Preservation*, Birkhäuser, Boston, Basel, Berlin, 2000.

[3] Anastassiou, G.A., *Generalized Shisha-Mond type inequality*, Anals of Univ. in Oradea, Fascicola Mathematica, Tom XV(2008), 141–143

[4] Anastassiou, G.A., *Intelligent Mathematics: Computational Analysis*, Springer, 2011

[5] Brudnyĭ, Yu. A., Krugljak, N. Ya., *Interpolation Functors and Interpolation Spaces*, Vol. I, North-Holland, Amsterdam, 1991

[6] Cheney, E.W., Introduction to Approximation Theory, AMS Chelsea Publishing, 1966

[7] Gonska, H.H., *On approximation in spaces of continuous functions*, Bull. Austral. Math. Soc., 28(1983), 411–432

[8] Gonska, H. H., Piţul, P., Raşa, I., *On Peano's form of the Taylor remainder, Voronovskaja's theorem and the commutator of positive linear operators*, Numerical Analysis and Approximation Theory (Proc. Int. Conf. Cluj-Napoca 2006, ed. by O. Agratini and P. Blaga), 55-80

[9] Holhoş, A., *The rate of convergence of positive linear operators in weighted spaces*, Automat. Comput. Appl. Math., 17(2008), no. 2, 239–246

[10] Mitjagin, B. S. , Semenov, E. M., *Lack of interpolation of linear operators in spaces of smooth functions*, Math. USSR-Izv. 11(1977), 1229-1266

[11] Păltănea, R., *Approximation theory using positive linear operators*, Birkhäuser, 2004

[12] Păltănea, R., *Representation of the K-functional* $K\left(f, C[a,b], C^1[a,b], \cdot\right)$ *- a new approach*, Bulletin of the Transilvania University of Braşov, Vol **3(52)** (2010), 93-100.

[13] Peetre, J., *Exact interpolation theorems for Lipschitz continuous functions*, Ricerce Math. 18(1969), 239-259

[14] Talpău Dimitriu, M., *Estimates with optimal constants using Peetre's K-functionals*, Carpathian J. Math. **26** (2010), No. 2, 158 - 169