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GAUSS-WEINGARTEN AND FRENET EQUATIONS IN THE THEORY OF THE HOMOGENEOUS LIFT TO THE 2-OSCULATOR BUNDLE OF A FINSLER METRIC

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Abstract

In this article we present a study of the subspaces of the manifold Osc^2M, the total space of the 2-osculator bundle of a real manifold M. We obtain the induced connections of the canonical N-linear metric connection determined by the homogeneous prolongation of a Finsler metric to the manifold Osc^2M. We present the Gauss-Weingarten equations of the associated 2-osculator submanifold. We construct a Frenet frame and we determine the Frenet equations of a curve from the manifold Osc^2M.

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1 Indroduction

The Sasaki N-prolongation $\mathbb{G}$ to the 2-osculator bundle without the null section $\widetilde{Osc^2M}$ $= Osc^2M\setminus\{0\}$ of a Finslerian metric g_{ab} on the real manifold M given by

$$\mathbb{G} = g_{ab}\left(x, y^{(1)}\right) dx^a \otimes dx^b + g_{ab}\left(x, y^{(1)}\right) \delta y^{(1)a} \otimes \delta y^{(1)b} + g_{ab}\left(x, y^{(1)}\right) \delta y^{(2)a} \otimes \delta y^{(2)b} \quad (*)$$

is a Riemannian structure on $\widetilde{Osc^2M}$, which depends only on the metric g_{ab}.

The tensor $\mathbb{G}$ is not invariant with respect to the homothetis on the fibres of $\widetilde{Osc^2M}$, because $\mathbb{G}$ is not homogeneous with respect to the variable $y^{(1)a}$.

In this paper, we use a new kind of prolongation $\mathring{\mathbb{G}}$ to $\widetilde{Osc^2M}$, ([7]), which depends only on the metric g_{ab}. Thus, $\mathring{\mathbb{G}}$ determines on the manifold $\widetilde{Osc^2M}$ a Riemannian structure which is 0-homogeneous on the fibres of Osc^2M.

Some geometrical properties of $\mathring{\mathbb{G}}$ are studied: the canonical N-linear metric connection, the induced linear connections , Gauss-Weingarten and Frenet equations.

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2 Preliminaries

As far we know the general theory of submanifolds (in particular the Finsler submanifolds or the complex Finsler submanifolds) is far from being settled ([9], [3], [10], [11]). In [8] and [9] R.Miron and M. Anastasiei give the theory of subspaces in generalized Lagrange spaces. Also, in [6] and [5] R. Miron presented the theory of subspaces in higher order Finsler and Lagrange spaces respectively.

If $\check{M}$ is an immersed manifold in manifold M, a nonlinear connection on Osc^2M induce a nonlinear connection $\check{N}$ on $Osc^2\check{M}$.

The d-tensor $\mathbb{G}$ from (*) is not homogeneous with respect to the variable $y^{(1)a}$. This in an incovenient from the point of view of analytical mechanics. Moreover, the physical dimensions of the terms of $\mathbb{G}$ are not the same. This disavantaj was corected by Gh. Atanasiu. He taked a new kind of prolongation $\overset{\circ}{\mathbb{G}}$ to $\widetilde{Osc^2M}$ of the fundamental tensor of a Finsler space, [1], which depends only on the metric g_{ab}. Thus, $\overset{\circ}{\mathbb{G}}$ determines on the manifold $\widetilde{Osc^2M}$ a Riemannian structure which is 0-homogeneous on the fibres of Osc^2M and p is a positive constant required by applications in order that the physical dimensions of the terms of $\overset{\circ}{\mathbb{G}}$ be the same. He proved that there exist metrical N-linear connections with respect to the metric tensor $\overset{\circ}{\mathbb{G}}$.

We take this canonical N-linear metric connection D on the manifold Osc^2M and obtain the induced tangent and normal connections and the relative covariant derivation in the algebra of d-tensor fields .

In this paper we get the Gauss-Weingarten formulae of submanifold $Osc^2\check{M}$ for the homogeneos lift $\overset{\circ}{\mathbb{G}}$ and we construct a Frenet frame and we determine the Frenet equations of a curve from the manifold Osc^2M.

Let us consider the Finsler space $F^n = (M, F)$ ([9]) with the fundamental function $F : TM = OscM \to \mathbb{R}$ and the fundamental tensor $g_{ab}\left(x, y^{(1)}\right)$ on $\widetilde{OscM}$, given by

$$g_{ab}\left(x, y^{(1)}\right) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^{(1)a}\partial y^{(1)b}}, \tag{2.1}$$

where $g_{ab}\left(x, y^{(1)}\right)$ is positively defined on $\widetilde{OscM}$.

The canonical 2-spray of F^n is given by

$$\frac{d^2x^a}{dt^2} + 2G^a\left(x, \frac{dx}{dt}\right) = 0$$

where

$$G^a = \frac{1}{2}\gamma^a_{bc}\left(x, y^{(1)}\right) y^{(1)b} y^{(1)c} \tag{2.2}$$

where $\gamma^a_{bc}\left(x, y^{(1)}\right)$ are the Christoffels symbols of the metric tensor $g_{ab}\left(x, y^{(1)}\right)$. The canonical nonlinear connection N of the space F^n has the dual coefficients [5]

$$\underset{(1)}{M}{}^a{}_b = \frac{\partial G^a}{\partial y^{(1)b}},\ \underset{(2)}{M}{}^a{}_b = \frac{1}{2}\left\{\Gamma \underset{(1)}{M}{}^a{}_b + \underset{(1)}{M}{}^a{}_c \underset{(1)}{M}{}^c{}_b\right\}, \tag{2.3}$$

where $\Gamma = y^{(1)a}\frac{\partial}{\partial x^a} + 2y^{(2)a}\frac{\partial}{\partial y^{(1)a}}$.

We have the next decomposition

$$T_w Osc^2 M = N_0(w) \oplus N_1(w) \oplus V_2(w), \forall w \in Osc^2 M. \tag{2.4}$$

The adapted basis to (2.4) is given by $\left\{\frac{\delta}{\delta x^a}, \frac{\delta}{\delta y^{(1)a}}, \frac{\partial}{\partial y^{(2)a}}\right\}$, $(a = 1, .., n)$ and its dual basis is $\left(dx^a, \delta y^{(1)a}, \delta y^{(2)a}\right)$, where

$$\begin{cases} \frac{\delta}{\delta x^a} = \frac{\partial}{\partial x^a} - \underset{(1)}{N}{}^b{}_a \frac{\delta}{\delta y^{(1)b}} - \underset{(2)}{N}{}^b{}_a \frac{\partial}{\partial y^{(2)b}} \\ \frac{\delta}{\delta y^{(1)a}} = \frac{\partial}{\partial y^{(1)a}} - \underset{(1)}{N}{}^b{}_a \frac{\partial}{\partial y^{(2)b}} \end{cases} \tag{2.5}$$

and

$$\begin{cases} \delta y^{(1)a} = dy^{(1)a} + \underset{(1)}{M}{}^a{}_b dx^b \\ \delta y^{(2)a} = dy^{(2)a} + \underset{(1)}{M}{}^a{}_b \delta y^b + \underset{(2)}{M}{}^a{}_b \delta y^{(2)b} \end{cases} \tag{2.6}$$

We use the next notations:

$$\delta_a = \frac{\delta}{\delta x^a}, \delta_{1a} = \frac{\delta}{\delta y^{(1)a}} \ \dot{\partial}_{2a} = \frac{\partial}{\partial y^{(2)a}}.$$

Proposition 2.1. *The Lie brakets of the vector fields* $\left\{\frac{\delta}{\delta x^a}, \frac{\delta}{\delta y^{(1)a}}, \frac{\partial}{\partial y^{(2)a}}\right\}$ *are given by*

$$\begin{aligned} [\delta_b, \delta_c] &= \underset{(01)}{R}{}^a_{bc}\delta_{1a} + \underset{(02)}{R}{}^a_{bc}\dot{\partial}_{2a}, \\ [\delta_b, \delta_{1c}] &= \underset{(11)}{B}{}^a_{bc}\delta_{1a} + \underset{(12)}{B}{}^a_{bc}\dot{\partial}_{2a}, \\ \left[\delta_b, \dot{\partial}_{2c}\right] &= \underset{(21)}{B}{}^a_{bc}\delta_{1a} + \underset{(22)}{B}{}^a_{bc}\dot{\partial}_{2a}, \\ [\delta_{1b}, \delta_{1c}] &= \underset{(12)}{R}{}^a_{bc}\dot{\partial}_{2a}, \\ \left[\delta_{1b}, \dot{\partial}_{2c}\right] &= \underset{(21)}{B}{}^a_{bc}\dot{\partial}_{2a}, \end{aligned} \tag{2.7}$$

where

$$\begin{aligned}
\underset{(01)}{R}{}^{a}_{bc} &= \delta_c \underset{1}{N}{}^a{}_b - \delta_b \underset{1}{N}{}^a{}_c, \\
\underset{(02)}{R}{}^{a}_{bc} &= \delta_c \underset{2}{N}{}^a{}_b - \delta_b \underset{2}{N}{}^a{}_c + \underset{1}{N}{}^a{}_f \underset{(01)}{R}{}^{f}_{bc}, \\
\underset{(11)}{B}{}^{a}_{bc} &= \delta_{1c} \underset{1}{N}{}^a{}_b, \ \underset{(12)}{B}{}^{a}_{bc} = \delta_{1c} \underset{2}{N}{}^a{}_b - \delta_b \underset{1}{N}{}^a{}_c + \underset{1}{N}{}^a{}_f \underset{(11)}{B}{}^{f}_{bc}, \qquad (2.8) \\
\underset{(21)}{B}{}^{a}_{bc} &= \dot{\partial}_{2c} \underset{1}{N}{}^a{}_b, \ \underset{(22)}{B}{}^{a}_{bc} = \dot{\partial}_{2c} \underset{2}{N}{}^a{}_b + \underset{1}{N}{}^a{}_f \underset{(21)}{B}{}^{f}_{bc}, \\
\underset{(12)}{R}{}^{a}_{bc} &= \delta_{1c} \underset{1}{N}{}^a{}_b - \delta_{1b} \underset{1}{N}{}^a{}_c.
\end{aligned}$$

The fundamental tensor g_{ab} determines on the manifold $\widetilde{Osc^2 M}$ the homogeneous tensor field $\overset{0}{\mathbb{G}}$, [1],

$$\begin{aligned}
\overset{0}{\mathbb{G}} = g_{ab}\left(x, y^{(1)}\right) dx^a \otimes dx^b + \underset{(1)}{g}{}_{ab}\left(x, y^{(1)}\right) \delta y^{(1)a} \otimes \delta y^{(1)b} + \\
+ \underset{(2)}{g}{}_{ab}\left(x, y^{(1)}\right) \delta y^{(2)a} \otimes \delta y^{(2)b}, \qquad (2.9)
\end{aligned}$$

where

$$\underset{(1)}{g}{}_{ab}\left(x, y^{(1)}\right) = \frac{p^2}{\left\|y^{(1)}\right\|^2} g_{ab}\left(x, y^{(1)}\right),$$

$$\underset{(2)}{g}{}_{ab}\left(x, y^{(1)}\right) = \frac{p^4}{\left\|y^{(1)}\right\|^4} g_{ab}\left(x, y^{(1)}\right),$$

$$\left\|y^{(1)}\right\|^2 = g_{ab} y^{(1)a} y^{(1)b}.$$

This is homogeneous tensor field with respect to $y^{(1)a}$, $y^{(2)a}$ and p is a positive constant required by applications in order that the physical dimensions of the terms of $\overset{\circ}{\mathbb{G}}$ be the same.

Let $\check{M}$ be a real, m-dimensional manifold, immersed in M through the immersion $i : \check{M} \to M$. Localy, i can be given in the form

$$x^a = x^a\left(u^1, ..., u^m\right), \qquad rank \left\| \frac{\partial x^a}{\partial u^\alpha} \right\| = m.$$

The indices $a, b, c,$....run over the set $\{1, ..., n\}$ and $\alpha, \beta, \gamma, ...$ run on the set $\{1, ..., m\}$. We assume $1 \leq m < n$. We take the immersed submanifold $Osc^2\check{M}$ of the manifold $Osc^2 M$, by the immersion $Osc^2 i : Osc^2\check{M} \to Osc^2 M$. The parametric equations of the submanifold $Osc^2\check{M}$ are

$$\begin{cases} x^a = x^a\left(u^1, ..., u^m\right), rank \left\| \frac{\partial x^a}{\partial u^\alpha} \right\| = m \\ y^{(1)a} = \frac{\partial x^a}{\partial u^\alpha} v^{(1)\alpha} \\ 2y^{(2)a} = \frac{\partial y^{(1)a}}{\partial u^\alpha} v^{(1)\alpha} + 2\frac{\partial y^{(1)a}}{\partial v^{(1)\alpha}} v^{(2)\alpha}, \end{cases} \tag{2.10}$$

where

$$\begin{cases} \frac{\partial x^a}{\partial u^\alpha} = \frac{\partial y^{(1)a}}{\partial v^{(1)\alpha}} = \frac{\partial y^{(2)a}}{\partial v^{(2)\alpha}} \\ \frac{\partial y^{(1)a}}{\partial u^\alpha} = \frac{\partial y^{(2)a}}{\partial v^{(1)\alpha}}. \end{cases}$$

The restriction of the fundamental function F to the submanifold $\widetilde{Osc\check{M}}$ is

$$\check{F}\left(u, v^{(1)}\right) = F\left(x\left(u\right), y\left(u, v^{(1)}, v^{(2)}\right)\right)$$

and we call $\check{F}^m = \left(\check{M}, \check{F}\right)$ the **induced Finsler subspaces** of F^n and $\check{F}$ the **induced fundamental function**.

Let $B^a_\alpha(u) = \frac{\partial x^a}{\partial u^\alpha}$ and $g_{\alpha\beta}$ the induced fundamental tensor,

$$g_{\alpha\beta}\left(u, v^{(1)}\right) = g_{ab}\left(x\left(u\right), y\left(u, v^{(1)}\right)\right) B^a_\alpha B^b_\beta. \tag{2.11}$$

We obtain a system of d-vectors $\{B^a_\alpha, B^a_{\bar{\alpha}}\}$ wich determines a moving frame $\mathcal{R} = \left\{\left(u, v^{(1)}, v^{(2)}\right); B^a_\alpha\left(u\right), B^a_{\bar{\alpha}}\left(u, v^{(1)}, v^{(2)}\right)\right\}$ in Osc^2M along to the submanifold $Osc^2\check{M}$.

Its dual frame will be denoted by $\mathcal{R}^* = \left\{B^\alpha_a\left(u, v^{(1)}, v^{(2)}\right), B^{\bar{\alpha}}_a\left(u, v^{(1)}, v^{(2)}\right)\right\}$. This is also defined on an open set $\check{\pi}^{-1}\left(\check{U}\right) \subset Osc^2\check{M}$, $\check{U}$ being a domain of a local chart on the submanifold $\check{M}$.

The conditions of duality are given by:

$$B^a_\beta B^\alpha_a = \delta^\alpha_\beta, \quad B^a_\beta B^{\bar{\alpha}}_a = 0, \quad B^\alpha_a B^a_{\bar{\beta}} = 0, \quad B^{\bar{\alpha}}_a B^a_{\bar{\beta}} = \delta^{\bar{\alpha}}_{\bar{\beta}}$$

$$B^a_\alpha B^\alpha_b + B^a_{\bar{\alpha}} B^{\bar{\alpha}}_b = \delta^a_b.$$

The restriction of the of the nonlinear connection N to $\widetilde{Osc^2\check{M}}$ uniquely determines an induced nonlinear connection $\check{N}$ on $\widetilde{Osc^2\check{M}}$ with the dual coeficients ([2],[13])

$$\begin{aligned} \check{\underset{1}{M}}{}^\alpha{}_\beta &= B^\alpha_a\left(B^a_{0\beta} + \underset{1}{M}{}^a{}_b B^b_\beta\right), \\ \check{\underset{2}{M}}{}^\alpha{}_\beta &= B^\alpha_a\left(\frac{1}{2}\frac{\partial B^a_{\delta\gamma}}{\partial u^\beta} v^{(1)\delta} v^{(1)\gamma} + B^a_{\delta\beta} v^{(2)\delta} + \underset{1}{M}{}^a{}_b B^b_{0\beta} + \underset{2}{M}{}^a{}_b B^b_\beta\right), \end{aligned} \tag{2.12}$$

where $\underset{1}{M}{}^a{}_b$, $\underset{2}{M}{}^a{}_b$ are the dual coeficients of the N.

The cobasis $\left(dx^i, \delta y^{(1)a}, \delta y^{(2)a}\right)$ restricted to $Osc^2\check{M}$ is uniquely represented in the moving frame $\mathcal{R}$ in the following form ([2], [12]):

$$\begin{cases} dx^a = B^a_\beta du^\beta \\ \delta y^{(1)a} = B^a_\alpha \delta v^{(1)\alpha} + B^a_{\bar{\alpha}} \underset{(1)}{K}{}^{\bar{\alpha}}_\beta du^\beta \\ \delta y^{(2)a} = B^a_\alpha \delta v^{(2)\alpha} + B^a_{\bar{\beta}} \underset{(1)}{K}{}^{\bar{\beta}}_\alpha \delta v^{(1)\alpha} + B^a_{\bar{\beta}} \underset{(2)}{K}{}^{\bar{\beta}}_\alpha du^\alpha \end{cases} \tag{2.13}$$

where

$$\underset{(1)}{K}{}^{\bar{\alpha}}_\beta = B^{\bar{\alpha}}_a \left(B^a_{0\beta} + \underset{(1)}{M}{}^a_b B^b_\beta \right)$$

$$\underset{(2)}{K}{}^{\bar{\alpha}}_\beta = B^{\bar{\alpha}}_a \left(\frac{1}{2} \frac{\partial B^a_{\delta\gamma}}{\partial u^\beta} v^{(1)\delta} v^{(1)\gamma} + B^b_{\delta\beta} v^{(2)\delta} + \underset{(1)}{M}{}^a_b B^b_{0\beta} + \underset{(2)}{M}{}^a_b B^b_\beta - \right. \tag{2.14}$$

$$\left. - B^{\bar{\alpha}}_f B^\gamma_d \left(B^f_\gamma + \underset{(1)}{M}{}^f_b B^b_\gamma \right) \left(B^d_{0\beta} + \underset{(1)}{M}{}^d_g B^g_\beta \right) \right)$$

are mixed d-tensor fields.

A linear connection D on the manifold Osc^2M is called **metrical N-linear connection** with respect to $\mathring{\mathbb{G}}$, if $D\mathring{\mathbb{G}} = 0$ and D preserves by parallelism the distributions N_0, N_1 and V_2. The coefficients of the N-linear connections $D\Gamma(N)$ will be denoted with $\left(\underset{(i0)}{\overset{V_i}{L}}{}^a_{bc}, \underset{(i1)}{\overset{V_i}{C}}{}^a_{bc}, \underset{(i2)}{\overset{V_i}{C}}{}^a_{bc} \right), (i = 0, 1, 2)$.

Theorem 2.2. *([1]) There exist metrical N-linear connections $D\Gamma(N)$ on $\widetilde{Osc^2M}$, with respect to the homogeneous prolongation $\mathring{\mathbb{G}}$, wich depend only on the metric $g_{ab}\left(x, y^{(1)}\right)$. One of these connections has the "horizontal" coefficients*

$$\underset{(00)}{\overset{H}{L}}{}^a_{bc} = \frac{1}{2} g^{ad} \left(\delta_b g_{cd} + \delta_c g_{bd} - \delta_d g_{bc} \right)$$

$$\underset{(10)}{\overset{V_1}{L}}{}^a_{bc} = \frac{1}{2} \underset{(1)}{g}{}^{ad} \left(\delta_b \underset{(1)}{g}{}_{cd} + \delta_c \underset{(1)}{g}{}_{bd} - \delta_d \underset{(1)}{g}{}_{bc} \right) \tag{2.15}$$

$$\underset{(20)}{\overset{V_2}{L}}{}^a_{bc} = \frac{1}{2} \underset{(2)}{g}{}^{ad} \left(\delta_b \underset{(2)}{g}{}_{cd} + \delta_c \underset{(2)}{g}{}_{bd} - \delta_d \underset{(2)}{g}{}_{bc} \right)$$

the "1-vertical" coefficients

$$\overset{H}{\underset{(01)}{C}}{}^{a}_{bc} = \frac{1}{2} g^{ad} \left(\delta_{1b} g_{dc} + \delta_{1c} g_{bd} - \delta_{1d} g_{bc}\right)$$

$$\overset{V_1}{\underset{(11)}{C}}{}^{a}_{bc} = \frac{1}{2} \underset{(1)}{g}{}^{ad} \left(\delta_{1b} \underset{(1)}{g}{}_{cd} + \delta_{1c} \underset{(1)}{g}{}_{bd} - \delta_{1d} \underset{(1)}{g}{}_{bc}\right) \tag{2.16}$$

$$\overset{V_2}{\underset{(21)}{C}}{}^{a}_{bc} = \frac{1}{2} \underset{(2)}{g}{}^{ad} \left(\delta_{1b} \underset{(2)}{g}{}_{cd} + \delta_{1c} \underset{(2)}{g}{}_{bd} - \delta_{1d} \underset{(2)}{g}{}_{bc}\right)$$

and the "2-vertical" coefficients

$$\overset{H}{\underset{(02)}{C}}{}^{a}_{bc} = \overset{V_1}{\underset{(12)}{C}}{}^{a}_{bc} = \overset{V_2}{\underset{(22)}{C}}{}^{a}_{bc} = 0. \tag{2.17}$$

*It is called the **canonical N-linear metric connection**.*

This linear connection will be used throughout this paper.

For this N-linear connection, we have the operators $\overset{V_i}{D}, (i = 0, 1, 2; V_0 = H)$ which are given by the following relations

$$\overset{V_i}{D} X^a = dX^a + \overset{V_i}{\omega}{}^a_b X^b,\ \forall X \in \mathcal{F}\left(\widetilde{Osc^2 M}\right), \tag{2.18}$$

where

$$\overset{H}{\omega}{}^a_b = \overset{H}{\underset{(00)}{L}}{}^{a}_{bc} dx^c + \overset{H}{\underset{(01)}{C}}{}^{a}_{bc} \delta y^{(1)c} + \overset{H}{\underset{(02)}{C}}{}^{a}_{bc} \delta y^{(2)c}$$

$$\overset{V_1}{\omega}{}^a_b = \overset{V_1}{\underset{(10)}{L}}{}^{a}_{bc} dx^c + \overset{V_1}{\underset{(11)}{C}}{}^{a}_{bc} \delta y^{(1)c} + \overset{V_1}{\underset{(12)}{C}}{}^{a}_{bc} \delta y^{(2)c} \tag{2.19}$$

$$\overset{V_2}{\omega}{}^a_b = \overset{V_2}{\underset{(20)}{L}}{}^{a}_{bc} dx^c + \overset{V_2}{\underset{(21)}{C}}{}^{a}_{bc} \delta y^{(1)c} + \overset{V_2}{\underset{(22)}{C}}{}^{a}_{bc} \delta y^{(2)c}.$$

We call these operators the **horizontal,** 1- and 2-**vertical covariant differentials**. The 1-forms $\overset{H}{\omega}{}^a_b, \overset{V_1}{\omega}{}^a_b, \overset{V_2}{\omega}{}^a_b$ will be called the **horizontal**, 1- and 2-**vertical 1-form.** From (2.17) we get that the horizontal, 1- and 2- vertical 1-form are

$$\overset{H}{\omega}{}^a_b = \overset{H}{\underset{(00)}{L}}{}^{a}_{bc} dx^c + \overset{H}{\underset{(01)}{C}}{}^{a}_{bc} \delta y^{(1)c} + \overset{H}{\underset{(02)}{C}}{}^{a}_{bc} \delta y^{(2)c}$$

$$\overset{V_1}{\omega}{}^a_b = \overset{V_1}{\underset{(10)}{L}}{}^{a}_{bc} dx^c + \overset{V_1}{\underset{(11)}{C}}{}^{a}_{bc} \delta y^{(1)c} + \overset{V_1}{\underset{(12)}{C}}{}^{a}_{bc} \delta y^{(2)c}$$

$$\overset{V_2}{\omega}{}^a_b = \overset{V_2}{\underset{(20)}{L}}{}^{a}_{bc} dx^c + \overset{V_2}{\underset{(21)}{C}}{}^{a}_{bc} \delta y^{(1)c} + \overset{V_2}{\underset{(22)}{C}}{}^{a}_{bc} \delta y^{(2)c}.$$

3 The relative covariant derivatives

Let $D\Gamma(N)$, the canonical N-linear metric connection of the manifold Osc^2M. A classical method to determine the laws of derivation on a Finsler submanifold is the type of the coupling([5],[6],[8],[9]).

Definition 3.1. *We call a **coupling** of the canonical N-linear metric connection D to the induced nonlinear connection $\check{N}$ along $Osc^2\check{M}$ the operators $\overset{V_i}{\check{D}}$,$(i=0,1,2;$ $V_0=H)$ defined by the operators $\overset{V_i}{D}$,$(i=0,1,2;V_0=H)$ (2.18) with the property*

$$\overset{V_i}{\check{D}}X^a=\overset{V_i}{D}X^a,(i=0,1,2;V_0=H)\ (modulo\ 2.13) \tag{3.1}$$

Here

$$\overset{V_i}{\check{D}}X^a=dX^a+\overset{V_i}{\breve{\omega}}{}^a_bX^b,\ \forall X\in\mathcal{F}\left(\widetilde{Osc^2M}\right). \tag{3.2}$$

The 1-forms $\underset{(i)}{\breve{\omega}}{}^a_b$,$(i=0,1,2)$ are the **connection 1-forms of the coupling** Ď.

Theorem 3.2. *The coupling of the N-linear connection D to the induced nonlinear connection $\check{N}$ along $\widetilde{Osc^2\check{M}}$ is locally given by the set of coefficients $\check{D}\Gamma\left(\check{N}\right)=\left(\underset{(i0)}{\overset{V_i}{\check{L}}}{}^a_{b\delta},\underset{(i1)}{\overset{V_i}{\check{C}}}{}^a_{b\delta},\underset{(i2)}{\overset{V_i}{\check{C}}}{}^a_{b\delta}\right)$, $(i=0,1,2;V_0=H)$ where*

$$\begin{aligned}
&\underset{(i0)}{\overset{V_i}{\check{L}}}{}^a_{b\delta}=\underset{(i0)}{\overset{V_i}{L}}{}^a_{bd}B^d_\delta+\underset{(i1)}{\overset{V_i}{C}}{}^a_{bd}B^d_{\bar{\delta}}\underset{(1)}{K}{}^{\bar{\delta}}_\delta\\
&\underset{(i1)}{\overset{V_i}{\check{C}}}{}^a_{b\delta}=\underset{(i1)}{\overset{V_i}{C}}{}^a_{bd}B^d_\delta\\
&\underset{(i2)}{\overset{V_i}{\check{C}}}{}^a_{b\delta}=0,(i=0,1,2;V_0=H).
\end{aligned} \tag{3.3}$$

Proof. From (3.1), (3.2), (2.18), and (2.13) we obtain

$$\begin{aligned}
&\underset{(i0)}{\overset{V_i}{\check{L}}}{}^a_{b\delta}=\underset{(i0)}{\overset{V_i}{L}}{}^a_{bd}B^d_\delta+\underset{(i1)}{\overset{V_i}{C}}{}^a_{bd}B^d_{\bar{\delta}}\underset{(1)}{K}{}^{\bar{\delta}}_\delta+\underset{(i2)}{\overset{V_i}{C}}{}^a_{b\delta}B^d_{\bar{\delta}}\underset{(2)}{K}{}^{\bar{\delta}}_\delta\\
&\underset{(i1)}{\overset{V_i}{\check{C}}}{}^a_{b\delta}=\underset{(i1)}{\overset{V_i}{C}}{}^a_{bd}B^d_\delta+\underset{(i2)}{\overset{V_i}{C}}{}^a_{bd}B^d_{\bar{\delta}}\underset{(1)}{K}{}^{\bar{\delta}}_\delta\\
&\underset{(i2)}{\overset{V_i}{\check{C}}}{}^a_{b\delta}=\underset{(i2)}{\overset{V_i}{C}}{}^a_{bd}B^d_\delta,(i=0,1,2;V_0=H).
\end{aligned}$$

and from (2.17) we get (3.3). □

Definition 3.3. *We call the* ***induced tangent connection*** *on* $\widetilde{Osc^2\check{M}}$ *by the canonical N-linear metric connection D, the couple of the operators* $\overset{V_i}{D}{}^{\top}$ *, $(i = 0, 1, 2; V_0 = H)$ which are defined by*

$$\overset{V_i}{D}{}^{\top} X^{\alpha} = B^{\alpha}_{b} \overset{V_i}{\check{D}} X^{b}, \quad \textit{for } X^{a} = B^{a}_{\gamma} X^{\gamma} \tag{3.4}$$

where

$$\overset{V_i}{D}{}^{\top} X^{\alpha} = dX^{\alpha} + X^{\beta} \overset{V_i}{\omega}{}^{\alpha}_{\beta} \tag{3.5}$$

and $\overset{V_i}{\omega}{}^{\alpha}_{\beta}$ *, $(i = 0, 1, 2; V_0 = H)$ are called the* ***tangent connection 1-forms***.

We have

Theorem 3.4. *The tangent connections 1-forms are as follows:*

$$\overset{V_i}{\omega}{}^{\alpha}_{\beta} = \underset{(i0)}{\overset{V_i}{L}}{}^{\alpha}_{\beta\delta} du^{\delta} + \underset{(i1)}{\overset{V_i}{C}}{}^{\alpha}_{\beta\delta} \delta v^{(1)\delta} + \underset{(i2)}{\overset{V_i}{C}}{}^{\alpha}_{\beta\delta} \delta v^{(2)\delta}, \tag{3.6}$$

where

$$\begin{aligned} \underset{(i0)}{\overset{V_i}{L}}{}^{\alpha}_{\beta\delta} &= B^{\alpha}_{d} \left(B^{d}_{\beta\delta} + B^{f}_{\beta} \underset{(i0)}{\overset{V_i}{\check{L}}}{}^{d}_{f\delta} \right), \\ \underset{(i1)}{\overset{V_i}{C}}{}^{\alpha}_{\beta\delta} &= B^{\alpha}_{d} B^{f}_{\beta} \underset{(i1)}{\overset{V_i}{\check{C}}}{}^{d}_{f\delta}, \\ \underset{(i2)}{\overset{V_i}{C}}{}^{\alpha}_{\beta\delta} &= 0, (i = 0, 1, 2; V_0 = H). \end{aligned} \tag{3.7}$$

Proof. From (3.2),(3.5) and (3.4) we have

$$\begin{aligned} \underset{(i0)}{\overset{V_i}{L}}{}^{\alpha}_{\beta\delta} &= B^{\alpha}_{d} \left(B^{d}_{\beta\delta} + B^{f}_{\beta} \underset{(i0)}{\overset{V_i}{\check{L}}}{}^{d}_{f\delta} \right), \\ \underset{(i1)}{\overset{V_i}{C}}{}^{\alpha}_{\beta\delta} &= B^{\alpha}_{d} B^{f}_{\beta} \underset{(i1)}{\overset{V_i}{\check{C}}}{}^{d}_{f\delta}, \\ \underset{(i2)}{\overset{V_i}{C}}{}^{\alpha}_{\beta\delta} &= B^{\alpha}_{d} B^{f}_{\beta} \underset{(i2)}{\overset{V_i}{\check{C}}}{}^{d}_{f\delta}, (i = 0, 1, 2; V_0 = H). \end{aligned}$$

and from (2.17) we get (3.7). ☐

Definition 3.5. *We call the* ***induced normal connection*** *on* $\widetilde{Osc^2\check{M}}$ *by the canonical N-linear metric connection D, the couple of the operators* $\overset{V_i}{D}{}^{\perp}$ *, $(i = 0, 1, 2; V_0 = H)$ which are defined by*

$$\overset{V_i}{D}{}^{\perp} X^{\overline{\alpha}} = B^{\alpha}_{b} \overset{V_i}{\check{D}} X^{b} \quad \textit{for } X^{a} = B^{a}_{\bar{\gamma}} X^{\bar{\gamma}} \tag{3.8}$$

where

$$\overset{V_i}{D}{}^{\perp} X^{\overline{\alpha}} = dX^{\overline{\alpha}} + X^{\overline{\beta}} \overset{V_i}{\omega}{}^{\overline{\alpha}}_{\overline{\beta}} \tag{3.9}$$

and $\overset{V_i}{\omega}{}^{\overline{\alpha}}_{\overline{\beta}}$ *, $(i = 0, 1, 2; V_0 = H)$ are called the* ***normal connection 1-forms****.*

We have

Theorem 3.6. *The normal connections 1-forms are as follows:*

$$\overset{V_i}{\omega}{}^{\overline{\alpha}}_{\overline{\beta}} = \underset{(i0)}{\overset{V_i}{L}}{}^{\overline{\alpha}}_{\overline{\beta}\delta} du^{\delta} + \underset{(i1)}{\overset{V_i}{C}}{}^{\overline{\alpha}}_{\overline{\beta}\delta} \delta v^{(1)\delta} + \underset{(i2)}{\overset{V_i}{C}}{}^{\overline{\alpha}}_{\overline{\beta}\delta} \delta v^{(2)\delta} \tag{3.10}$$

where

$$\begin{aligned} \underset{(i0)}{\overset{V_i}{L}}{}^{\bar{\alpha}}_{\bar{\beta}\delta} &= B^{\bar{\alpha}}_{d} \left(\frac{\delta B^{d}_{\bar{\beta}}}{\delta u^{\delta}} + B^{f}_{\bar{\beta}} \underset{(i0)}{\overset{V_i}{\check{L}}}{}^{d}_{f\delta} \right) \\ \underset{(i1)}{\overset{V_i}{C}}{}^{\bar{\alpha}}_{\bar{\beta}\delta} &= B^{\bar{\alpha}}_{d} \left(\frac{\partial B^{d}_{\bar{\beta}}}{\partial u^{\delta}} + B^{f}_{\bar{\beta}} \underset{(i1)}{\overset{V_i}{\check{C}}}{}^{d}_{f\delta} \right) \\ \underset{(i2)}{\overset{V_i}{C}}{}^{\bar{\alpha}}_{\bar{\beta}\delta} &= 0, (i = 0, 1, 2; V_0 = H) \end{aligned} \tag{3.11}$$

Proof. From (3.2),(3.8),(3.9) and (2.13) we obtain

$$\underset{(i0)}{\overset{V_i}{L}}{}^{\bar{\alpha}}_{\bar{\beta}\delta} = B^{\bar{\alpha}}_{d} \left(\frac{\delta B^{d}_{\bar{\beta}}}{\delta u^{\delta}} + B^{f}_{\bar{\beta}} \underset{(i0)}{\overset{V_i}{\check{L}}}{}^{d}_{f\delta} \right)$$

$$\underset{(i1)}{\overset{V_i}{C}}{}^{\bar{\alpha}}_{\bar{\beta}\delta} = B^{\bar{\alpha}}_{d} \left(\frac{\delta B^{d}_{\bar{\beta}}}{\delta v^{(1)\delta}} + B^{f}_{\bar{\beta}} \underset{(i1)}{\overset{V_i}{\check{C}}}{}^{d}_{f\delta} \right)$$

$$\underset{(i2)}{\overset{V_i}{C}}{}^{\bar{\alpha}}_{\bar{\beta}\delta} = B^{\bar{\alpha}}_{d} \left(\frac{\partial B^{d}_{\bar{\beta}}}{\partial v^{(2)\delta}} + B^{f}_{\bar{\beta}} \underset{(i2)}{\overset{V_i}{\check{C}}}{}^{d}_{f\delta} \right),$$

$(i = 0, 1, 2; V_0 = H)$ and from (3.3) and $\frac{\partial B^{d}_{\bar{\beta}}}{\partial v^{(1)\delta}} = \frac{\partial B^{d}_{\bar{\beta}}}{\partial v^{(2)\delta}} = 0$ we have (3.11). □

Now, we can define the relative (or mixed) covariant derivatives $\overset{V_i}{\nabla}, (i = 0, 1, 2; V_0 = H)$.

Theorem 3.7. *The relative covariant (mixed) derivatives in the algebra of mixed d-tensor fields are the operators $\overset{V_i}{\nabla}, (i = 0, 1, 2; V_0 = H)$ for which the following properties hold:*

$$\overset{V_i}{\nabla} f = df, \quad \forall f \in \mathcal{F}\left(Osc^2 \check{M}\right)$$

$$\overset{V_i}{\nabla} X^{a} = \overset{V_i}{\check{D}} X^{a}, \quad \overset{V_i}{\nabla} X^{\alpha} = \overset{V_i}{D}{}^{\top} X^{\alpha}, \quad \overset{V_i}{\nabla} X^{\bar{\alpha}} = \overset{V_i}{D}{}^{\perp} X^{\bar{\alpha}}, \ (i = 0, 1, 2; V_0 = H)$$

$\overset{V_i}{\check{\omega}}{}^a_b, \overset{V_i}{\omega}{}^\alpha_\beta, \overset{V_i}{\omega}{}^{\overline{\alpha}}_{\overline{\beta}}$ are called the **connection 1-forms** of $\overset{V_i}{\nabla}, (i = 0, 1, 2; V_0 = H)$.

4 The Gauss-Weingarten formulae

In the theory of the submanifolds we are interesed in finding the moving equations of the moving frame $\mathcal{R}$ along $Osc^2\check{M}$.

These equations, called also Gauss-Weingarten formulae, are obtained when the relative covariant derivatives of the vector fields from $\mathcal{R}$ are expressed again in the frame $\mathcal{R}$.

Thus we have

Theorem 4.1. *The following Gauss-Weingarten formulae hold:*

$$\overset{V_i}{\nabla} B^a_\alpha = B^a_{\overline{\delta}} \overset{V_i}{\Pi}{}^{\overline{\delta}}_\alpha, \tag{4.1}$$

$$\overset{V_i}{\nabla} B^a_{\overline{\alpha}} = -B^a_\delta \overset{V_i}{\Pi}{}^\delta_{\overline{\alpha}}, \tag{4.2}$$

where

$$\begin{aligned} \overset{V_i}{\Pi}{}^{\overline{\delta}}_\alpha &= \underset{(0)}{\overset{V_i}{H}}{}_\alpha{}^{\overline{\delta}}{}_\beta du^\beta + \underset{(1)}{\overset{V_i}{H}}{}_\alpha{}^{\overline{\delta}}{}_\beta \delta v^{(1)\beta} + \underset{(2)}{\overset{V_i}{H}}{}_\alpha{}^{\overline{\delta}}{}_\beta \delta v^{(2)\beta} \\ \overset{V_i}{\Pi}{}^\alpha_{\overline{\delta}} &= g^{\alpha\sigma}\delta_{\overline{\delta}\overline{\sigma}} \overset{V_i}{\Pi}{}^{\overline{\sigma}}_\sigma, \end{aligned} \tag{4.3}$$

and the d-tensors

$$\begin{aligned} \underset{(0)}{\overset{V_i}{H}}{}_\alpha{}^{\overline{\delta}}{}_\beta &= B^{\overline{\delta}}_d \left(B^d_{\alpha\beta} + B^f_\alpha \underset{(i0)}{\overset{V_i}{\check{L}}}{}^d_{f\beta} \right) \\ \underset{(1)}{\overset{V_i}{H}}{}_\alpha{}^{\overline{\delta}}{}_\beta &= B^{\overline{\delta}}_d B^f_\alpha \underset{(i1)}{\overset{V_i}{\check{C}}}{}^d_{f\beta} \\ \underset{(2)}{\overset{V_i}{H}}{}_\alpha{}^{\overline{\delta}}{}_\beta &= B^{\overline{\delta}}_d B^f_\alpha \underset{(i2)}{\overset{V_i}{\check{C}}}{}^d_{f\beta} \end{aligned} \tag{4.4}$$

are the ***fundamental d-tensors of the second order*** *of manifold* $\widetilde{Osc^2 M}$, $(i = 0, 1, 2, V_0 = H)$.

Proof. From (2.15),(2.16) and (2.17) we have

$$\overset{H}{\nabla} B^a_\alpha = B^a_{\alpha|0\beta} du^\beta + B^a_\alpha \overset{(1)}{|}_{0\beta} \delta v^{(1)\delta} + B^a_\alpha \overset{(2)}{|}_{0\beta} \delta v^{(2)\delta}$$

$$= \left(\frac{\delta B^a_\alpha}{\delta u^\beta} + \underset{(00)}{\overset{H}{\check{L}}}{}^a_{b\beta} B^b_\alpha - \underset{(00)}{\overset{H}{L}}{}^\delta_{\alpha\beta} B^a_\delta \right) du^\beta +$$

$$+ \left(\frac{\delta B^a_\alpha}{\delta v^{(1)\beta}} + \underset{(p1)}{\check{C}}{}^a_{b\beta} B^b_\alpha - \underset{(p1)}{C}{}^\delta_{\alpha\beta} B^a_\delta \right) \delta v^{(1)\beta} +$$

$$+ \left(\frac{\partial B^a_\alpha}{\partial v^{(2)\beta}} + \underset{(02)}{\check{C}}{}^a_{b\beta} B^b_\alpha - \underset{(02)}{C}{}^\delta_{\alpha\beta} B^a_\delta \right) \delta v^{(2)\beta}$$

$$= B^a_{\alpha\beta} du^\beta + B^b_\alpha \left(\underset{(00)}{\overset{H}{\check{L}}}{}^a_{b\beta} du^\beta + \underset{(01)}{\overset{H}{\check{C}}}{}^a_{b\beta} \delta v^{(1)\beta} + \underset{(02)}{\overset{H}{\check{C}}}{}^a_{b\beta} \delta v^{(2)\beta} \right) -$$

$$-B^a_\delta \left[\left[B^\delta_d \left(B^d_{\alpha\beta} + B^f_\alpha \underset{(00)}{\overset{H}{\check{L}}}{}^d_{f\beta} \right) du^\beta + B^\delta_d B^f_\alpha \underset{(01)}{\overset{H}{\check{C}}}{}^d_{f\beta} \delta v^{(1)\beta} + \right] \right.$$

$$\left. + B^\delta_d B^f_\alpha \underset{(02)}{\overset{H}{\check{C}}}{}^d_{f\beta} \delta v^{(2)\beta} \right].$$

Using (4.3) we get the relation (4.1) for $V_0 = H$.

Now, by applying $\overset{H}{\nabla}$ to the both sides of the equations $g_{ab} B^a_\alpha B^b_{\bar{\beta}} = 0$ one get

$$g_{ab} B^a_{\bar{\delta}} \overset{H}{\Pi}{}^{\bar{\delta}}_\alpha B^b_{\bar{\beta}} + g_{ab} B^a_\alpha \overset{H}{\Pi} B^b_{\bar{\beta}} = 0.$$

Multiplying these relation with B^α_d we obtain

$$g_{bd} \overset{H}{\nabla} B^b_{\bar{\beta}} - B^a_{\bar{\delta}} B^{\bar{\delta}}_d g_{ab} \overset{H}{\nabla} B^b_{\bar{\beta}} = -B^\alpha_d \delta_{\bar{\beta}\bar{\gamma}} \overset{H}{\Pi}{}^{\bar{\gamma}}_\alpha.$$

But $B^a_{\bar{\delta}} B^{\bar{\delta}}_d g_{ab} \overset{H}{\nabla} B^b_{\bar{\beta}} = 0$. Consequently, we obtain the relations (4.2) for $V_0 = H$.

Analogously, for the operators $\overset{V_i}{\nabla}, (i = 1, 2)$ one gets the other relations. □

5 Curves in the manifold $\mathrm{Osc}^2 M$

In this section we construct a Frenet frame and determine the Frenet equations for a curve in the manifold $Osc^2 M$.

The start point of these researchs is the Bejancu and Farran results in case of vertical bundle of TM ([3]). We construct a Frenet frame and derive all the Frenet equations for a curve in the manifold $\widetilde{Osc^2M}$. This enables us to state a fundamental theorem for curves in manifold $\widetilde{Osc^2M}$.

Let $c : t \to (x^a(t))$ a smooth curve in M, t a real parameter and $s(t)$ a parameter change. On the manifold Osc^2M with the local coordinates $\left(x^a, y^{(1)a}, y^{(2)a}\right)$, the curve c induce a curve $\mathcal{C}$ with the property
$\left(x^a(t), y^{(1)a}(t) = \frac{dx^a}{dt}, y^{(2)a}(t) = \frac{1}{2}\frac{d^2x^a}{dt^2}\right)$.

If we change the parameter on $\mathcal{C}$ we have

$$\begin{aligned} y^{(1)a}(s) &= \frac{ds}{dt}\frac{dx^a}{ds} = v^{(1)}\frac{dx^a}{ds}, \\ y^{(2)a}(s) &= \frac{1}{2}\frac{d}{dt}\left(v^{(1)}\frac{dx^a}{ds}\right) = \frac{1}{2}\left(\frac{d^2s}{dt^2}\frac{dx^a}{ds} + \left(v^{(1)}\right)^2\frac{d^2x^a}{ds^2}\right) \\ &= \frac{1}{2}\left(v^{(2)}\frac{dx^a}{ds} + \left(v^{(1)}\right)^2\frac{d^2x^a}{ds^2}\right), \end{aligned} \tag{5.1}$$

where we noted

$$v^{(1)} = \frac{ds}{dt}, \quad v^{(2)} = \frac{d^2s}{dt^2}.$$

Thus, on $\mathcal{C}$, we may consider the parameters $\left(s, v^{(1)}, v^{(2)}\right)$, and from the above calcul we get the parametric equations in s:

$$\begin{cases} x^a = x^a(s), \\ y^{(1)a} = v^{(1)}\frac{dx^a}{ds}, \\ 2y^{(2)a} = v^{(2)}\frac{dx^a}{ds} + \left(v^{(1)}\right)^2\frac{d^2x^a}{ds^2}. \end{cases}$$

We use notation $x'^a = \frac{dx^a}{ds}$ and $x''^a = \frac{d^2x^a}{ds^2}$, thus the above equations become:

$$\begin{cases} x^a = x^a(s), \\ y^{(1)a} = v^{(1)}x'^a, \\ 2y^{(2)a} = v^{(2)}x'^a + \left(v^{(1)}\right)^2 x''^a. \end{cases}$$

The tangent vectors along $\mathcal{C}$ are $\left\{\frac{\partial}{\partial s}, \frac{\partial}{\partial v^{(1)}}, \frac{\partial}{\partial v^{(2)}}\right\}$, and their connection with the tangent vectors $\left\{\frac{\partial}{\partial x^a}, \frac{\partial}{\partial y^{(1)a}}, \frac{\partial}{\partial y^{(2)a}}\right\}$ is obtained by using the Jacobi matrix of these transformations

$$\begin{aligned} \frac{\partial}{\partial s} &= x'^a\frac{\partial}{\partial x^a} + v^{(1)}x''^a\frac{\partial}{\partial y^{(1)a}} + \left(v^{(1)2}x'''^a + v^{(2)}x''^a\right)\frac{\partial}{\partial y^{(2)a}}, \\ \frac{\partial}{\partial v^{(1)}} &= x'^a\frac{\partial}{\partial y^{(1)a}} + v^{(1)}x''^a\frac{\partial}{\partial y^{(2)a}}, \\ \frac{\partial}{\partial v^{(2)}} &= x'^a\frac{\partial}{\partial y^{(2)a}}. \end{aligned}$$

Let (M, F) a Finsler space and $g_{ab}(x, y^{(1)}) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^{(1)a}\partial y^{(1)b}}$ the fundamental tensor and $s\,(t) = \int_0^t F\left(x(t), y^{(1)}(t)\right) dt$ the arc length parameter on $\mathcal{C}$. We have $\frac{ds}{dt} = F\left(x(t), y^{(1)}(t)\right)$ along of curve $\mathcal{C}$. A a consequence of homogeneity of F it fallows that $v^{(1)} = \frac{ds}{dt} = F\left(x(t), y^{(1)}(t)\right) = F\left(x(s), v^{(1)}y^{(1)}(s)\right) = v^{(1)}F\left(x(s), y^{(1)}(s)\right)$. We deduce that $F\left(x(s), y^{(1)}(s)\right) = 1$, which is true on $\mathcal{C}$ restrictioned to the manifold $Osc^1M \equiv TM$.

Let $\mathbb{G}$ the Sasaki prolongation of the fundamental tensor g_{ab} to the manifold $\widetilde{Osc^2}M$, (*),

$$\mathbb{G} = g_{ab}\left(x, y^{(1)}\right) dx^a \otimes dx^b + g_{ab}\left(x, y^{(1)}\right) \delta y^{(1)a} \otimes \delta y^{(1)b} + \\ + g_{ab}\left(x, y^{(1)}\right) \delta y^{(2)a} \otimes \delta y^{(2)b}$$

and $\mathrm{D}\Gamma\,(N)$, the canonical N-linear metric connection on the manifold Osc^2M, with the coefficients

$$\begin{aligned} &\underset{(i0)}{L}{}^a_{bc} = \tfrac{1}{2}g^{ad}\left(\delta_b g_{cd} + \delta_c g_{bd} - \delta_d g_{bc}\right) \\ &\underset{(i1)}{C}{}^a_{bc} = \tfrac{1}{2}g^{ad}\left(\delta_{1b} g_{cd} + \delta_{1c} g_{bd} - \delta_{1d} g_{bc}\right) \quad (i = 0, 1, 2) \\ &\underset{(i2)}{C}{}^a_{bc} = 0, \end{aligned} \tag{5.2}$$

and $\underset{(i)}{\nabla}$, $(i = 0, 1, 2)$ the covariant derivatives operators.

Let

$$g_b{}^a{}_c = \frac{1}{2}g^{ad}\frac{\partial g_{bd}}{\partial y^{(1)c}} \tag{5.3}$$

and $B\Gamma\,(N)$, the N-linear connection of Berwald type with the coefficients

$$\left(\overset{b}{\underset{(00)}{L}}{}^a_{bc}, \overset{b}{\underset{(10)}{L}}{}^a_{bc}, \overset{b}{\underset{(20)}{L}}{}^a_{bc}, \overset{b}{\underset{(01)}{C}}{}^a_{bc}, \overset{b}{\underset{(11)}{C}}{}^a_{bc}, \overset{b}{\underset{(21)}{C}}{}^a_{bc}, \overset{b}{\underset{(02)}{C}}{}^a_{bc}, \overset{b}{\underset{(12)}{C}}{}^a_{bc}, \overset{b}{\underset{(22)}{C}}{}^a_{bc}\right) \tag{5.4}$$

where

$$\begin{aligned} &\overset{b}{\underset{(00)}{L}}{}^a_{bc} = \underset{(00)}{L}{}^a_{bc} && \overset{b}{\underset{(10)}{L}}{}^a_{bc} = \underset{(11)}{B}{}^a_{cb} && \overset{b}{\underset{(20)}{L}}{}^a_{bc} = \underset{(22)}{B}{}^a_{cb}, \\ &\overset{b}{\underset{(01)}{C}}{}^a_{bc} = 0 && \overset{b}{\underset{(11)}{C}}{}^a_{bc} = \underset{(11)}{C}{}^a_{bc}, && \overset{b}{\underset{(21)}{C}}{}^a_{bc} = 0, \\ &\overset{b}{\underset{(02)}{C}}{}^a_{bc} = 0, && \overset{b}{\underset{(12)}{C}}{}^a_{bc} = 0, && \overset{b}{\underset{(22)}{C}}{}^a_{bc} = \underset{(22)}{C}{}^a_{bc} = 0, \end{aligned} \tag{5.5}$$

where $\underset{(00)}{L}{}^{a}_{bc}$, $\underset{(11)}{C}{}^{a}_{bc}$,$\underset{(22)}{C}{}^{a}_{bc}$ and $\underset{(11)}{B}{}^{a}_{bc}$,$\underset{(22)}{B}{}^{a}_{bc}$ are given by the formulas (5.2) and (2.8),

$$\underset{(11)}{B}{}^{a}_{bc} = \delta_{1c}\underset{1}{N}{}^{a}{}_{b},\ \underset{(12)}{B}{}^{a}_{bc} = \delta_{1c}\underset{2}{N}{}^{a}{}_{b} - \delta_{b}\underset{1}{N}{}^{a}{}_{c} + \underset{1}{N}{}^{a}{}_{f}\underset{(11)}{B}{}^{f}_{bc},$$

$$\underset{(12)}{R}{}^{a}_{bc} = \delta_{1c}\underset{1}{N}{}^{a}{}_{b} - \delta_{1b}\underset{1}{N}{}^{a}{}_{c}.$$

Then, we have

$$\mathbb{G}\left(\frac{\partial}{\partial v^{(2)}},\frac{\partial}{\partial v^{(2)}}\right) = g_{ab}\left(x\left(t\right),\dot{x}\left(t\right)\right)\frac{dx^{a}}{dt}\frac{dx^{b}}{dt}. \tag{5.6}$$

If we take the paramater s, it fallows that

$$\mathbb{G}\left(\frac{\partial}{\partial v^{(2)}},\frac{\partial}{\partial v^{(2)}}\right) = g_{ab}x'^{a}x'^{b} = F^{2}(x(s),y^{(1)}(s)) = 1. \tag{5.7}$$

We study geometric objects along curve $\mathcal{C}$ in points $\left(x^{a}\left(s\right),y^{(1)a}\left(s\right),y^{(2)a}\left(s\right)\right)$ where $v^{(1)}\neq 0$.

An other consequence of homogeneity of F is that g_{ab}, g^{ab} and $\overset{H}{\underset{(00)}{L}}{}^{a}_{bc}$ (2.15) (or $\underset{(i0)}{L}{}^{a}_{bc}$, $(i = 0,1,2)$ from (5.2)) are positive homogeneous of degree zero, while G^{a}, G^{a}_{b} and $g_{b}{}^{a}{}_{c}$, (5.3), are positive homogeneous of degrees $2,1$ and -1. The functions G^{a} (2.2) are

$$G^{a}\left(x,y^{(1)}\right) = \frac{1}{4}g^{ab}\left(x,y^{(1)}\right)\left(\frac{\partial^{2}F^{*}}{\partial y^{(1)b}\partial x^{c}}y^{(1)c} - \frac{\partial F^{*}}{\partial x^{b}}\right)\left(x,y^{(1)}\right),\ F^{*} = F^{2}.$$

We deduce:

$$\begin{aligned} g_{ab}\left(x\left(s\right),v^{(1)}x'\left(s\right)\right) &= g_{ab}\left(x\left(s\right),x'\left(s\right)\right),\\ g^{ab}\left(x\left(s\right),v^{(1)}x'\left(s\right)\right) &= g^{ab}\left(x\left(s\right),x'\left(s\right)\right),\end{aligned} \tag{5.8}$$

$$x'^{b}\left(s\right)\underset{(00)}{L}{}^{a}_{bc}\left(x\left(s\right),v^{(1)}x'\left(s\right)\right) = x'^{b}\left(s\right)\underset{(00)}{L}{}^{a}_{bc}\left(x\left(s\right),x'\left(s\right)\right) = G^{a}_{c}\left(x\left(s\right),x'\left(s\right)\right), \tag{5.9}$$

$$x'^{b}\left(s\right)G^{a}_{b}\left(x\left(s\right),v^{(1)}x'\left(s\right)\right) = v^{(1)}x'^{b}\left(s\right)G^{a}_{b}\left(x\left(s\right),x'\left(s\right)\right) = 2v^{(1)}G^{a}\left(x\left(s\right),x'\left(s\right)\right), \tag{5.10}$$

$$x'^{b}\left(s\right)g_{b}{}^{a}{}_{c}\left(x\left(s\right),x'\left(s\right)\right) = 0,$$

$$g_{b}{}^{a}{}_{c}\left(x\left(s\right),v^{(1)}x'\left(s\right)\right) = \tfrac{1}{v^{(1)}}g_{b}{}^{a}{}_{c}\left(x\left(s\right),x'\left(s\right)\right). \tag{5.11}$$

Remark 5.1. *Here and in the sequel we use the vector notations $x\left(s\right)$ and $x'\left(s\right)$ to represent the vectors $\left(x^{0}\left(s\right),...,x^{m}\left(s\right)\right)$ and $\left(x'^{0}\left(s\right),...,x'^{m}\left(s\right)\right)$, respectively. Also, the components of a geometric object $T^{abc...}_{def...}$ at the point $\left(x\left(s\right),x'\left(s\right)\right)$ we denote them by $T^{abc...}_{def...}\left(s\right)$.*

We say that a vector field $X \in \mathcal{X}\left(\widetilde{Osc^2 M}\right)$ along $Osc^2\mathcal{C}$ is **projectable on** $\mathcal{C}$ if locally at any point $\left(x(s), v^{(1)}x'(s), y^{(2)a}(s)\right) \in \widetilde{Osc^2\mathcal{C}}$ we have

$$X\left(x(s), v^{(1)}x'(s), y^{(2)a}(s)\right) = X^{(0)a}(s)\frac{\delta}{\delta x^a}\left(x(s), v^{(1)}x'(s), y^{(2)a}(s)\right) \tag{5.12}$$

$$+X^{(1)a}(s)\frac{\delta}{\delta y^{(1)a}}\left(x(s), v^{(1)}x'(s), y^{(2)a}(s)\right)+X^{(2)a}(s)\frac{\partial}{\partial y^{(2)a}}\left(x(s), v^{(1)}x'(s), y^{(2)a}(s)\right),$$

or, equivalently, the local components of X at any point of $\mathcal{C}$ depend only on the arc length parameter s of $\mathcal{C}$. The above name is justified because X given by (5) on $Osc^2\mathcal{C}$ defines a vector field X^* on $\mathcal{C}$ by the formula

$$X^*(x(s)) = X^{(2)a}(s)\frac{\partial}{\partial x^a}(x(s)).$$

Thus $X^*(x(s))$ can be considered as the projection of the vector field $X\left(x(s), v^{(1)}x'(s), y^{(2)a}(s)\right)$ on the tangent space TM at $x(s) \in \mathcal{C}$. As an example, $\frac{\partial}{\partial y^{(2)a}}$ is a projectable vector field. Also we shall see later that a Frenet frame for a curve on the manifold $\widetilde{Osc^2 M}$ contains only projectable vector fields.

Let D$\Gamma(N)$, the canonical N-linear metric connection (5.2) and B$\Gamma(N)$, the N-liniar connection on Berwald type from (5.4), and $\overset{c}{\underset{(i)}{\nabla}}, \overset{b}{\underset{(i)}{\nabla}}$, $(i = 0, 1, 2)$ are the covarant derivatives of these N-linear connections.

Proposition 5.2. *The covariant derivatives of any projectable vector field X in the direction of $\frac{\delta}{\delta v^{(1)}}$ or $\frac{\partial}{\partial v^{(2)}}$ with respect to $D\Gamma(N)$ and $B\Gamma(N)$ vanish identically on $\widetilde{Osc^2\mathcal{C}}$, that is we have*

$$\begin{aligned}\left(\nabla_{\frac{\delta}{\delta v^{(1)}}}X\right)\left(x(s), v^{(1)}x'(s), y^{(2)a}(s)\right) &= 0,\\ \left(\nabla_{\frac{\partial}{\partial v^{(2)}}}X\right)\left(x(s), v^{(1)}x'(s), y^{(2)a}(s)\right) &= 0, \forall s \in (-\varepsilon, \varepsilon),\end{aligned}$$

where $\underset{(i)}{\nabla}$ are $\overset{c}{\underset{(i)}{\nabla}}$ or $\overset{b}{\underset{(i)}{\nabla}}$, $(i = 0, 1, 2)$.

Proof. We have

$$\left(\underset{(i)}{\nabla}_{\frac{\delta}{\delta v^{(j)}}}X\right)\left(x(s), v^{(1)}x'(s), y^{(2)a}(s)\right) =$$

$$X^{(0)a}(s)\overset{(j)}{|}_{0\text{-}}\frac{\delta}{\delta x^a} + X^{(1)a}(s)\overset{(j)}{|}_{1\text{-}}\frac{\delta}{\delta y^{(1)a}} + X^{(2)a}(s)\overset{(j)}{|}_{2\text{-}}\frac{\partial}{\partial y^{(2)a}},$$

where

$$X^{(i)a}(s) \overset{(j)}{|}_{i\_} = X^{(i)b}(s)\, x'^{c} \underset{(ij)}{C}{}_{bc}^{a} = 0,$$

since for the above N-linear connections we have either

$$\underset{(ij)}{C}{}_{bc}^{a} = g_{b}{}^{a}{}_{c} \text{ or } \underset{(ij)}{C}{}_{bc}^{a} = 0,$$

where $i \in \{0,1,2\}, j \in \{1,2\}, \frac{\delta}{\delta v^{(2)}} = \frac{\partial}{\partial v^{(2)a}}$. □

Hence, in particular, we have

$$\underset{(i)\frac{\delta}{\delta v^{(1)}}}{\nabla} \frac{\partial}{\partial y^{(2)a}} = 0, \underset{(i)\frac{\partial}{\partial v^{(2)a}}}{\nabla} \frac{\partial}{\partial y^{(2)a}} = 0,$$

wich enables us to state that *the "vertical" covariant derivatives along* $\mathcal{C}$ with respect to the canonical N-linear metric connection and of the Berwald type (5.4) do not provide any Frenet frame for $\mathcal{C}$. Hence we have to proceed with the horizontal covariant derivatives along $\mathcal{C}$.

From (2.5), we get

$$\frac{\partial}{\partial s} = \frac{dx^a}{ds}\frac{\delta}{\delta x^a} + v^{(1)}\left[\frac{d^2x^a}{ds^2} + 2G^a(s)\right]\frac{\delta}{\delta y^{(1)a}} + v^{(2)}\left[\frac{d^2x^a}{ds^2} + 2\widetilde{G}^a\left(s, v^{(1)}\right)\right]\frac{\partial}{\partial y^{(2)a}}, \quad (5.13)$$

where $\widetilde{G}^a\left(s, v^{(1)}\right) = \frac{1}{2v^{(2)}}\left(\frac{dx^b}{ds}\underset{2}{M}{}_b^a + \frac{d^2x^b}{ds^2}G_b^a(s)\,v^{(1)} + \frac{d^3x^a}{ds^3}\left(v^{(1)}\right)^2\right)$.

The canonical N-linear metric connection is the best choice for studying the geometry of curves in the manifold $\widetilde{Osc^2M}$. First, by direct calculation we get

$$\underset{(i)\frac{\partial}{\partial s}}{\nabla} \frac{\partial}{\partial v^{(2)}} = \left(\frac{d^2x^a}{ds^2} + 2G^a(s)\right)\frac{\partial}{\partial y^{(2)a}}, \quad (5.14)$$

On the other hand, using (5.7) and taking into account that $D\Gamma(N)$ is a metric N-linear connection we obtain

$$\mathbb{G}\left(\underset{(i)\frac{\partial}{\partial s}}{\nabla} \frac{\partial}{\partial v^{(2)}}, \frac{\partial}{\partial v^{(2)}}\right) = 0, (i = 0,1,2). \quad (5.15)$$

As a consequence of (5.15) we may set

$$\underset{(i)\frac{\partial}{\partial s}}{\nabla} \frac{\partial}{\partial v^{(2)}} = k_1 N_1, \quad (5.16)$$

where $N_1 \in V_2T\left(Osc^2\mathcal{C}\right)^{\perp}$, and

$$k_1 = \left\|\underset{(i)\frac{\partial}{\partial s}}{\nabla} \frac{\partial}{\partial v^{(2)}}\right\|, \quad (5.17)$$

for $i=0,1,2$ and $\|X\| = \mathbb{G}(X,X)$, $\forall X \in \mathcal{X}\left(\widetilde{Osc^2}M\right)$. By (5.14) we infer that

$$k_1 = \left\{g_{ab}(s)\left(x"^a + 2G^a(s)\right)\left(x"^b + 2G^b(s)\right)\right\}^{1/2} \tag{5.18}$$

and it call the **geodesic curvature (first curvature) function** of $\mathcal{C}$. If $k_1(s) \neq 0, \forall s \in (-\varepsilon,\varepsilon)$ we call

$$N_1 = \frac{1}{k_1(s)}\left(x"^a + 2G^a(s)\right)\frac{\partial}{\partial y^{(2)a}} = N_1(s)\frac{\partial}{\partial y^{(2)a}}, \tag{5.19}$$

the **principal (first) normal** of $\mathcal{C}$. Clearly, N_1 is a projectable vector field along $\mathcal{C}$. Actually, this is a consequence of the following general result.

Proposition 5.3. *The covariant derivatives of a projectable vector field X along $\mathcal{C}$ with respect to the canonical N-linear metric connection in the direction of $\frac{\partial}{\partial s}$ is a projectable vector field too, given by*

$$\begin{aligned} \underset{(i)\frac{\partial}{\partial s}}{\nabla} X \;=\; & \left[\tfrac{dX^{(0)a}}{ds} + X^{(0)b}S^a_b(s)\right]\tfrac{\delta}{\delta x^a} + \\ & + \left[\tfrac{dX^{(1)a}}{ds} + X^{(1)b}S^a_b(s)\right]\tfrac{\delta}{\delta y^{(1)a}} + \\ & + \left[\tfrac{dX^{(2)a}}{ds} + X^{(2)b}S^a_b(s)\right]\tfrac{\partial}{\partial y^{(2)a}}, (i=0,1,2) \end{aligned} \tag{5.20}$$

where

$$S^a_b(s) = G^a_b(s) + \left(x"^c + 2G^c(s)\right)g_b{}^a{}_c(s). \tag{5.21}$$

Proof. The assertion fallows by direct calculation using (5.13), (5.9) and (5.11). □

Next, suppose that $n+1>2$. Since $D\Gamma(N)$ is the canonical N-linear metric connection, from $\mathbb{G}(N_1,N_1)=1$ and $\mathbb{G}\left(\frac{\partial}{\partial v^{(1)}}, N_1\right)=0$ using (5.16) we deduce that

$$\mathbb{G}\left(\underset{(i)\frac{\partial}{\partial s}}{\nabla} N_1, N_1\right) = 0 \;,\; \mathbb{G}\left(\underset{(i)\frac{\partial}{\partial s}}{\nabla} N_1, \frac{\partial}{\partial v^{(2)}}\right) = -k_1, (i=0,1,2).$$

Hence we may set

$$\underset{(i)\frac{\partial}{\partial s}}{\nabla} N_1 = -k_1\frac{\partial}{\partial v^{(2)}} + N, \tag{5.22}$$

where $N \in V_2Osc^2\mathcal{C}^{\perp}$.

Thus we may define the next function

$$k_2 = \left\| k_1(s)\frac{\partial}{\partial v^{(2)}} + \underset{(i)\frac{\partial}{\partial s}}{\nabla} N_1 \right\|.$$

Then by straightforward calculation using (5.19) and (5.20) we infer that

$$k_2 = \left\{ g_{ab}(s) \left(k_1 x'^a + N_1'^a + N_1^c S_c^a \right) \left(k_1 x'^b + N_1'^b + N_1^d S_d^a \right) \right\}^{1/2}. \tag{5.23}$$

The function k_2 is called the **second curvature function** of $\mathcal{C}$.

If $k_2(s) \neq 0, \forall (-\varepsilon, \varepsilon)$ we define the vector field

$$N_2 = \frac{1}{k_2(s)} \left(k_1(s) \frac{\partial}{\partial v^{(2)}} + \underset{(i)\frac{\partial}{\partial s}}{\nabla} N_1 \right).$$

Hence, (5.22) becomes

$$\underset{(i)\frac{\partial}{\partial s}}{\nabla} N_1 = -k_1 \frac{\partial}{\partial v^{(2)}} + k_2(s) N_2(s), (i = 0, 1, 2).$$

We suppose inductively that there exist orthonormal projectable vector fields $\left\{ N_0 = \frac{\partial}{\partial v^{(2)}}, N_1, .., N_j \right\}$ and nowhere zero curvature functions $\{k_1, k_2, .., k_j\}$, $1 \leq j \leq n$, such that the following equations hold

$$\begin{aligned} &(F_1)\ \underset{(i)\frac{\partial}{\partial s}}{\nabla} N_0 = k_1 N_1, \\ &(F_2)\ \underset{(i)\frac{\partial}{\partial s}}{\nabla} N_1 = -k_1 N_0 + k_2 N_2, \\ &\dots\ \dots\ \dots\ \dots\ \dots\ \dots \\ &(F_j)\ \underset{(i)\frac{\partial}{\partial s}}{\nabla} N_{j-1} = -k_{j-1} N_{j-2} + k_j N_j, (i = 0, 1, 2). \end{aligned} \tag{5.24}$$

Then by using the Proposition 5.3 and following a proof similar to that of the Finsler case (cf. Bejancu, Farran [3], p.156), for any $j < n$ we obtain

$$(F_{j+1})\ \underset{(i)\frac{\partial}{\partial s}}{\nabla} N_j = -k_j N_{j-1} + k_{j+1} N_{j+1},$$

where

$$k_{j+1} = \left\{ g_{ab}(s) \left(k_j N_{j-1}^a + N_j'^a + N_j^c S_c^a \right) \left(k_j N_{j-1}^b + N_j'^b + N_j^d S_d^b \right) \right\}^{1/2}. \tag{5.25}$$

Moreover, the system of vector fields $\{N_0, N_1, .., N_j\}$ is an orthonormal set of projectable vector fields along $\mathcal{C}$.

If $j = n$, then $\{N_0, N_1, .., N_j\}$ is an orthonormal basis of $\mathcal{X}V_2T\left(Osc^2\mathcal{C}\right)^{\perp}$. As $\underset{(i)\frac{\partial}{\partial s}}{\nabla} N_n$ is orthogonal to N_n, the equation (F_{j+1}) becomes

$$(F_{n+1}):\ \underset{(i)\frac{\partial}{\partial s}}{\nabla} N_n = -k_n N_{n-1}, (i = 0, 1, 2).$$

We call the system of vector fields $\{N_0, N_1, .., N_n\}$ and the equations $\{(F_1), ..., (F_{n+1})\}$ the **Frenet frame** and the **Frenet equations** along $\mathcal{C}$ respectively. If $k_j = 0$ on $(-\varepsilon, \varepsilon)$, for some $j < n$, then we can not define N_j. Thus the equation (F_j) becomes

$$(F_j)' : \underset{(i)\frac{\partial}{\partial s}}{\nabla} N_{j-1} = -k_{j-1} N_{j-2}, (i = 0, 1, 2).$$

Thus, in the case in wich there exist nowhere zero curvature functions $\{k_1, k_2, .., k_{j-1}\}$ on $(-\varepsilon, \varepsilon)$ and k_j is everywhere zero on $(-\varepsilon, \varepsilon)$, then we have constructed the Frenet frame $\{N_0, N_1, .., N_{j-1}\}$ satisfying the Frenet equations $\left\{(F_1), ..., (F_{j-1}), (F_j)'\right\}$. We obtain the following fundamental theorem for curves in the manifold $\widetilde{Osc^2 M}$.

Theorem 5.4. *Let $\left(x_0, y_0^{(1)}, y_0^{(2)}\right) = \left(x_0^a, y_0^{(1)a}, y_0^{(2)a}\right)$ be a fixed point of the manifold $\widetilde{Osc^2 M}$, $\{V_0, V_1, .., V_n\}$ an orthonormal basis of $V_2 T\left(\widetilde{Osc^2 M}\right)$ and $k_1, k_2, .., k_n : (-\varepsilon, \varepsilon) \to R$ be everywhere positive smooth functions. Then exists a unique curve $\mathcal{C}$ on given by equations $x^a = x^a(s)$, $y^{(1)a} = y^{(1)a}(s)$, $y^{(2)a} = y^{(2)a}(s)$, $s \in (-\varepsilon, \varepsilon)$, where s is the arc length parameter of $\mathcal{C}$, such that $\left(x^a(0), y^{(1)a}(0), y^{(2)a}(0)\right) = \left(x_0^a, y_0^{(1)a}, y_0^{(2)a}\right)$ and $k_1, k_2, .., k_n$ are the curvature functions of $\mathcal{C}$ with respect to the Frenet frame $\{N_0, N_1, .., N_n\}$ wich satisfies $N_h(0) = V_h$, $h \in \{0, ..., n\}$.*

Proof. We can use the Theorem 2.1,[3], p.158 □

Remark 5.5. *1. If we consider the homogenneous lift $\overset{0}{\mathbb{G}}$, (2.9), we obtain*

$$\overset{0}{\mathbb{G}}\left(\frac{\partial}{\partial v^{(2)}}, \frac{\partial}{\partial v^{(2)}}\right) = p^4$$

and (5.15) becomes

$$\mathbb{G}\left(\nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial v^{(2)}}, \frac{\partial}{\partial v^{(2)}}\right) = 0.$$

2. The curvature geodesic functions become

$$\begin{aligned} \overset{0}{k}_{j+1} &= \left\{ \underset{(2)}{g}_{ab}(s)\left(k_j N_{j-1}^a + N_j'^a + N_j^c S_c^a\right)\left(k_j N_{j-1}^b + N_j'^b + N_j^d S_d^b\right)\right\}^{1/2} \\ &= p^2 k_{j+1,}, (j = 0, 1, ..., n-1). \end{aligned}$$

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