Bulletin of the *Transilvania* University of Braşov • Vol 8(57), No. 1 - 2015
Series III: Mathematics, Informatics, Physics, 43-56

FIRST ORDER JETS OF BUNDLES OVER A MANIFOLD ENDOWED WITH A SUBFOLIATION

Adelina MANEA[1]

Abstract

Let (E, π, M) be a bundle over the manifold (M, F_1, F_2), where (F_1, F_2) is a subfoliation on M. We define the (F_1, F_2)-1-jet manifold, $J^1_{F_{21}}\pi$. If $J^1\pi$ is the 1-jet manifold of π, then there is a diffeomorphism between $J^1\pi$ and the total space of the fibre bundle $J^t_{F_1}\pi \times_E J^1_{F_{21}}\pi \times_E J^l_{F_2}\pi$, where $J^t_{F_1}\pi$ and $J^l_{F_2}\pi$ are the transversal 1-jet manifold and the leafwise 1-jet manifold of the bundle π, with respect to foliations $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively.

2000 *Mathematics Subject Classification:* 55R10, 53C12, 58A20
Key words: fiber bundles, foliated manifold, leafwise, jets.

1 Introduction

Generally speaking, jets are equivalence classes of maps between manifolds, maps which have the same derivative until a specificated order. The language of jets has appeared as a concise way of describing phenomena associated with the derivatives of maps, so it is an appropiate language for many physical theories (mechanics, field theories). Jet spaces constitute a natural geometric environment also for differential equations and for equations of mathematical physics, particularly those associated with the calculus of variations, [5], [7], [12]. We refer to [2], [10], [13], for an introduction to jets.

In this paper we investigate the first order jets of bundles in a particular case, when the base space admits a subfoliation. Foliations, subfoliations and l-flags of foliations on manifolds also could have physical interpretations. Geometrical and cohomologycal aspects of such manifolds are investigated in [3], [9], [14], [15] and [16]. Tangent manifold of a Finsler space, big-tangent manifold, are examples of manifolds endowed with subfoliations, [1], [4], [17]. Bundles over foliated manifolds are studied in [11], [8] and the k-jets of origin 0 of differentialble mappings from $\mathbf{R}$ to a manifold endowed with an l-flag of foliations are studied in [6].

The paper is organized as follows. Section 2 contains the basic theoretical aspects about subfoliations and first order jets of bundles. In subsections 2.1 we

[1]Faculty of Mathematics and Informatics, *Transilvania* University of Braşov, Romania, e-mail: amanea28@yahoo.com

determine a basis adapted to a subfoliation on a Riemannian manifold, which will be used in the next section. In section 3 we consider a bundle (E, π, M) over the manifold (M, F_1, F_2), where (F_1, F_2) is a subfoliation on M and we define leafwise F_2-jet manifold, the $F_{21}-1$-jet manifold and the transversal F_1-jet manifold of the bundle π. The main result, Theorem 1, proves that the 1-jet manifold of π, $J^1\pi$, is diffeomorphic with the total space of the fibre bundle $J^t_{F_1}\pi \times_E J^1_{F_{21}}\pi \times_E J^l_{F_2}\pi$, where $J^t_{F_1}\pi$ and $J^l_{F_2}\pi$ are the transversal 1-jet manifold and the leafwise 1-jet manifold of the bundle π, with respect to foliations $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively, since $J^1_{F_{21}}\pi$ is the $F_{21} - 1$-jet manifold of π with respect to subfoliation (F_1, F_2). In the last section we particularised the general result from previous sections in the case of bundles over a big-tangent manifold of a Riemannian manifold.

2 Preliminaries

In this section we present the notions of foliation and subfoliation following [14], [3], then the first order jets manifold of a bundle, [10].

2.1 Subfoliations

A q-codimensional foliation $\mathcal{F}$ of an m-dimensional manifold M is a partition of M into $(m-q)$-dimensional submanifolds, called *leaves*. The set of vector fields tangent to leaves form an integrable subbundle F of TM, called the structural bundle of $(M, \mathcal{F})$. The transversal bundle $QF = TM/F$ is exactly the normal bundle of F in TM when M is a Riemannian manifold.

On the foliated manifold $(M, \mathcal{F})$ there is an adapted atlas whose coordinate system on the open set $V \subset M$ is $\left(x^i\right) = (x^a, x^u)$, where $a = \overline{1, q}$, $u = \overline{q+1, m}$, such that the points in the same leaf $\mathcal{L} \cap V$ have their first q coordinates equal, and are distinguished by their last $(m-q)$ coordinates. Locally, the structural bundle F is spanned by $\left\{\frac{\partial}{\partial x^u}\right\}_u$.

A (q_1, q_2)-*codimensional subfoliation* on M is a couple (F_1, F_2) of integrable subbundles F_k of TM of dimension $m - q_k$, $k = 1, 2$, and F_2 being at the same time a subbundle of F_1, [3]. Such a subfoliation determines two foliations on M: a $(m - q_1)$-dimensional foliation $\mathcal{F}_1$ with structural bundle F_1 and a $(m - q_2)$-dimensional foliation $\mathcal{F}_2$ with structural bundle F_2. Moreover, every leaf of $\mathcal{F}_1$ has a $d = (q_2 - q_1)$-codimensional foliated structure determined by F_2, with transversal bundle $QF_{21} = F_1/F_2$.

We denote by $QF_k = TM/F_k$ the transversal bundle of foliation $\mathcal{F}_k$ and by p_k the canonical projection on QF_k.

Let (M, g) be a Riemannian m-dimensional manifold, and (F_1, F_2) a (q_1, q_2)-codimensional subfoliation on it. Then, QF_k is isomorphic with the normal bundle of F_k and we have the following decompositions:

$$TM = QF_1 \oplus F_1, \quad TM = QF_2 \oplus F_2, \quad F_1 = QF_{21} \oplus F_2. \tag{1}$$

We also have the isomorphism $QF_2 \cong QF_1 \oplus QF_{21}$.

The first equality of (1) produces a double grading of forms on M of bidegree (p,q), with QF_1-degree p and F_1-degree q, since the last relation (1) leads to a double gradind (r,s), $q=r+s$, of F_1-degree q into QF_{21}-degree r and F_2-degree s. The exterior diferential admits the decomposition

$$d=d_{10}^{\mathcal{F}_1}+d_{2,-1}^{\mathcal{F}_1}+d_{01}^{\mathcal{F}_1},\quad d_{01}^{\mathcal{F}_1}=d_{10}^{\mathcal{F}_{21}}+d_{2,-1}^{\mathcal{F}_{21}}+d_{01}^{\mathcal{F}_{21}}, \tag{2}$$

where $d_{01}^{\mathcal{F}_1}$ means the exterior derivative along the leaves of $\mathcal{F}_1$ in M, and $d_{01}^{\mathcal{F}_{21}}$ means the exterior derivative along the leaves of $\mathcal{F}_2$ in $\mathcal{L}$, for every leaf $\mathcal{L}$ of $\mathcal{F}_1$.

From the classical theory of foliated manifolds, there is an atlas $\{(U,\varphi)\}$ adapted to (F_1,F_2), with local adapted coordinates

$$(x^i,x^a,x^u)_{1\leq i\leq q_1<a\leq q_2<u\leq m},$$

such that in every domain U, leaves of $\mathcal{F}_1$ are defined by fixing the first q_1 coordinates and the leaves of $\mathcal{F}_2$ are defined by $x^i=const.$ and $x^a=const.$

In this paper, the indices will take the following values: $i,i_1,..=\overline{1,q_1}$; $a,a_1,..=\overline{q_1+1,q_2}$; $u,u_1,...=\overline{q_2+1,m}$ and, begining with the next section, $\alpha,\alpha_1,..=\overline{1,n}$.

For two adapted local charts $(U,(x^i,x^a,x^u))$, $(\bar{U},(\bar{x}^{i_1},\bar{x}^{a_1},\bar{x}^{u_1})$ whose domains overlap, in $U\cap\bar{U}$, there are the following relations:

$$\frac{\partial x^i}{\partial\bar{x}^{a_1}}=\frac{\partial x^i}{\partial\bar{x}^{u_1}}=\frac{\partial x^a}{\partial\bar{x}^{u_1}}=0,$$

so the change rules for local coordinates are

$$\bar{x}^{i_1}=\bar{x}^{i_1}(x^i),\quad \bar{x}^{a_1}=\bar{x}^{a_1}(x^i,x^a),\quad \bar{x}^{u_1}=\bar{x}^{u_1}(x^i,x^a,x^u). \tag{3}$$

For such an adapted chart $(U,(x^i,x^a,x^u))$, the local coordinates on the plaque $U\cap\mathcal{F}_2$ are (x^u), so the bundle F_2 is locally spanned by $\left\{\frac{\partial}{\partial x^u}\right\}_{q_2<u\leq m}$. Let us denote

$$\frac{\delta}{\delta x^a}=p_2\left(\frac{\partial}{\partial x^a}\right),$$

the projection of vector field $\frac{\partial}{\partial x^a}$ on the normal bundle QF_2, for every $a=\overline{q_1+1,q_2}$. Since $\frac{\delta}{\delta x^a}-\frac{\partial}{\partial x^a}$ belongs to F_2, there are the local differentiable functions $t_a^u\in\Omega^0(U)$ such that

$$\frac{\delta}{\delta x^a}=\frac{\partial}{\partial x^a}-t_a^u\frac{\partial}{\partial x^u}, \tag{4}$$

where we use the Einstein convention for summation.

Local coordinates on the plaque $U\cap\mathcal{F}_1$ are (x^a,x^u), so the bundle F_1 is locally spanned by $\left\{\frac{\partial}{\partial x^a},\frac{\partial}{\partial x^u}\right\}_{q_1<a\leq q_2<u\leq m}$. Let us denote

$$\frac{\delta}{\delta x^i}=p_1\left(\frac{\partial}{\partial x^i}\right),$$

the projection of $\frac{\partial}{\partial x^i}$ on the normal bundle QF_1, for every $i=\overline{1,q_1}$.

Since $\frac{\delta}{\delta x^i}-\frac{\partial}{\partial x^i}$ belongs to F_1, there are the local differentiable functions $t_i^a, t_i^u \in \Omega^0(U)$ such that

$$\frac{\delta}{\delta x^i}=\frac{\partial}{\partial x^i}-t_i^a\frac{\partial}{\partial x^a}-t_i^u\frac{\partial}{\partial x^u}. \tag{5}$$

The functions t_a^u, t_i^a, t_i^u are satisfying $g\left(\frac{\delta}{\delta x^a},\frac{\partial}{\partial x^u}\right)=0$, $g\left(\frac{\delta}{\delta x^i},\frac{\partial}{\partial x^k}\right)=0$, $\forall q_1<k\leq m$, respectively.

The coordinate transformation (3) leads to the following transformation for t_a^u, t_i^a, t_i^u:

$$\bar{t}_{a_1}^{u_1}\frac{\partial x^u}{\partial \bar{x}^{u_1}}=\frac{\partial x^u}{\partial \bar{x}^{a_1}}+t_a^u\frac{\partial x^a}{\partial \bar{x}^{a_1}}, \tag{6}$$

$$\bar{t}_{i_1}^{a_1}\frac{\partial x^a}{\partial \bar{x}^{a_1}}=\frac{\partial x^a}{\partial \bar{x}^{i_1}}+t_i^a\frac{\partial x^i}{\partial \bar{x}^{i_1}}, \tag{7}$$

$$\bar{t}_{i_1}^{a_1}\frac{\partial x^u}{\partial \bar{x}^{a_1}}+\bar{t}_{i_1}^{u_1}\frac{\partial x^u}{\partial \bar{x}^{u_1}}=\frac{\partial x^u}{\partial \bar{x}^{i_1}}+t_i^u\frac{\partial x^i}{\partial \bar{x}^{i_1}}, \tag{8}$$

since we have

$$\frac{\delta}{\delta \bar{x}^{a_1}}=\frac{\partial x^a}{\partial \bar{x}^{a_1}}\frac{\delta}{\delta x^a},\quad \frac{\delta}{\delta \bar{x}^{i_1}}=\frac{\partial x^i}{\partial \bar{x}^{i_1}}\frac{\delta}{\delta x^i}.$$

We obtained in this way the local basis

$$\left\{\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^a},\frac{\partial}{\partial x^u}\right\}, \tag{9}$$

of TM, adapted to (F_1,F_2), where the vector fields $\{\frac{\delta}{\delta x^i}\}_i$ spanned a complementary distribution to the structural distribution of $\mathcal{F}_1$ in TM, and $\{\frac{\delta}{\delta x^i},\frac{\delta}{\delta x^a}\}_{i,a}$ spanned a complementary distribution to the structural distribution of $\mathcal{F}_2$ in TM.

2.2 First order jet manifold of a bundle

Let (E,π,M) be a fiber bundle, where $\pi: E\to M$ is a surjective submersion, M is a m-dimensional differentiable manifold and the fibre dimension is equal to n (so, E is a $(m+n)$-dimensional manifold). For a local chart $\left(V,\left(x^i\right)\right)$ in M, the adapted coordinate system in $\pi^{-1}(V)\subset E$ is $\left(x^i,y^\alpha\right)$, where $i=\overline{1,m}$, $\alpha=\overline{1,n}$. We shall use the same notation x^i for the coordinate functions x^i from M and $x^i\circ\pi$ from the manifold E. A local section of the bundle π in $x\in M$ is a map $\Phi: V\to E$, $x\in V\subset M$ such that $\Phi\circ\pi=1_V$. The set of all local sections of π in x is denoted by $\Gamma_x(\pi)$. In [10] is defined the 1-jet of a local section as follows:

Definition 1. *We said that two local sections $\Phi,\Psi\in\Gamma_x(\pi)$ are **1-equivalent** at x if $\Phi(x)=\Psi(x)$ and if in some adapted coordinate system $\left(x^i,y^\alpha\right)$ around $\Phi(x)$,*

$$\frac{\partial\left(y^\alpha\circ\Phi\right)}{\partial x^i}(x)=\frac{\partial\left(y^\alpha\circ\Psi\right)}{\partial x^i}(x), \tag{10}$$

*for $i=\overline{1,m}$ and $\alpha=\overline{1,n}$. The equivalence class containing Φ is called the **1-jet** of the section Φ at x and is denoted $j_x^1\Phi$.*

Obviously, the conditions (10) have geometrical meaning and for $\Phi, \Psi \in \Gamma_x(\pi)$ which satisfy $\Phi(x) = \Psi(x)$, the equality $j_x^1\Phi = j_x^1\Psi$ is equivalent with $\Phi_{*|T_xM} = \Psi_{*|T_xM}$, where Φ_* is the linear tangent map of the map Φ.

The *1-jet manifold* of π is the set

$$J^1\pi = \left\{ j_x^1\Phi \mid x \in M, \Phi \in \Gamma_x(\pi) \right\}.$$

Given an atlas of adapted charts (U, z) on E, where $z = \left(x^i, y^\alpha\right)$, the collection of charts $\left(U^1, z^1\right)$ is a $(m + n + mn)$-dimensional C^∞-atlas on $J^1\pi$, where

$$U^1 = \left\{ j_x^1\Phi \in J^1\pi \mid \Phi(x) \in U \right\},$$

and the functions

$$z^1 = \left(x^i, y^\alpha, y_i^\alpha\right), \tag{11}$$

are defined by $x^i\left(j_x^1\Phi\right) = x^i(x)$, $y^\alpha\left(j_x^1\Phi\right) = \Phi(x)$, $y_i^\alpha\left(j_x^1\Phi\right) = \frac{\partial(y^\alpha\circ\Phi)}{\partial x^i}(x)$. Moreover, $\left(J^1\pi, \pi_1, M\right)$ and $\left(J^1\pi, \pi_{1,0}, E\right)$ are bundles, where the surjective submersions $\pi_1 : J^1\pi \to M$, $\pi_{1,0} : J^1\pi \to E$ are defined by $\pi_1\left(j_x^1\Phi\right) = x$ and $\pi_{1,0}\left(j_x^1\Phi\right) = \Phi(x)$.

In the following we shall consider the bundles over a manifold endowed with a 2-flag (also called subfoliation).

3 First order jets of a bundle, adapted to a subfoliation of the base space

Let (M, F_1, F_2) be an m-dimensional manifold with (q_1, q_2)- subfoliation (F_1, F_2), and (E, π, M) a bundle over M of rank n, so that $\dim E = n + m$. For a local adapted chart $\left(V, \left(x^i, x^a, x^u\right)\right)$ in M, we have an adapted local chart $(U, (x^i, x^a, x^u, y^\alpha))$ in E and an adapted local chart $\left(U^1, \left(x^i, x^a, x^u, y^\alpha, y_i^\alpha, y_a^\alpha, y_u^\alpha\right)\right)$ in the 1-jet manifold $J^1\pi$. The last one is exactly the chart (11), where we replace the indice i taking values from 1 to n with the indices $i = \overline{1, q_1}$, $a = \overline{q_1 + 1, q_2}$, $u = \overline{q_2 + 1, m}$.

3.1 Leafwise F_2-1-jet manifold

Definition 2. *We say that two local sections $\Phi, \Psi \in \Gamma_x(\pi)$ are **leafwise** F_2**-equivalent** at $x \in M$ if $\Phi(x) = \Psi(x)$ and if, in some adapted coordinate system $\left(x^i, x^a, x^u, y^\alpha\right)$ around $\Phi(x)$*

$$\frac{\partial(y^\alpha \circ \Phi)}{\partial x^u}(x) = \frac{\partial(y^\alpha \circ \Psi)}{\partial x^u}(x), \tag{1}$$

*for every $u = \overline{q_2 + 1, m}$. The equivalence class containing Φ is called the **leafwise** F_2**-1-jet** of Φ and it is denoted by $j_x^{l,F_2}\Phi$.*

Remark 1. *The conditions* (1) *do not depend upon the choice of charts. So, let us consider* $\left(\tilde{x}^{i_1}, \tilde{x}^{a_1}, \tilde{x}^{u_1}, \tilde{y}^{\alpha_1}\right)$ *be another coordinate system around* $\Phi(x)$. *Then we have*

$$\frac{\partial\left(\tilde{y}^{\alpha_1}\circ\Phi\right)}{\partial\tilde{x}^{u_1}}(x)=\frac{\partial\tilde{y}^{\alpha_1}}{\partial x^u}(\Phi(x))\cdot\frac{\partial x^u}{\partial\tilde{x}^{u_1}}(x)+\frac{\partial\tilde{y}^{\alpha_1}}{\partial y^\alpha}(\Phi(x))\cdot\frac{\partial\left(y^\alpha\circ\Phi\right)}{\partial x^u}(x)\cdot\frac{\partial x^u}{\partial\tilde{x}^{u_1}}(x),$$

using the relationship $x^u\circ\Phi=x^u$ *between similarly-named coordinate functions on* E *and* M. *The result follows now from* $\Phi(x)=\Psi(x)$ *and relations* (1).

Remark 2. *We can say that* $\Phi,\Psi\in\Gamma_x(\pi)$ *are leafwise* F_2*-equivalent at* x *if they are 1-equivalent at* x *(Definition 1) in the leaf of* $\mathcal{F}_2$ *which contains the point* x.

Proposition 1. *Let* $\Phi,\Psi\in\Gamma_x(\pi)$ *be two local sections such that* $\Phi(x)=\Psi(x)$. *Then* $j_x^{l,F_2}\Phi=j_x^{l,F_2}\Psi$ *if and only if* $\Phi_{*|F_{2,x}}=\Psi_{*|F_{2,x}}$.

Proof: In adapted coordinates $\left(x^i,x^a,x^u\right)$ around $x\in M$, we have $F_{2,x}=span\left\{\frac{\partial}{\partial x^u}(x)\right\}$,

$$\Phi_{*,x}\left(\frac{\partial}{\partial x^u}(x)\right)=\frac{\partial\left(y^\alpha\circ\Phi\right)}{\partial x^u}(x)\cdot\frac{\partial}{\partial y^\alpha}(\Phi(x)),$$

and a similar expression for $\Psi_{*,x}\left(\frac{\partial}{\partial x^u}(x)\right)$. The equality $j_x^{l,F_2}\Phi=j_x^{l,F_2}\Psi$ is equivalent with $\Phi(x)=\Psi(x)$ and relations (1), which implies $\Phi_{*,x}\left(\frac{\partial}{\partial x^u}(x)\right)=\Psi_{*,x}\left(\frac{\partial}{\partial x^u}(x)\right)$. □

Remark 3. *If* $j_x^1\Phi=j_x^1\Psi$, *then* $j_x^{l,F_2}\Phi=j_x^{l,F_2}\Psi$. *Indeed, if* Φ *and* Ψ *are local sections 1-equivalent at* x, *then* $\Phi_{*|T_xM}=\Psi_{*|T_xM}$, *which assures* $\Phi_{*|F_{2,x}}=\Psi_{*|F_{2,x}}$, *so* $j_x^{l,F_2}\Phi=j_x^{l,F_2}\Psi$ *by Proposition 1. The converse is not true:* $j_x^{l,F_2}\Phi=j_x^{l,F_2}(\Phi+\Omega)$ *for every local section* $\Omega\in\Gamma_x(\pi)$ *which is basic with respect to foliation* $\mathcal{F}_2$ *(that means* $\frac{\partial(y^\alpha\circ\Omega)}{\partial x^u}(x)=0$*), but* $j_x^1\Phi\neq j_x^1(\Phi+\Omega)$.

Let $\mathcal{A}_E=\left\{\left(U,z=\left(x^i,x^a,x^u,y^\alpha\right)\right)\right\}$ be an adapted atlas on E. The induced coordinate system $\left(U^{l,F_2},z^{l,F_2}\right)$ on the set

$$J_{F_2}^l\pi=\left\{j_x^{l,F_2}\Phi\mid x\in M,\Phi\in\Gamma_x(\pi)\right\},$$

is defined by:

$$U^{l,F_2}=\left\{j_x^{l,F_2}\Phi\in J^{l,F_2}\pi\mid\Phi(x)\in U\right\},\ z^{l,F_2}=\left(x^i,x^a,x^u,y^\alpha,z_u^\alpha\right),\qquad(2)$$

with $x^i\left(j_x^{l,F_2}\Phi\right)=x^i(x)$, $x^a\left(j_x^{l,F_2}\Phi\right)=x^a(x)$, $x^u\left(j_x^{l,F_2}\Phi\right)=x^u(x)$, $y^\alpha\left(j_x^{l,F_2}\Phi\right)=\Phi(x)$, and $z_u^\alpha\left(j_x^{l,F_2}\Phi\right)=\frac{\partial(y^\alpha\circ\Phi)}{\partial x^u}(x)$, respectively.

It is easy to verify that the collection of all charts $\left(U^{l,F_2},z^{l,F_2}\right)$ is a $(m+n+n(m-q_2))$-dimensional C^∞-atlas on $J^{l,F_2}\pi$. The maps

$$\pi_1^l:J_{F_2}^l\pi\to M;\quad\pi_1^l\left(j_x^{l,F_2}\Phi\right)=x,$$

$$\pi^l_{1,0} : J^l_{F_2}\pi \to E; \quad \pi^l_{1,0}\left(j^{l,F_2}_x \Phi\right) = \Phi(x),$$

for every $j^{l,F_2}_x \Phi \in J^l_{F_2}\pi$, are the correspondent of maps π_1, $\pi_{1,0}$ from subsection 2.2. The Remark 3 assures that the map

$$\pi^l : J^1\pi \to J^l_{F_2}\pi; \quad \pi^l\left(j^1_x\Phi\right) = j^{l,F_2}_x\Phi,$$

is well-defined. Moreover, there are the relations

$$\pi^l_{1,0} \circ \pi^l = \pi_{1,0}, \tag{3}$$

$$\pi^l_1 \circ \pi^l = \pi_1; \quad \pi \circ \pi^l_{1,0} = \pi^l_1.$$

Proposition 2. *The functions $\pi^l_{1,0}$, π^l are surjective submersions.*

Proof: Let $w \in E$ and $x = \pi(w) \in M$. There is a local section $\Phi \in \Gamma_x(\pi)$ such that $\Phi(x) = w$ and then $\pi^l_{1,0}\left(j^{l,F_2}_x\Phi\right) = w$. Hence, $\pi^l_{1,0}$ is a surjective map. Using the local coordinates (U, z) around $w = \Phi(x)$, and $\left(U^{l,F_2}, z^{l,F_2}\right)$ around $j^{l,F_2}_x\Phi$, the composite map $z \circ \pi^l_{1,0} \circ \left(z^{l,F_1}\right)^{-1}$ is just the projection $pr_1 : z^{l,F_2}\left(U^{l,F_2}\right) \subset R^{n+m} \times R^{m(n-q_2)} \to R^{n+m}$, so its rank is equal to $n+m$. Then $\pi^l_{1,0}$ is a submersion. A similar argument is used for the map π^l, so it is also a submersion. For an arbitrary $j^{l,F_2}_x\Phi \in J^{l,F_2}\pi$, the 1-jet of the section Φ satisfies $\pi^l\left(j^1_x\Phi\right) = j^{l,F_2}_x\Phi$. We obtained that π^l is surjective. □

Remark 4. *Taking into account the Proposition 2, $\left(J^l_{F_2}\pi, \pi^l_{1,0}, E\right)$ is a fibered manifold, and the relation* (3) *implies that π^l is a surjective bundles morphism.*

Example 1. *If π is the trivial bundle $(M \times \boldsymbol{R}, \pi, M)$, then it is known, [10], that there is a canonical diffeomorphism $\varphi : J^1\pi \to T^*M \times \mathbf{R}$ defined by $\varphi(j^1_x\Phi) = (d\bar{\Phi}_x, \bar{\Phi}(x))$, where $\bar{\Phi} = pr_2 \circ \Phi$, for $\Phi \in \Gamma_x(\pi)$, and $pr_2 : M \times \boldsymbol{R} \to \boldsymbol{R}$. The map*

$$\tau : J^l_{F_2}\pi \to F^*_2 \times \boldsymbol{R}, \quad \tau(j^{l,F_2}_x\Phi) = (d^{\mathcal{F}_{21}}_{01}\bar{\Phi}_x, \bar{\Phi}(x)),$$

is also a diffeomorphism, where F^*_2 is the dual of bundle F_2, and the operator $d^{\mathcal{F}_{21}}_{01}$ is introduced in (2).

3.2 The F_{21}-1-jet manifold of the bundle π

Let (E, π, M) be a bundle with the base space endowed with a subfoliation manifold like in previous section.

Definition 3. *We say that two local sections $\Phi, \Psi \in \Gamma_x(\pi)$ are F_{21}**-1-equivalent** at $x \in M$ if $\Phi(x) = \Psi(x)$ and if, in some adapted coordinate system $\left(x^i, x^a, x^u, y^\alpha\right)$ around $\Phi(x)$,*

$$\frac{\delta\left(y^\alpha \circ \Phi\right)}{\delta x^a}(x) = \frac{\delta\left(y^\alpha \circ \Psi\right)}{\delta x^a}(x), \tag{4}$$

for every $a = \overline{q_1 + 1, q_2}$, (where $\frac{\delta}{\delta x^a}$ is introduced in (4)*). The equivalence class containing Φ is called the F_{21}**-1-jet** of Φ and it is denoted by $j^{1,F_{21}}_x\Phi$.*

Remark 5. *By a straightforward computation, we see that the conditions* (4) *have geometrical meaning.*

Proposition 3. *Let* $\Phi, \Psi \in \Gamma_x(\pi)$ *be two local sections such that* $\Phi(x) = \Psi(x)$. *Then* $j_x^{1,F_{21}}\Phi = j_x^{1,F_{21}}\Psi$ *if and only if* $\Phi_{*|QF_{21,x}} = \Psi_{*|QF_{21,x}}$.

Proof: In adapted coordinates (x^i, x^a, x^u) around $x \in M$, we have $QF_{21,x} = span\left\{\frac{\delta}{\delta x^a}(x)\right\}$,

$$\Phi_{*,x}\left(\frac{\delta}{\delta x^a}(x)\right) = \frac{\delta(y^\alpha \circ \Phi)}{\delta x^a}(x) \cdot \frac{\partial}{\partial y^\alpha}(\Phi(x)),$$

and a similar expression for $\Psi_{*,x}\left(\frac{\delta}{\delta x^a}(x)\right)$. The equality $j_x^{1,F_{21}}\Phi = j_x^{1,F_{21}}\Psi$ is equivalent with $\Phi(x) = \Psi(x)$ and relation (4), which implies

$$\Phi_{*,x}\left(\frac{\delta}{\delta x^a}(x)\right) = \Psi_{*,x}\left(\frac{\delta}{\delta x^a}(x)\right).$$

□

The set of all F_{21}-1-jets of π,

$$J^1_{F_{21}}\pi = \left\{ j_x^{1,F_{21}}\Phi \mid x \in M, \Phi \in \Gamma_x(\pi) \right\},$$

has a natural structure of $(m + n + n(q_2 - q_1))$-dimensional C^∞-differentiable manifold given by an atlas whose charts are $(U^{F_{21}}, z^{F_{21}})$ where

$$U^{F_{21}} = \left\{ j_x^{1,F_{21}}\Phi \in J^1_{F_{21}}\pi \mid \Phi(x) \in U \right\},\ z^{F_{21}} = \left(x^i, x^a, x^u, y^\alpha, z^\alpha_a\right), \tag{5}$$

with $x^i\left(j_x^{1,F_{21}}\Phi\right) = x^i(x)$, $x^a\left(j_x^{1,F_{21}}\Phi\right) = x^a(x)$, $x^u\left(j_x^{1,F_{21}}\Phi\right) = x^u(x)$, $y^\alpha\left(j_x^{1,F_{21}}\Phi\right) = \Phi(x)$, and $z^\alpha_a\left(j_x^{1,F_{21}}\Phi\right) = \frac{\delta(y^\alpha \circ \Phi)}{\delta x^a}(x)$, respectively, where $(U, (x^i, x^a, x^u, y^\alpha))$ is a local chart on E. The manifold $J^1_{F_{21}}\pi$ could be also called the (F_1, F_2)-*1-jet manifold*. Moreover, there are the following maps who give to $J^1_{F_{21}}\pi$ some structures of fibered manifold:

$$\pi_1^{F_{21}} : J^1_{F_{21}}\pi \to M; \quad \pi_1^{F_{21}}\left(j_x^{1,F_{21}}\Phi\right) = x,$$

$$\pi_{1,0}^{F_{21}} : J^1_{F_{21}}\pi \to E; \quad \pi_{1,0}^{F_{21}}\left(j_x^{1,F_{21}}\Phi\right) = \Phi(x),$$

for $j_x^{1,F_{21}}\Phi \in J^1_{F_{21}}\pi$. They are the correspondent of maps π_1, $\pi_{1,0}$ from subsection 2.2.

The maps

$$\pi^{F_{21}} : J^1\pi \to J^1_{F_{21}}\pi\,; \ \pi^{F_{21}}\left(j_x^1\Phi\right) = j_x^{1,F_{21}}\Phi,$$

are well-defined. Indeed, from Proposition 3, the equality $j^1_x\Phi = j^1_x\Psi$ implies $j_x^{1,F_{21}}\Phi = j_x^{1,F_{21}}\Psi$. Moreover, there are the following relations

$$\pi_{1,0}^{F_{21}} \circ \pi^{F_{21}} = \pi_{1,0}, \tag{6}$$

$$\pi_1^{F_{21}} \circ \pi^{F_{21}} = \pi_1; \quad \pi \circ \pi_{1,0}^{F_{21}} = \pi_1^{F_{21}}.$$

From the similar reasons as in Proposition 2, the functions $\pi_{1,0}^{F_{21}}$, $\pi^{F_{21}}$ are surjective submersions.

3.3 Transversal F_1-1-jet manifold

Definition 4. *We say that two local sections $\Phi, \Psi \in \Gamma_x(\pi)$ are* ***transversal F_1-1-equivalent*** *at $x \in M$ if $\Phi(x) = \Psi(x)$ and if, in some adapted coordinate system $\left(x^i, x^a, x^u, y^\alpha\right)$ around $\Phi(x)$*

$$\frac{\delta\left(y^\alpha \circ \Phi\right)}{\delta x^i}(x) = \frac{\delta\left(y^\alpha \circ \Psi\right)}{\delta x^i}(x), \tag{7}$$

for every $i = \overline{1, q_1}$, (where $\frac{\delta}{\delta x^i}$ is introduced in (9)). The equivalence class containing Φ is called the ***transversal F_1-jet*** *of Φ and it is denoted by $j_x^{t,F_1}\Phi$.*

Remark 6. *By a straightforward calculation, the conditions (7) have geometrical meaning.*

Proposition 4. *Let $\Phi, \Psi \in \Gamma_x(\pi)$ be two local sections such that $\Phi(x) = \Psi(x)$. Then $j_x^{t,F_1}\Phi = j_x^{t,F_1}\Psi$ if and only if $\Phi_{*|QF_{1,x}} = \Psi_{*|QF_{1,x}}$.*

Proof: In adapted coordinates $\left(x^i, x^a, x^u\right)$ around $x \in M$, we have $QF_{1,x} = span\left\{\frac{\delta}{\delta x^i}(x)\right\}$,

$$\Phi_{*,x}\left(\frac{\delta}{\delta x^i}(x)\right) = \frac{\delta\left(y^\alpha \circ \Phi\right)}{\delta x^i}(x) \cdot \frac{\partial}{\partial y^\alpha}(\Phi(x)),$$

and a similar expression for $\Psi_{*,x}\left(\frac{\delta}{\delta x^a}(x)\right)$. The equality $j_x^{t,F_1}\Phi = j_x^{t,F_1}\Psi$ is equivalent with $\Phi(x) = \Psi(x)$ and relation (7), which implies $\Phi_{*,x}\left(\frac{\delta}{\delta x^a}(x)\right) = \Psi_{*,x}\left(\frac{\delta}{\delta x^a}(x)\right)$. □

The set of all transversal F_1-jets of π,

$$J_{F_1}^t\pi = \left\{j_x^{t,F_1}\Phi \mid x \in M, \Phi \in \Gamma_x(\pi)\right\},$$

has a natural structure of $(m + n + n \cdot q_1)$)-dimensional C^∞-differentiable manifold given by an atlas whose charts are $\left(U^{t,F_1}, z^{t,F_1}\right)$, where

$$U^{t,F_1} = \left\{j_x^{t,F_1}\Phi \in J_{F_1}^t\pi \mid \Phi(x) \in U\right\}, \; z^{t,F_1} = \left(x^i, x^a, x^u, y^\alpha, z_i^\alpha\right), \tag{8}$$

with $x^i\left(j_x^{t,F_1}\Phi\right) = x^i(x)$, $x^a\left(j_x^{t,F_1}\Phi\right) = x^a(x)$, $x^u\left(j_x^{t,F_1}\Phi\right) = x^u(x)$, $y^\alpha\left(j_x^{t,F_1}\Phi\right) = \Phi(x)$, and $z_i^\alpha\left(j_x^{t,F_1}\Phi\right) = \frac{\delta(y^\alpha \circ \Phi)}{\delta x^i}(x)$, respectively, where $\left(U, \left(x^i, x^a, x^u, y^\alpha\right)\right)$ is a local chart on E. Moreover, there are the following maps which give to $J_{F_1}^t\pi$ some structures of fibered manifold:

$$\pi_1^{t,F_1} : J_{F_1}^t\pi \to M; \quad \pi_1^{t,F_1}\left(j_x^{t,F_1}\Phi\right) = x,$$

$$\pi_{1,0}^{t,F_1} : J_{F_1}^t\pi \to E; \quad \pi_{1,0}^{t,F_1}\left(j_x^{t,F_1}\Phi\right) = \Phi(x),$$

for $j_x^{t,F_1}\Phi \in J_{F_1}^t\pi$. They are the correspondent of maps π_1, $\pi_{1,0}$ from subsection 2.2.

The maps

$$\pi^{t,F_1} : J^1\pi \to J^t_{F_1}\pi\,; \quad \pi^{t,F_1}\left(j^1_x\Phi\right) = j^{t,F_1}_x\Phi,$$

are well-defined. Indeed, from Proposition 4, the equality $j^1_x\Phi = j^1_x\Psi$ implies $j^{t,F_1}_x\Phi = j^{t,F_1}_x\Psi$. Moreover, there are the following relations

$$\pi^{t,F_1}_{1,0} \circ \pi^{t,F_1} = \pi_{1,0}, \tag{9}$$

$$\pi^{t,F_1}_1 \circ \pi^{t,F_1} = \pi_1; \pi \circ \pi^{t,F_1}_{1,0} = \pi^{t,F_1}_1.$$

From the similar reasons as in Proposition 2, the functions $\pi^{t,F_1}_{1,0}$, π^{t,F_1} are surjective submersions.

3.4 A decomposition theorem for $J^1\pi$

This subsection contains the main result of the paper, a diffeomorphism between the 1-jet manifold of π and the total space of the fibered product $J^t_{F_1}\pi \times_E J^1_{F_{21}}\pi \times_E J^l_{F_2}\pi$.

From the Propositions 1, 3 and 4, it results:

Proposition 5. *Let $\Phi, \Psi \in \Gamma_x(\pi)$ be two local sections of bundle π. Then the following assertions are equivalent:*

a) $j^1_x\Phi = j^1_x\Psi$;

b) $j^{l,F_2}_x\Phi = j^{l,F_2}_x\Psi$, $j^{t,F_1}_x\Phi = j^{t,F_1}_x\Psi$ *and* $j^{1,F_{21}}_x\Phi = j^{1,F_{21}}_x\Psi$.

Given a bundle (E, π, M) over the Riemannian manifold (M, g) endowed with a subfoliation (F_1, F_2), there are four bundles of first order jets of π, with the base space E: $\left(J^1\pi, \pi_{1,0}, E\right)$, $\left(J^l_{F_2}\pi, \pi^{l,F_2}_{1,0}, E\right)$, $\left(J^1_{F_{21}}\pi, \pi^{F_{21}}_{1,0}, E\right)$, and $\left(J^t_{F_1}\pi, \pi^{t,F_1}_{1,0}, E\right)$, respectively. The fibered product of the last three bundles has the total space

$$\begin{gathered} J^t_{F_1}\pi \times_E J^1_{F_{21}}\pi \times_E J^l_{F_2}\pi = \\ = \left\{\left(j^{t,F_1}_x\Phi, j^{1,F_{21}}_x\Psi, j^{l,F_2}_x\xi\right) \mid \pi^{t,F_1}_{1,0}\left(j^{t,F_1}_x\Phi\right) = \pi^{F_{21}}_{1,0}\left(j^{1,F_{21}}_x\Psi\right) = \pi^{l,F_2}_{1,0}(j^{l,F_2}_x\xi)\right\} = \\ = \left\{\left(j^{t,F_1}_x\Phi, j^{1,F_{21}}_x\Psi, j^{l,F_2}_x\xi\right) \mid \Phi(x) = \Psi(x) = \xi(x)\right\}, \end{gathered} \tag{10}$$

and the projection map $\pi^{t,F_1}_{1,0} \times_E \pi^{F_{21}}_{1,0} \times_E \pi^{l,F_2}_{1,0} : J^t_{F_1}\pi \times_E J^1_{F_{21}}\pi \times_E J^l_{F_2}\pi \to E$ defined by

$$\left(\pi^{t,F_1}_{1,0} \times_E \pi^{F_{21}}_{1,0} \times_E \pi^{l,F_2}_{1,0}\right)\left(j^{t,F_1}_x\Phi, j^{1,F_{21}}_x\Psi, j^{l,F_2}_x\xi\right) = \Phi(x) = \Psi(x) = \xi(x).$$

If $\left\{\left(U^{t,F_1}, \left(x^i, x^a, x^u, y^\alpha, z^\alpha_i\right)\right)\right\}$, $\left\{\left(U^{F_{21}}, \left(x^i, x^a, x^u, y^\alpha, z^\alpha_a\right)\right)\right\}$ and $\left\{\left(U^{l,F_2}, \left(x^i, x^a, x^u, y^\alpha, z^\alpha_u\right)\right)\right\}$ are atlases on the manifolds $J^t_{F_1}\pi$, $J^1_{F_{21}}\pi$ and $J^l_{F_2}\pi$, respectively, then $\left\{\left(U^{t,F_1} \times U^{F_{21}} \times U^{l,F_2}, \left(x^i, x^a, x^u, y^\alpha, z^\alpha_i, z^\alpha_a, z^\alpha_u\right)\right)\right\}$ is a C^∞ -atlas on $J^t_{F_1}\pi \times_E J^1_{F_{21}}\pi \times_E J^t_{F_1}\pi$. Now, we can give the main result of this paper:

Theorem 1. *The map*

$$\mu : J^1\pi \to J^t_{F_1}\pi \times_E J^1_{F_{21}}\pi \times_E J^l_{F_2}\pi,$$

defined by

$$\mu\left(j^1_x\Phi\right) = \left(j^{t,F_1}_x\Phi, j^{1,F_{21}}_x\Phi, j^{l,F_2}_x\Phi\right),$$

is a diffeomorphism between the 1-jet manifold $J^1\pi$ and $J^t_{F_1}\pi \times_E J^1_{F_{21}}\pi \times_E J^l_{F_2}\pi$.

Proof: First of all, we have to remark that from Proposition 5, the map μ is well-defined and injective. We shall prove that it is surjective, too. For an arbitrary triple $\left(j^{t,F_1}_x\Phi, j^{1,F_{21}}_x\Psi, j^{l,F_2}_x\xi\right) \in J^t_{F_1}\pi \times_E J^1_{F_{21}}\pi \times_E J^l_{F_2}\pi$, we search for a local section $\Omega \in \Gamma_x(\pi)$ transversal F_1-1-equivalent at x to Φ, F_{21}-1-equivalent at x to Ψ and leafwise F_2-1-equivalent at x to ξ.

If we have $\Phi : U_1 \to E$, $\Psi : U_2 \to E$ and $\xi : U_3 \to E$ with $x \in U_1 \cap U_2 \cap U_3 \subset M$, then we can define the local section $\Omega : U \to E$, $x \in U \subset U_1 \cap U_2 \cap U_3$, by its local representation in $\left(\pi^{-1}(U), \left(x^i, x^a, x^u, y^\alpha\right)\right)$:

$$x^i \circ \Omega = x^i; \quad x^a \circ \Omega = x^a; \quad x^u \circ \Omega = x^u,$$

$$\begin{aligned} y^\alpha \circ \Omega \quad &= \left(\frac{\delta\left(y^\alpha \circ \Phi\right)}{\delta x^i}(x) + t^a_i(x) \cdot \frac{\partial\left(y^\alpha \circ \Psi\right)}{\partial x^a}(x) + t^u_i(x) \cdot \frac{\partial\left(y^\alpha \circ \xi\right)}{\partial x^u}(x)\right) \cdot x^i \mid_U \\ &+ \left(t^u_a(x) \cdot \frac{\partial\left(y^\alpha \circ \xi\right)}{\partial x^u}(x) + \frac{\delta\left(y^\alpha \circ \Psi\right)}{\delta x^a}(x)\right) \cdot x^a \mid_U + \frac{\partial\left(y^\alpha \circ \xi\right)}{\partial x^u}(x) \cdot x^u \mid_U, \end{aligned}$$

and it is easy to see that it satisfies the required conditions. This proved that μ is a surjective map.

The map μ is diffeomorfism. Indeed, if $\left(U^1, z^1 = \left(x^i, x^a, x^u, y^\alpha, y^\alpha_k\right)_{k=\overline{1,n}}\right)$ is a local chart around $j^1_x\Phi \in J^1\pi$, and $\Big(U^{t,F_1} \times U^{F_{21}} \times U^{l,F_2}, z = (x^i, x^a, x^u, y^\alpha, z^\alpha_i, z^\alpha_a, z^\alpha_u)\Big)$, is a local chart around $\mu\left(j^1_x\Phi\right)$, then, for every $\left(\zeta^i, \zeta^a, \zeta^u, \zeta^\alpha, \zeta^\alpha_k\right) \in z^1\left(U^1\right) \subset R^{q_1} \times R^{q_2-q_1} \times R^{n-q_2} \times R^n \times R^{mn}$, we obtain $\left(z \circ \mu \circ (z^1)^{-1}\right)\left(\zeta^i, \zeta^a, \zeta^u, \zeta^\alpha, \zeta^\alpha_k\right) = (\zeta^i, \zeta^a, \zeta^u, \zeta^\alpha, \zeta^\alpha_i - t^a_i(x)\zeta^\alpha_a - t^u_i(x)\zeta^\alpha_u, \zeta^\alpha_a - t^u_a(x)\zeta^\alpha_u, \zeta^\alpha_u)$, where $\zeta^\alpha_k = (\zeta^\alpha_i, \zeta^\alpha_a, \zeta^\alpha_u) \in R^{nq_1} \times R^{n(q_2-q_1)} \times R^{n(m-q_2)}$. □

4 First order jets of bundles over a big-tangent manifold

An example of a manifold which admits a subfoliation is the big-tangent manifold of a Riemannian manifold (M, g). Let us briefly recall some elementary notions about the geometry of big-tangent manifold $\mathcal{T}M$. For more see [17].

Let M be an n-dimensional smooth manifold and we consider the associated big tangent bundle $TM \oplus T^*M$. The total space of the big-tangent bundle, called *big-tangent manifold*, is a $3n$-dimensional smooth manifold denoted here by $\mathcal{T}M$.

The points of $\mathcal{T}M$ are triples (x, y, p), $x \in M$, $y \in T_xM$, $p \in T_x^*M$, and one has local coordinates (x^i, y^i, p_i), where $i = 1, \ldots, n = \dim M$, (x^i) are local coordinates on M, (y^i) are vector coordinates and (p_i) are covector coordinates. The change rules of these coordinates are:

$$\widetilde{x}^i = \widetilde{x}^i(x^j) \,,\, \widetilde{y}^i = \frac{\partial \widetilde{x}^i}{\partial x^j} y^j \,,\, \widetilde{p}_i = \frac{\partial x^j}{\partial \widetilde{x}^i} p_j. \tag{1}$$

Also, for the big-tangent manifold $\mathcal{T}M$ we have the following projections

$$p : \mathcal{T}M \to M \,,\, p_1 : \mathcal{T}M \to TM \,,\, p_2 : \mathcal{T}M \to T^*M$$

on M and on the total spaces of tangent and cotangent bundle, respectively.

As usual, we denote by $V = V(\mathcal{T}M)$ the vertical bundle on the big-tangent manifold $\mathcal{T}M$ and it has the decomposition

$$V = V_1 \oplus V_2, \tag{2}$$

where $V_1 = p_{1*}^{-1}(V(TM))$, $V_2 = p_{2*}^{-1}(V(T^*M))$ and have the local frames $\{\frac{\partial}{\partial y^i}\}$, $\{\frac{\partial}{\partial p_i}\}$, respectively. Since V, V_1, V_2 are integrable bundles and V_1, V_2 are subbundles of V, on the big tangent manifold $\mathcal{T}M$ there are two $(n, 2n)$-codimensional subfoliations: (V, V_1) and (V, V_2).

The subbundles V_1, V_2 are structural bundles of the vertical foliations $\mathcal{V}_1$, $\mathcal{V}_2$ of $\mathcal{T}M$ by fibers of p_2, p_1, respectively, and $\mathcal{T}M$ has a multi-foliate structure [17]. So, as usual, for tangent bundle and like in foliation theory, the geometry of the big-tangent manifold $\mathcal{T}M$ may be developed by considering a *horizontal bundle* H such that

$$T(\mathcal{T}M) = H \oplus V = H \oplus V_1 \oplus V_2. \tag{3}$$

According to subsection 2.1, $F_1 = V$, $QF_1 = H$, $F_2 = V_2$, $QF_{21} = V_1$.

An adapted basis to subfoliation (V, V_1) could be found considering g a Riemannian metric on M, see [17]. In this case, the Levi-Civita connection Γ on M with local coefficients Γ^i_{jk} locally span a complement of the vertical distribution. Then a horizontal bundle on $\mathcal{T}M$ has local bases

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - y^k \Gamma^j_{ik} \frac{\partial}{\partial y^j} + p_k \Gamma^k_{ij} \frac{\partial}{\partial p_j}, \tag{4}$$

corresponding to (5).

The adapted basis to subfoliation (V, V_1) is $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial p_i}\}$ and let $\{dx^i, \delta y^i, \delta p_i\}$ be its corresponding cobasis. Then, the formula

$$G = g_{ij}(x) dx^i \otimes dx^j + g_{ij}(x) \delta y^i \otimes \delta y^j + g^{ij}(x) \delta p_i \otimes \delta p_j, \tag{5}$$

defines a metric on the big-tangent manifold $\mathcal{T}M$, which is non degenerate on V and called the *Sasaki-type metric*. Here $(g^{ij}(x))$ denotes the inverse matrix of $(g_{ij}(x))$.

The first equality of (3) produces a double grading of forms and multivectors on $\mathcal{T}M$ of bidegree or type (p, q) that means H-degree p and V-degree q. The exterior diferential admits the decomposition (2), which becames:

$$d = d_{1,0} + d_{0,1} + d_{2,-1}, \quad d_{0,1} = d_{0,1,0} + d_{0,0,1}, \tag{6}$$

where $d_{0,1}$ means the exterior differential along the leaves of V.

Finaly, we consider the Riemannian manifold $(\mathcal{TM}, G)$ endowed with the subfoliation (V, V_2) to be the base space of a bundle π. According to Theorem 1, the 1-jet manifold of π is diffeomorphic with the fiberd product

$$J^t_V \pi \times_{\mathcal{T}M} J^1_{V_1} \pi \times_{\mathcal{T}M} J^l_{V_2} \pi.$$

In particular, let π be the trivial bundle $(\mathcal{T}M \times \mathbf{R}, \pi, \mathcal{T}M)$. Then, see Example 1, the maps

$$\varphi_2 : J^l_{V_2} \pi \to V_2^* \times \mathbf{R}, \varphi_2(j^{l,V_2}_w \Phi) = (d_{0,0,1} \bar{\Phi}, \bar{\Phi}(w)),$$

$$\varphi_1 : J^l_{V_1} \pi \to V_1^* \times \mathbf{R}, \varphi_1(j^{1,VV_2}_w \Phi) = (d_{0,1,0} \bar{\Phi}, \bar{\Phi}(w)),$$

are diffeomorphisms, where V_i^* is the dual of V_i, for $i = 1, 2$, $w = (x, y, p) \in \mathcal{T}M$, Φ a local section of π and $\bar{\Phi} = pr_2 \circ \Phi$, with projection $pr_2 : \mathcal{T}M \times \mathbf{R} \to \mathbf{R}$.

References

[1] Bejancu, A. and Farran, H. R., *Foliations and Geometrical Structures*, Mathematics and Its Applications Vol. **580**, Springer, Dordrecht, 2006.

[2] Kolar, I., Michor, P. W. and Slovak, J., *Natural operations in differential geometry*, Springer-Verlag, Berlin Heidelberg, 1993.

[3] Cordero, L. A. and Masa, X., *Characteristic classes of subfoliations*, Annales de l'Institut Fourier, **31** (1981), 61-86.

[4] Ida, C. and Popescu, P., *Vertical Liouville foliations on the big-tangent manifold of a Finsler space*, Preprint available to arXiv: 1402.6099v1.

[5] Krasil'shchik, J. and Verbovetsky, A., *Geometry of jet spaces and integrable systems*, Journal of Geometry and Physics, 1 **61(9)** (2010).

[6] Lebtahi, L., *Lie algebra on the transverse bundle of a decreasing family of foliations*, Journal of Geometry and Physics, **60** (2010) 122-130.

[7] McLean, M. and Norris, L. K., *Covariant field theory on frame bundles of fibered manifolds*, Journal of Mathematical Physics, **40** (2000), 6808-6823.

[8] Manea, A., *First order jets of bundles over a foliated manifold*, Modern Trends in Geometry and Topology, Proceedings of the 7-th Intl. Workshop on Differential Geometry and Applications, Deva, Sept. 2005, 273-281.

[9] Molino, P., *Riemannian foliations,* Progr. Math., Birkhäuser, Vol **73**, 1988.

[10] Saunders, D. J., *The Geometry of jet bundles*, London Math. Soc. Lecture Note Series, **142**, Cambridge U.P., 1989.

[11] Sardanashvily, G., *Differential geometry of fibre bundles over foliated manifolds*, arXiv: math-ph/010820, 2001.

[12] Sardanashvily, G., *Axiomatic classical (prequantum) field theory. Jet formalism*, arXiv:hep-th/0612182, 2006.

[13] Sardanashvily, G., *Fibre bundles, jet manifolds and Lagrangian theory. Lectures for theoreticians*, arXiv: math-ph/0908.1886, 2009.

[14] Tondeur, Ph., *Foliations on Riemannian manifolds,* Universitext, Springer-Verlag, New-York, 1988.

[15] Vaisman, I., *Varietes riemanniennes feuilletees,* Czechosl. Math. J., **21** (1971), 46-75.

[16] Vaisman, I., *Cohomology and differential forms,* Marcel Dekker Inc., New York, 1973.

[17] Vaisman, I., *Geometry on big-tangent manifolds.* Available to arXiv: 1303.0658v1. math.DG. 2013.