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ON THE SEMI-REAL QUATERNIONIC BERTRAND CURVES OF AW(k)-TYPE IN E_1^3

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Abstract

In this paper, we analyze the harmonic curvature conditions of AW(k)-type $(1 \leq k \leq 3)$ semi-real quaternionic curves and semi-real quaternionic Bertrand curves with $k \neq 0$ and $r \neq 0$ in E_1^3 and we give some theorems and results on these curves.

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1 Introduction

The quaternion was introduced by Hamilton. His initial attempt to generalize the complex numbers by introducing a three-dimensional objects failed in the sense that the algebra he constructed for these three-dimensional object did not have the desired properties. On the 16th October 1843 Hamilton discovered that the appropriate generalization is one in which the scalar(real) axis is left unchanged whereas the vector(imaginary) axis is supplemented by adding two further vector axis. Besides, there are three different types of quaternions, namely real, complex and dual quaternions. A real quaternion is defined as $q = q_0 + q_1e_1 + q_2e_2 + q_3e_3$ is composed of four units $(1, e_1, e_2, e_3)$ where e_1, e_2, e_3 are orthogonal unit spatial vectors, $q_i\,(i = 0, 1, 2, 3)$ are real numbers and this quaternion may be written as a linear combination of a real part(scalar) and vectorial part(a spatial vector). Quaternions find uses in both theoretical and applied mathematics, in particular for calculations involving three-dimensional rotations such as in three-dimensional computer graphics and computer vision. They can be used alongside

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other methods, such as Euler angles and matrices, or as an alternative to them depending on the application. Baharathi and Nagaraj represented the curves by unit quaternions in E^3 and E^4 and called these curves quaternionic curves [11]. They studied the differential geometry of space curves and introduced Frenet frames and formulae by using quaternions. After them, many mathematicians have studied quaternionic curves. Another issue we study in this paper is AW(k)-type curves. In [7], K. Arslan and A. West defined the notion of AW(k)-type submanifolds. Since then, many works have been done related to AW(k)-type submanifolds [8, 9, 10]. In [9], K. Arslan and the C. Özgür studied curves and surfaces of AW(k)-type. Further, many interesting results on curves of AW(k)-type have been obtained by many mathematicians (see [3, 4, 12, ?]). For example, in [3], Özgür and Gezgin studied a Bertrand curve of AW(k)-type and furthermore, they showed that there was no such Bertrand curve of AW(1)-type and was of AW(3)-type if and only if it was a right circular helix. In addition they studied weak AW(2)-type and AW(3)-type conical geodesic curves in E^3. Besides, in [4], Yoon studied curves of AW(k)-type in the Lie group G with a bi-invariant metric and he also characterized general helices in terms of AW(k)-type curve in the Lie group G. Finally, in [12], the curves of AW(k)- type in 3-dimensional null cone were investigated by Külahç i, Bektaş and Ergüt and in [14], Kızıltuğ and Yaylı studied quaternionic Mannheim curves of AW(k)-type in E^3.

In this paper, we have done a study on semi-real quaternionic Bertrand curves of AW(k)-type. Firstly, basic notions and properties of a quaternionic curve are reviewed. Later, we study quaternionic curves of AW(k)-type and quaternionic Bertrand curves of AW(k)-type in E_1^3, respectively.

2 Preliminaries

In this section, we give the basic elements of the theory of real and semi-real quaternions and quaternionic curves. First we recall the real quaternions. Let Q denotes the set of all real quaternions. A real quaternion is defined by $q = a\overrightarrow{e_1}+b\overrightarrow{e_2}+c\overrightarrow{e_3}+d$ where a, b, c, d are real numbers and $\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}$ are orthogonal unit spatial vectors in three dimensional space such that

$$\begin{aligned} \overrightarrow{e_1} \times \overrightarrow{e_2} = \overrightarrow{e_3} = -\overrightarrow{e_2} \times \overrightarrow{e_1} \\ \overrightarrow{e_2} \times \overrightarrow{e_3} = \overrightarrow{e_1} = -\overrightarrow{e_3} \times \overrightarrow{e_2} \\ \overrightarrow{e_3} \times \overrightarrow{e_1} = \overrightarrow{e_2} = -\overrightarrow{e_1} \times \overrightarrow{e_3} \\ e_1^2 + e_2^2 + e_3^2 = 1 \end{aligned}$$

and we can write a real quaternion as a linear combination of scalar part $S_q = d$ and vectorial part $V_q = a\overrightarrow{e_1} + b\overrightarrow{e_2} + c\overrightarrow{e_3}$. Using these basic products we can now expand the product of two quaternions as

$$p \times q = S_pS_q - \left\langle \overrightarrow{V_p}, \overrightarrow{V_q} \right\rangle + S_p\overrightarrow{V_q} + S_q\overrightarrow{V_p} + \overrightarrow{V_p} \wedge \overrightarrow{V_q} \quad \text{for every } p, q \in Q,$$

where $\langle , \rangle$ and $\wedge$ are inner product and cross product on E^3, respectively.

The conjugate of quaternion q is denoted by γq. In that case $\gamma q = -a\overrightarrow{e_1} - b\overrightarrow{e_2} - c\overrightarrow{e_3} + d$ for every $q \in Q$ which is called the "Hamiltonian conjugation".

The h-inner product of two quaternions is defined by

$$h(p,q) = \frac{1}{2}\left(p \times \gamma q + q \times \gamma p\right) \qquad \text{for every } p, q \in Q,$$

where h is the symmetric, non-degenerate, real valued and bilinear form [9, 6]. Now we can give the definition of the norm for every quaternion. The norm of any q real quaternion is denoted

$$\|q\|^2 = h(q,q) = q \times \gamma q = a^2 + b^2 + c^2 + d^2.$$

In this paper, we will study the semi-real spatial quaternionic curve in E_1^3. A semi-real quaternion is defined by $q = a\overrightarrow{e_1} + b\overrightarrow{e_2} + c\overrightarrow{e_3} + d$ such that

$$\begin{aligned} e_i \times e_i &= -\varepsilon(e_i) \qquad 1 \leqslant i \leqslant 3, \\ e_i \times e_j &= \varepsilon(e_i)\varepsilon(e_j)e_k \ \text{ in } E_1^3, \end{aligned}$$

where (ijk) is an even permutation of (123).

Notice here that we denote the set of all semi-real quaternions by Q_ν where ν is an index.

$$Q_\nu = \left\{ \begin{array}{c} q \mid q = a\overrightarrow{e_1} + b\overrightarrow{e_2} + c\overrightarrow{e_3} + d; \quad a, b, c, d \in \mathbb{R}, \\ \overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3} \in E_1^3, \ h_\nu(e_i, e_i) = \varepsilon(e_i), \quad 1 \leqslant i \leqslant 3. \end{array} \right\}$$

If e_i is a spacelike or timelike vector, then $\varepsilon(e_i) = +1$ or -1 respectively.

For $p = S_p + \overrightarrow{V_p}$ and $q = S_q + \overrightarrow{V_q}$, the multiplication of two semi-real quaternions p and q is defined as follows

$$p \times q = S_p S_q + \left\langle \overrightarrow{V_p}, \overrightarrow{V_q} \right\rangle + S_p\overrightarrow{V_q} + S_q\overrightarrow{V_p} + \overrightarrow{V_p} \wedge \overrightarrow{V_q} \ \text{ for every } \ p, q \in Q_\nu,$$

where we have used the scalar and cross products in E_1^3. Let symbol γ denotes the conjugate of a quaternion, $\gamma q = -a\overrightarrow{e_1} - b\overrightarrow{e_2} - c\overrightarrow{e_3} + d$ for every $q \in Q_\nu$. This helps to define the symmetric, non-degenerate, bilinear form h_ν as follows.

$$\begin{aligned} h_\nu &: Q_\nu \times Q_\nu \longrightarrow \mathbb{R}, \\ h_1(p,q) &= \frac{1}{2}\left[\varepsilon(p)\varepsilon(\gamma q)\left(p \times \gamma q\right) + \varepsilon(q)\varepsilon(\gamma p)\left(q \times \gamma p\right)\right] \quad \text{for } E_1^3, \end{aligned}$$

the norm of semi-real quaternion q is denoted by

$$\|q\|^2 = |h_\nu(q,q)| = |\varepsilon(q)\left(q \times \gamma q\right)| = \left|-a^2 - b^2 + c^2 + d^2\right|$$

for $p, q \in Q_\nu$, where if $h_\nu(p,q) = 0$ then p and q are called $h-$orthogonal. Also, q is called a spatial quaternion whenever $q + \gamma q = 0$ [2].

The three-dimensional semi-Euclidean space E_1^3 is identified with the space of spatial quaternions $\{q \in Q_\nu : q + \gamma q = 0\}$ in an obvious manner. Let $I = [0, 1]$ be an interval in real line $\mathbb{R}$ and $s \in I$ be parameter along the regular curve

$$\alpha : I \subset \mathbb{R} \to Q_\nu, \quad s \to \alpha(s) = \sum_{i=1}^{3} \alpha_i(s) \vec{e_i}$$

chosen such that the tangent $\alpha'(s)$ is unit, i.e., $\|\alpha'(s)\| = 1$ for all s. Then $\alpha(s)$ is called semi-real spatial quaternionic curve [11].

The Serret–Frenet formulas for semi-real quaternionic curves in E_1^3 are as follows:

Theorem 1. *Let $\alpha(s)$ be an arc-lengthed semi-real quaternionic curve with nonzero curvatures $\{k, r\}$ and $\{t(s), n(s), b(s)\}$ denotes the Frenet frame of the curve α. Then Frenet formulas are given by*

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_n k & 0 \\ -\varepsilon_t k & 0 & \varepsilon_n r \\ 0 & -\varepsilon_b r & 0 \end{bmatrix} \begin{bmatrix} t \\ n \\ b \end{bmatrix}, \tag{1}$$

where k is the principal curvature, r is torsion of α and $h(t,t) = \varepsilon_t$, $h(n,n) = \varepsilon_n$, $h(b,b) = \varepsilon_b$ [1].

Definition 1. *Let $\alpha : I \subset \mathbb{R} \longrightarrow E_1^3$ be a regular semi-real spatial quaternionic curve in E_1^3 with arc length parameter s and $\{k(s), r(s)\}$ are non-zero curvatures at the point $\alpha(s)$ of the curve α. In that case, harmonic curvature function of the curve α is [5]*

$$H : I \longrightarrow \mathbb{R}$$
$$H(s) = \frac{\varepsilon_n r(s)}{\varepsilon_t k(s)}.$$

3 Quaternionic Curves of AW(k)-type in E_1^3

Let $\alpha : I \to Q_\nu$ be an arc-lenght parametrized semi-real spatial quaternionic curve in E_1^3. The curve α is called a Frenet curve of osculating order 3 if its derivatives $\alpha'(s), \alpha''(s), \alpha'''(s)$ are linearly independent and $\alpha'(s), \alpha''(s), \alpha'''(s), \alpha''''(s)$ are no longer linearly independent for all $s \in I$. Each Frenet curve of order 3 is associated with an orthonormal $3-$frame $t(s), n(s), b(s)$ along $\alpha(s)$ (such that $\alpha'(s) = \vec{t}(s)$) known as the Frenet frame as well as the functions $k, r : I \to \mathbb{R}$ known as Frenet curvatures.

In this section, we consider quaternionic curves of AW(k)-type in E_1^3. First, we give definitions of weak AW(k)-type and AW(k)-type semi-real quaternionic curves and we obtain some results.

Proposition 1. *Let $\alpha : I \subset \mathbb{R} \longrightarrow E_1^3$ be a regular semi-real spatial quaternionic curve given by arc-length parameter s, thus we have*

$$\begin{aligned}
\alpha^{'}(s) &= \overrightarrow{t}(s) \\
\alpha^{''}(s) &= \varepsilon_n k \overrightarrow{n}(s) \\
\alpha^{'''}(s) &= -\varepsilon_t \varepsilon_n k^2 \overrightarrow{t}(s) + \varepsilon_n k^{'} \overrightarrow{n}(s) + \varepsilon_t \varepsilon_n k^2 H \overrightarrow{b}(s) \\
\alpha^{''''}(s) &= (-3\varepsilon_t \varepsilon_n k k^{'}) \overrightarrow{t}(s) + (-\varepsilon_t k^3 + \varepsilon_n k^{''} - \varepsilon_b k^3 H^2) \overrightarrow{n}(s) \\
&\quad + (k^{'} r + \varepsilon_t \varepsilon_n (2kk^{'} H + k^2 H^{'})) \overrightarrow{b}(s).
\end{aligned}$$

where H is harmonic curvature function of the curve $\alpha(s)$.

Notation 1. *Let us write*

$$N_1(s) = \varepsilon_n k \overrightarrow{n}(s), \tag{2}$$

$$N_2(s) = \varepsilon_n k^{'} \overrightarrow{n}(s) + \varepsilon_t \varepsilon_n k^2 H \overrightarrow{b}(s), \tag{3}$$

$$\begin{aligned} N_3(s) &= (-\varepsilon_t k^3 + \varepsilon_n k^{''} - \varepsilon_b k^3 H^2) \overrightarrow{n}(s) \\ &\quad + (k^{'} r + \varepsilon_t \varepsilon_n (2kk^{'} H + k^2 H^{'})) \overrightarrow{b}(s). \end{aligned} \tag{4}$$

Remark 1. *$\alpha^{'}(s), \alpha^{''}(s), \alpha^{'''}(s), \alpha^{''''}(s)$ are linearly dependent if and only if $N_1(s), N_2(s), N_3(s)$ are linearly dependent.*

Definition 2. *Semi-real spatial quaternionic curves in E_1^3 are*

(i) of type weak AW(2) if they satisfy

$$N_3(s) = h(N_3(s), N_2^*(s)) N_2^*(s), \tag{5}$$

(ii) of type weak AW(3) if they satisfy

$$N_3(s) = h(N_3(s), N_1^*(s)) N_1^*(s) \tag{6}$$

where

$$N_1^*(s) = \frac{N_1(s)}{\|N_1(s)\|}, \quad N_2^*(s) = \frac{N_2(s) - h(N_2(s), N_1^*(s)) N_1^*(s)}{\|N_2(s) - h(N_2(s), N_1^*(s)) N_1^*(s)\|} \tag{7}$$

(see, [9]).

Proposition 2. *Let $\alpha(s)$ be a semi-real spatial quaternionic curve in E_1^3. If α is of type weak AW(2) then*

$$\varepsilon_n k^{''} - \varepsilon_t k^3 - \varepsilon_b k^3 H^2 = 0. \tag{8}$$

Proposition 3. *Let $\alpha(s)$ be a semi-real spatial quaternionic curve in E_1^3. If α is of type weak AW(3) then*

$$k^{'} r + \varepsilon_t \varepsilon_n (2kk^{'} H + k^2 H^{'}) = 0. \tag{9}$$

Definition 3. *Semi-real spatial quaternionic curves in E_1^3 are [9]*

(i) of type AW(1) if they satisfy

$$N_3(s) = 0; \tag{10}$$

(ii) of type AW(2) if they satisfy

$$\|N_2(s)\|^2 N_3(s) = h(N_3(s), N_2(s))N_2(s); \tag{11}$$

(iii) of type AW(3) if they satisfy

$$\|N_1(s)\|^2 N_3(s) = h(N_3(s), N_1(s))N_1(s). \tag{12}$$

Proposition 4. *Let $\alpha(s)$ be a semi-real spatial quaternionic curve. Thus, $\alpha(s)$ is AW(1)-type curve if and only if*

$$\varepsilon_n k^{''} - \varepsilon_t k^3 - \varepsilon_b k^3 H^2 = 0 \tag{13}$$

and

$$k^{'} r + \varepsilon_t \varepsilon_n (2kk^{'} H + k^2 H^{'}) = 0. \tag{14}$$

Proof. Let $\alpha(s)$ be an AW(1)-type semi-real spatial quaternionic curve. So, we get equations (13) and (14) directly by equations (4) and (10) taking into account the linear independence of $\vec{n}$ and $\vec{b}$. The converse statement is trivial. Hence, this proof is complete. □

Corollary 1. *Let $\alpha(s)$ be a semi-real spatial quaternionic curve. Thus, $\alpha(s)$ is AW(1)-type curve if and only if α is of type weak AW(2) and weak AW(3).*

Proof. If $\alpha(s)$ is AW(1)-type curve, then equations (13) and (14) hold. So, it is seen that α provides conditions of type weak AW(2) and weak AW(3). The converse statement is trivial. □

Proposition 5. *Let $\alpha(s)$ be a semi-real spatial quaternionic curve. Thus, $\alpha(s)$ is AW(2)-type curve if and only if*

$$\varepsilon_t \varepsilon_n k^3 H^2 \left(3 \left(k^{'} \right)^2 - kk^{''} \right) + k^7 H^2 \left(1 + \varepsilon_t \varepsilon_b H^2 \right) + \varepsilon_t \varepsilon_n k^4 k^{'} H H^{'} = 0 \tag{15}$$

and

$$\varepsilon_t \varepsilon_n k k^{'} H \left(3 \left(k^{'} \right)^2 - kk^{''} \right) + k^5 k^{'} H \left(1 + \varepsilon_t \varepsilon_b H^2 \right) + \varepsilon_t \varepsilon_n k^2 \left(k^{'} \right)^2 H^{'} = 0. \tag{16}$$

Proof. Let $\alpha(s)$ be an AW(2)-type semi-real spatial quaternionic curve. From eq.(11), we have

$$\begin{aligned} \left(\left(k^{'} \right)^2 + k^4 H^2 \right) N_3(s) \quad = \quad & \Big[\varepsilon_n k^{'} \left(-\varepsilon_t k^3 + \varepsilon_n k^{''} - \varepsilon_b k^3 H^2 \right) \\ & + \varepsilon_t \varepsilon_n k^2 H \left(k^{'} r + \varepsilon_t \varepsilon_n (2kk^{'} H + k^2 H^{'}) \right) \Big] N_2(s) \end{aligned}$$

and herefrom we obtain equations (15) and (16) . The converse statement is trivial. This proof is complete. □

A curve $\alpha : I \to E_1^3$ with $k(s) \neq 0$ is called a general helix if the tangent lines of α make a constant angle with a fixed direction. α is also called cylindrical helix. It has been known that the curve $\alpha(s)$ is a cylindrical helix if and only if

$$\frac{r(s)}{k(s)} = const. \tag{17}$$

If both $k(s) \neq 0$ and $r(s)$ are constant, it is called a circular helix [13].

Corollary 2. *Let $\alpha(s)$ be a semi-real spatial quaternionic curve. If α is quaternionic cylindrical helix and α is of type AW(2) then*

$$\varepsilon_t \varepsilon_n \left(3\left(k^{'}\right)^2 - kk^{''}\right) + k^4\left(1 + \varepsilon_t \varepsilon_b c^2\right) = 0$$

where $c = \frac{r(s)}{k(s)}$ is constant.

Theorem 2. *Let $\alpha(s)$ be a semi-real spatial quaternionic curve. Thus, $\alpha(s)$ is AW(3)-type curve if and only if*

$$3k^3 k^{'} H + k^4 H^{'} = 0. \tag{18}$$

Proof. Let $\alpha(s)$ be an AW(3)-type semi-real spatial quaternionic curve. So, from eq.(12), we have

$$k^2 N_3(s) = \left(kk^{''} - \varepsilon_t \varepsilon_n k^4 - \varepsilon_n \varepsilon_b k^4 H^2\right) N_1(s)$$

which implies eq.(18). The converse statement is trivial. This proof is complete. □

Corollary 3. *Let α be a general helix of osculating order 3. Then α is of type AW(3) if and only if α is a circular helix.*

Proof. Suppose that α is a general helix of type AW(3). Combining equations (17) and (18) we find $k(s)$ and $r(s)$ are nonzero constants. Thus, α is circular helix. The converse statements is trivial. □

Corollary 4. *Let $\alpha(s)$ be a semi-real spatial quaternionic curve with $k(s) \neq 0$ and $r(s) \neq 0$. Then, α is of type AW(3) if and only if α is of type weak AW(3).*

Proof. If equations (9) and (18) are arranged under conditions of $k(s) \neq 0$ and $r(s) \neq 0$, we find that equations (9) and (18) are equivalent. Thus the proof is complete. □

Corollary 5. *Let $\alpha(s)$ be a semi-real spatial quaternionic curve with $k(s) \neq 0$ and $r(s) \neq 0$. If α is of type AW(1), then α is both of type AW(3) and type weak AW(3).*

Proof. If α is of type AW(1), then eq.(14) holds. This equation is equivalent to (9) and (18) under conditions of $k(s) \neq 0$ and $r(s) \neq 0$. Thus the proof is complete. □

4 Quaternionic Bertrand Curves of AW(k)-type in E_1^3

This section characterizes the curvatures of AW(k)-type semi-real spatial quaternionic Bertrand curves in E_1^3. We provided theorems and a conclusion for some AW(k)-type semi-real quaternionic Bertrand curves in E_1^3.

Definition 4. *A curve $\alpha : I \to E_1^3$ with $k(s) \neq 0$ is called a Bertrand curve if there exists a curve $\widetilde{\alpha} : I \to E_1^3$ such that the principal normal lines of α and $\widetilde{\alpha}$ at $s \in I$ are equal. In this case $\widetilde{\alpha}$ is called a Bertrand mate of α.*

Theorem 3. *The distance between the corresponding points of the semi-real spatial quaternionic Bertrand curves α and $\widetilde{\alpha}$ is constant in E_1^3.*

Proof. Suppose that α is a Bertrand curve. Then by the definition we can assume that

$$\widetilde{\alpha}(s) = \alpha(s) + \lambda(s)\vec{n}(s) \tag{19}$$

for some function $\lambda(s)$. By taking the derivative of (19) with respect to s and applying equations (1), we have

$$\tilde{t}(\widetilde{s})\frac{d\widetilde{s}}{ds} = (1 - \lambda(s)\varepsilon_t k)\,\vec{t}\,(s) + \lambda^{'}(s)\vec{n}(s) + \lambda(s)\varepsilon_n r\,\vec{b}\,(s)$$

where s and $\widetilde{s}$ are respectively arc-length parameter of curves α and $\widetilde{\alpha}$. Since $\widetilde{n}(s)$ is coincident with $n(s)$ in direction, we have $h\left(\tilde{t}(s), \vec{n}(s)\right) = 0$. Then, we get

$$\lambda^{'}(s) = 0.$$

This means that $\lambda\,(s)$ is a nonzero constant. On the other hand, from the distance function between two points, we have

$$d\left(\widetilde{\alpha}(s), \alpha(s)\right) = \left\|\alpha(s) - \widetilde{\alpha}(s)\right\| = \left\|\lambda\vec{n}(s)\right\| = |\lambda|\,.$$

Namely, $d\left(\widetilde{\alpha}(s), \alpha(s)\right) = constant$. Hence, the proof is completed. □

Theorem 4. *Let $\alpha : I \to Q$ be a semi-real spatial quaternionic curve with arc lenght parameter s. If $\widetilde{\alpha}$ with arc length parameter $\widetilde{s}$ is a Bertrand mate of α, then angle measurement between tangent vectors of curves α and $\widetilde{\alpha}$ at corresponding points is constant.*

Proof. To prove this theorem, we must show that $\left(h\left(\tilde{t}(s), t(s)\right)\right)^{'} = 0$.

$$\begin{aligned}\left(h\left(\tilde{t}(s), t(s)\right)\right)^{'} &= h\left(\left(\tilde{t}(s)\right)^{'}, t(s)\right) + h\left(\tilde{t}(s), \left(t(s)\right)^{'}\right)\\ &= h\left(\varepsilon_n \widetilde{k}\,(s)\,\widetilde{n}\,(s)\,, t(s)\right) + h\left(\tilde{t}(s), \varepsilon_n k\,(s)\,n\,(s)\right)\\ &= \varepsilon_n \widetilde{k}\,(s)\,h\left(\widetilde{n}\,(s)\,, t(s)\right) + \varepsilon_n k\,(s)\,h\left(\tilde{t}(s), n\,(s)\right),\end{aligned}$$

since $\widetilde{n}(s)$ is parallel to $\overrightarrow{n}(s)$ and $\overrightarrow{t}(s) \perp \overrightarrow{n}(s)$, then

$$h\left(\widetilde{n}(s), t(s)\right) = 0.$$

Also, since $\widetilde{n}(s)$ is parallel to $\overrightarrow{n}(s)$ and $\widetilde{t}(s) \perp \widetilde{n}(s)$, then

$$h\left(\tilde{t}(s), n(s)\right) = 0.$$

Thus, we have

$$\left(h\left(\tilde{t}(s), t(s)\right)\right)' = 0$$

and the proof is completed. □

Theorem 5. *Let $\alpha : I \to Q$ be unit speed semi-real spatial quaternionic curve with $k(s) \neq 0$. α is a Bertrand curve if and only if there exists a linear relation*

$$\lambda \varepsilon_t k(s) + \mu \varepsilon_t k(s) H = 1 \tag{20}$$

where λ, μ are non-zero constants and $k(s)$, $H(s)$ are the curvature functions of α.

Proof. Let $\{t(s), n(s), b(s)\}$ and $\{\widetilde{t}(\widetilde{s}), \widetilde{n}(\widetilde{s}), \widetilde{b}(\widetilde{s})\}$ denote the Frenet frames of the curve α and $\widetilde{\alpha}$, respectively and θ be angle between $t(s)$ and $\widetilde{t}(\widetilde{s})$. As $\left\{\widetilde{n}(\widetilde{s}), n(s)\right\}$ is a linearly dependent set, we can write

$$\widetilde{t}(\widetilde{s}) = \cosh\theta t(s) + \sinh\theta b(s). \tag{21}$$

If we take derivative of eq.(21) and consider that $\left\{\widetilde{n}(\widetilde{s}), n(s)\right\}$ is a linearly dependent set we can easily see that θ is a constant. Since α and $\widetilde{\alpha}$ are Bertrand curve mate we have

$$\widetilde{\alpha}(\widetilde{s}) = \alpha(s) + \lambda n(s). \tag{22}$$

If we take derivative of eq.(22) with respect to s, we get

$$\widetilde{t}(\widetilde{s})\frac{d\widetilde{s}}{ds} = \left(1 - \lambda\varepsilon_t k(s)\right) t(s) + \lambda\varepsilon_t k(s) H b(s). \tag{23}$$

Thus, from equations (21) and (23) we have

$$\frac{1 - \lambda\varepsilon_t k(s)}{\cosh\theta} = \frac{\lambda\varepsilon_t k(s) H}{\sinh\theta}$$

and from here we obtain

$$\coth\theta\lambda\varepsilon_t k(s) H + \lambda\varepsilon_t k(s) = 1.$$

If we take $\coth\theta\lambda = \mu$, we get

$$\lambda\varepsilon_t k(s) + \mu\varepsilon_t k(s) H = 1.$$

The converse statement is trivial. This proof is complete. □

Corollary 6. *α is a semi-real quaternionic Bertrand curve with $k(s) \neq 0$ and $r(s) \neq 0$ if and only if there exists a nonzero real number λ such that*

$$k'H + kH' - \lambda\varepsilon_t k^2 H' = 0. \tag{24}$$

Proof. By the theorem 5, α is a Bertrand curve if and only if there exist real numbers $\lambda \neq 0$ and μ such that $\lambda\varepsilon_t k(s) + \mu\varepsilon_t k(s) H = 1$. This is equivalent to the condition that there exists a real number $\lambda \neq 0$ such that $\frac{1-\lambda\varepsilon_t k(s)}{\varepsilon_t k(s) H}$ is constant. If we take derivative both sides of the last equality, we have eq.(24) . □

Theorem 6. *Let α be a semi-real quaternionic Bertrand curve with $k(s) \neq 0$ and $r(s) \neq 0$. If α is of AW(1)-type, then the following equation holds*

$$2k'H + \lambda\varepsilon_t k^2 H' = 0 \tag{25}$$

where λ is a non zero real number.

Proof. Since α is of AW(1), equations (13) and (14) hold and since α is a semi-real quaternionic Bertrand curve, eq.(24) holds. If these equations are considered, eq.(25) is obtained. □

Theorem 7. *Let α be a semi-real quaternionic Bertrand curve with $k(s) \neq 0$ and $r(s) \neq 0$. Then α is of AW(2)-type if and only if there is a non zero real number λ such that*

$$k'H'(3\lambda\varepsilon_t k - 2) - k''H + \varepsilon_t\varepsilon_n k^3 H\left(1 + \varepsilon_t\varepsilon_b H^2\right) = 0 \tag{26}$$

Proof. Since α is of type AW(2), equations (15) and (16) hold and since α is a semi-real quaternionic Bertrand curve, eq.(24) holds. If these equations are considered, eq.(26) is obtained. □

Theorem 8. *Let α be a semi-real quaternionic Bertrand curve with $k(s) \neq 0$ and $r(s) \neq 0$. Then α is of AW(3)-type if and only if*

$$2k'H + \lambda\varepsilon_t k^2 H' = 0. \tag{27}$$

Proof. Since α is of type AW(3), eq.(18) holds and since α is a semi-real spatial quaternionic Bertrand curve, eq.(24) holds. If eq.(18) is substituted in (24), then (27) is obtained. The converse statement is trivial. Thus, this proof is complete. □

Theorem 9. *Let α be a semi-real spatial quaternionic Bertrand curve with $k(s) \neq 0$ and $r(s) \neq 0$. If α is of weak AW(2)-type, then the following equation holds*

$$\varepsilon_n k^3 H\left(\varepsilon_t + \varepsilon_b H^2\right) + (1 - \lambda\varepsilon_t k)\left(kH'' + 2k'H'\right) = 0 \tag{28}$$

Proof. Since α is of type weak AW(2), eq.(8) holds and since α is a semi-real spatial quaternionic Bertrand curve, eq.(24) holds. Arranging eq.(8), we have

$$\varepsilon_n k^{''} = \varepsilon_t k^3 + \varepsilon_b k^3 H^2 \tag{29}$$

If we take derivative of (24), we get

$$k^{''}H + kH^{''} + 2k^{'}H^{'}\left(1 - \lambda\varepsilon_t k\right) - \lambda\varepsilon_t k^2 H^{''} = 0 \tag{30}$$

If eq.(29) is substituted in (30), then (28) is obtained. Thus the proof is completed. □

Theorem 10. *Let α be a semi-real spatial quaternionic Bertrand curve with $k(s) \neq 0$ and $r(s) \neq 0$. If α is of weak AW(3)-type, then following equation holds*

$$2k'H + \lambda\varepsilon_t k^2 H^{'} = 0 \tag{31}$$

Proof. Since α is of type weak AW(3), eq. (9) holds and since α is a semi-real spatial quaternionic Bertrand curve, eq.(24) holds. If eq.(9) is substituted in (24), then (31) is obtained. Thus the proof is completed. □

Corollary 7. *Let α be a semi-real spatial quaternionic Bertrand curve with $k(s) \neq 0$ and $r(s) \neq 0$. Thus, if α is of type AW(1), then α is both of type AW(3) and type weak AW(3).*

Proof. If α is a semi-real spatial quaternionic Bertrand curve of AW(1) type, then eq.(25) holds. This equation is equivalent to (27) and (31) under conditions of $k(s) \neq 0$ and $r(s) \neq 0$. Thus the proof is complete. □

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