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RETAINMENT AND REDISTRIBUTION - AN INFORMATION THEORETICAL APPROACH

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Abstract

A probabilistic model has been developed for the retainment and redistribution in domains where uncertainty is inherent. It has been shown that the model has evident connection with the negation transformation developed by Yager[7].

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1 Introduction

Retainment is a key concept in almost every institution or process human beings are part of. How much to retain and how much to redistribute has been key to major financial decisions that companies/governments have to focus on from time to time. Even for salaried individuals, the amount they are able to retain after spending on all financial obligations is very important for dealing with financial uncertainties. At the conclusion of every fiscal cycle, whatever profits companies have incurred can be utilized for a variety of purposes. One option is that a portion of it is retained and the rest is distributed among the shareholders/employees of the company. The retained portion is termed as retained earnings which is generally reserved for reinvestment back into the business. We consider an example. Mr. A owns a company that focuses on the development of sustainable technologies. We assume that the company has been operating for the last 6 years. During the first two years of operations, the company posted a net profit of 50,000 dollars and did not pay any dividend to its stakeholders. Next two years, the company made a profit of 80,000 dollars and paid out 10,000 dollars

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as dividend. Finally, in the last two years, the company made a profit of 100,000 dollars and paid out 20,000 dollars as dividend. The retention ratio for the first two years is 100%, 87.5% for the next two years and finally 80% for the final two years of operation under consideration. The retention ratio is going down but the net profit is rising every two years with dividends rising by a proportionally larger amount. Generally, companies/organizations which have been operating for longer periods of time will normally post lower retention ratios and may opt to pay investors consistent earnings. Once how much to retain has been agreed upon, how to distribute the profit among various shareholders of the company is another major concern. Profits can be distributed in accordance with the number of shares held by a particular individual or by a resolution on the distribution of profit. However, keeping the process of retainment and distribution free from any bias can be a major challenge for any organization/company. Anyway, biasness is inherent in behaviour, thinking and other attributes of individuals and organizations. Any workplace, that is free from any bias, signifies that everyone receives a fair treatment in terms of incentives, opportunities and other attributes. Fairness(unbiasness) distributions up a positive environment for both employee and employer. Implementing an equitable (fair) working environment is essential, but it has to be worked upon. Transparency in the hiring process(skill based hiring), providing inclusive incentives to employees, empowering employee and many other innovative practices may be needed for enabling a competitive but fair environment so that the companies and organizations end up with the right people in the right places doing the right things. In order to maintain fairness in all operations, an organization may have to rethink or reframe policies which may end up in the redistribution of incentives/pay packages and other facilities. Many countries (not all) focus a lot on the equitable redistribution of the income so that people get identical opportunities for accumulating wealth(for *e.g.* some redistribute tax revenues to ensure an equitable distribution of wealth). This concept of retainment and redistribution correlates with negation transformation developed by Yager[7] in many(if not all) ways. Yager developed the negation of a probability distribution keeping in mind that we can negate the happening of any uncertain event by negating(opposing) its probability and redistribute its probability equally among all the other possible outcomes(*i.e* without any bias). The Yager's model suggests that while negating any event, we should not retain anything and redistribute the whole probability among other outcomes. However, a lot of situations and happenings in life needs retainment. So we need a mathematical framework which gives due weightage to both retainment and redistribution. In the present work, we have developed a Retainment and Redistribution(RR) model which has many similarities with the negation transformation developed by Yager[7]. The analysis of the proposed model has been done from both the probabilistic and information theoretical points of view. Results show that the model is unbiased in the sense that no preference is given to any particular outcome regardless of whether we retain or redistribute.

2 Negation vs Retainment Redistribution(RR) model

Consider an $n \times n$ doubly stochastic matrix $A = (a_{ij})$ *i.e.* owe

$$\sum_{i=1}^{n} a_{ij} = \sum_{j=1}^{n} a_{ij} = 1.$$

In particular if $a_{ii} = 0$ for all $i = 1, 2, \ldots, n$ and if $a_{ij} = \frac{1}{n-1}$ for all $i \neq j$; $i, j = 1, 2, \ldots, n$, then given a probability distribution $P(n) = \{p_1, p_2, \ldots, p_n\}$, its negation[3],[4],[5],[7] is defined as $\overline{P}(n) = \{\overline{p}_1, \overline{p}_2, \ldots, \overline{p}_n\}$, where $\overline{p}_i = \sum\limits_{j=1}^{n} a_{ij} p_j$ for all $i = 1, 2, \ldots, n$. We can further write

$$\overline{p}_1 = \frac{1-p_1}{n-1} = 0 \cdot p_1 + \frac{1}{n-1} \cdot p_2 + \frac{1}{n-1} \cdot p_3 + \cdots + \frac{1}{n-1} \cdot p_n$$

$$\overline{p}_2 = \frac{1-p_2}{n-1} = \frac{1}{n-1} \cdot p_1 + 0 \cdot p_2 + \frac{1}{n-1} \cdot p_3 + \cdots + \frac{1}{n-1} \cdot p_n$$

$$\vdots$$

$$\overline{p}_n = \frac{1-p_n}{n-1} = \frac{1}{n-1} \cdot p_1 + \frac{1}{n-1} \cdot p_2 + \frac{1}{n-1} \cdot p_3 + \cdots + 0 \cdot p_n$$

which can be written in matrix form as

$$\overline{P}(n) = \begin{pmatrix} \overline{p}_1 \\ \overline{p}_2 \\ \cdot \\ \cdot \\ \cdot \\ \overline{p}_n \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{n-1} & \cdot & \cdot & \cdot & \frac{1}{n-1} \\ \frac{1}{n-1} & 0 & \cdot & \cdot & \cdot & \frac{1}{n-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{1}{n-1} & \frac{1}{n-1} & \cdot & \cdot & \cdot & 0 \end{pmatrix} \cdot \begin{pmatrix} p_1 \\ p_2 \\ \cdot \\ \cdot \\ \cdot \\ p_n \end{pmatrix}.$$

Here the distribution $\overline{P}(n) = \{\overline{p}_1, \overline{p}_2, \ldots, \overline{p}_n\}$ satisfies $0 \leq \overline{p}_i \leq 1;\ i \in \{1, 2, 3, ..., n\}$ and $\sum_{i=1}^{n} \overline{p}_i = 1$. Consider the probability distribution $P(5) = \left\{\frac{1}{2}, \frac{1}{2}, 0, 0, 0\right\}$ and the corresponding negation $\overline{P}(5) = \left\{\frac{1}{8}, \frac{1}{8}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right\}$. The first entry in $\overline{P}(5)$ comes as a result of equal contribution(distribution) from the second entry in $P(5)$. Similarly, the second entry in $\overline{P}(5)$ comes as a result of equal contribution from the first entry in $P(5)$. Now the third entry $\frac{1}{4}$ in $\overline{P}(5)$ comes as a result of equal contributions from the first and second entries in $P(5)$. Similar is the case with the fourth and fifth entry in $\overline{P}(5)$ It is clear that the redistribution of probabilities is totally UNBIASED *i.e.* no preference is given to any of alternatives. However, Yager's model is based on redistribution of probabilities only and it may not be applicable in situations where how much to retain or how much to distribute is crucial and directly affects the uncertainty in coming times. Suppose half of each probability is retained and the remaining is equally distributed among the other alternatives, then the revised probabilities are

$$\tilde{p}_1 = \frac{p_1}{2} + \frac{p_2 + p_3 + \cdots + p_n}{2(n-1)} = \frac{p_1}{2} + \frac{\overline{p}_1}{2}$$

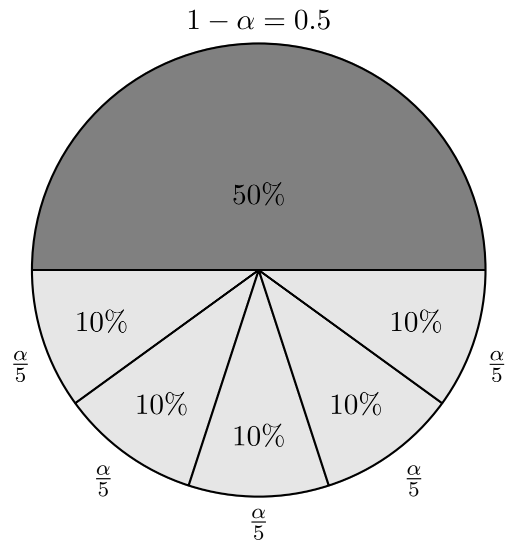

Figure 1: Retainment and redistribution for $\alpha = 0.5$

$$\tilde{p}_2 = \frac{p_2}{2} + \frac{p_1 + p_3 + \cdots + p_n}{2(n-1)} = \frac{p_2}{2} + \frac{\overline{p}_2}{2}$$

$$\vdots$$

$$\tilde{p}_n = \frac{p_n}{2} + \frac{p_1 + p_2 + \cdots + p_{n-1}}{2(n-1)} = \frac{p_n}{2} + \frac{\overline{p}_n}{2}.$$

Here the distribution $\{\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n\}$ satisfies $0 \leq \sum_{i=1}^{n} \tilde{p}_i \leq 1$ and $\sum_{i=1}^{n} \tilde{p}_i = 1$.

In general, if we retain the $(1-\alpha)$ component of p_i and distribute the remaining equally among the others, then

$$\tilde{p}_i = (1-\alpha)p_i + \alpha\overline{p}_i \qquad ; \quad 0 \leq \alpha \leq 1$$

. Further we can write

$$\begin{aligned}
\tilde{p}_1 &= (1-\alpha)p_1 + \tfrac{\alpha}{n-1}p_2 + \tfrac{\alpha}{n-1}p_3 + \cdots + \tfrac{\alpha}{n-1}p_n = (1-\alpha)p_1 + \alpha\overline{p}_1 \\
\tilde{p}_2 &= \tfrac{\alpha}{n-1}p_1 + (1-\alpha)p_1 + \tfrac{\alpha}{n-1}p_3 + \cdots + \tfrac{\alpha}{n-1}p_n = (1-\alpha)p_2 + \alpha\overline{p}_2
\end{aligned}$$

$$\vdots$$

$$\tilde{p}_n = \tfrac{\alpha}{n-1}p_1 + \tfrac{\alpha}{n-1}p_2 + \tfrac{\alpha}{n-1}p_3 + \cdots + (1-\alpha)p_n = (1-\alpha)p_n + \alpha\overline{p}_n,$$

which can be written in matrix form as

$$\tilde{P}(n) = \begin{pmatrix} \tilde{p}_1 \\ \tilde{p}_2 \\ \tilde{p}_3 \\ . \\ . \\ . \\ \tilde{p}_n \end{pmatrix} = \begin{pmatrix} 1-\alpha & \frac{\alpha}{n-1} & \frac{\alpha}{n-1} & . & . & . & \frac{\alpha}{n-1} \\ \frac{\alpha}{n-1} & 1-\alpha & \frac{\alpha}{n-1} & . & . & . & \frac{\alpha}{n-1} \\ \frac{\alpha}{n-1} & \frac{\alpha}{n-1} & 1-\alpha & . & . & . & \frac{\alpha}{n-1} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ \frac{\alpha}{n-1} & \frac{\alpha}{n-1} & \frac{\alpha}{n-1} & . & . & . & 1-\alpha \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ . \\ . \\ . \\ p_n \end{pmatrix}.$$

Also
$0 \leq p_i \leq 1$ for all $i \Rightarrow 0 \leq \frac{1-p_i}{n-1} \leq \frac{1}{n-1}$ for all i
$\Rightarrow 0 \leq \overline{p}_i \leq \frac{1}{n-1}$ for all i
$\Rightarrow 0 \leq (1-\alpha)p_i + \alpha\overline{p}_i \leq (1-\alpha) + \frac{\alpha}{n-1}$
$\Rightarrow 0 \leq \tilde{p}_i \leq (1-\alpha) + \frac{\alpha}{n-1}$ for all i. Also $\sum_{i=1}^{n} \tilde{p}_i = 1$. Therefore, the distribution $\tilde{P}(n) = \{\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_n\}$ characterizes the RR Model, with entries representing the convex combination of retained and redistributed terms. In the next section, we will compare the uncertainties embedded in $P(n)$, $\overline{P}(n)$ and $\tilde{P}(n)$.

3 Uncertainty and information

3.1 Jensen inequality

Theorem 3.1 [6] Let $f : G \rightarrow \mathbb{R}$ be convex such that G is an interval in $\mathbb{R}$ and $x_1, x_2, ..., x_n$ are in G. For non-negative real numbers $\zeta_1, \zeta_2, ..., \zeta_n$ satisfying $\sum_{i=1}^{n} \zeta_i = 1$, we have

$$f(\zeta_1 x_1 + \zeta_2 x_2 + ... + \zeta_n x_n) \leq \zeta_1 f(x_1) + \zeta_2 f(x_2) + ... + \zeta_n f(x_n). \tag{1}$$

If f is a concave function, then the above inequality will be reversed. Here the term $\zeta_1 x_1 + \zeta_2 x_2 + ... + \zeta_n x_n$ represents a convex combination of $x_1, x_2, ..., x_n$ ($\sum_{i=1}^{n} \zeta_i = 1$). From

$$\tilde{p}_i = (1-\alpha)p_i + \alpha\overline{p}_i \ ; \ 0 \ \leq \ \alpha \ \leq \ 1,$$

it is clear that every entry of the distribution $\{\overline{p}_1, \overline{p}_2, ..., \overline{p}_n\}$ can be written as the convex combination of $(n-1)$ entries of the distribution $\{p_1, p_2, ..., p_n\}$ with weights not all equal. We consider

$$\tilde{p}_1 = (1-\alpha).p_1 + \left(\frac{\alpha}{n-1}\right) p_2 + \left(\frac{\alpha}{n-1}\right) p_3 + ... + \left(\frac{\alpha}{n-1}\right) p_n.$$

Now replacing $x_1, x_2, x_3, ..., x_n$ by $p_1, p_2, p_3, ..., p_n$, respectively and $\zeta_1, \zeta_2, \zeta_3, ..., \zeta_n$ by $1-\alpha, \frac{\alpha}{n-1}, \frac{\alpha}{n-1}, ..., \frac{\alpha}{n-1}$ in (1), we obtain
$f\left((1-\alpha)p_1 + \frac{\alpha p_2}{n-1} + \frac{\alpha p_3}{n-1} + ... + \frac{\alpha p_n}{n-1}\right) \leq (1-\alpha)\, f(p_1) + \left(\frac{\alpha}{n-1}\right) f(p_2) + \left(\frac{\alpha}{n-1}\right) f(p_3) + ... + \left(\frac{\alpha}{n-1}\right) f(p_n)$

$$\Rightarrow f(\tilde{p}_1) \leq (1-\alpha)\, f(p_1) + \left(\frac{\alpha}{n-1}\right) [f(p_2) + f(p_3) + ... + f(p_n)] \ .$$

Similarly

$$f(\tilde{p}_2) \leq (1-\alpha)\, f(p_2) + \left(\frac{\alpha}{n-1}\right)[f(p_1) + f(p_3) + \ldots + f(p_n)]$$

$$\vdots$$

$$f(\tilde{p}_n) \leq \left(\frac{\alpha}{n-1}\right)[f(p_1) + f(p_2) + \ldots + f(p_{n-1})] + (1-\alpha)\, f(p_n).$$

Adding the above inequalities, we obtain

$$\sum_{i=1}^{n} f(\tilde{p}_i) \leq (1-\alpha)\sum_{i=1}^{n} f(p_i) + \alpha \sum_{i=1}^{n} f(p_i)$$

which finally gives

$$\sum_{i=1}^{n} f(\tilde{p}_i) \leq \sum_{i=1}^{n} f(p_i). \tag{2}$$

Also we can write $\tilde{p}_i = (1-\alpha)\, p_i + \alpha\overline{p}_i$
$\Rightarrow f(\tilde{p}_i) = f\left((1-\alpha)p_i + \alpha\overline{p}_i\right) \leq (1-\alpha)f(p_i) + \alpha f(\overline{p}_i)$ by virtue of convexity of f. Finally, we obtain

$$\sum_{i=1}^{n} f(\tilde{p}_i) \leq (1-\alpha)\sum_{i=1}^{n} f(p_i) + \alpha \sum_{i=1}^{n} f(\overline{p}_i). \tag{3}$$

Special Cases : **(a)** $f(x) = -x\log x \Rightarrow f^{''}(x) = -(1/x)$.
Now (2) gives $-\sum_{i=1}^{n} \tilde{p}_i \log \tilde{p}_i \geq -\sum_{i=1}^{n} p_i \log p_i$. Therefore the Shannon entropy [1],[2] associated with RR model represented by $\tilde{P}$ is greater than the entropy associated with P. Also from (3), we obtain

$$-\sum_{i=1}^{n} \tilde{p}_i \log \tilde{p}_i \geq -(1-\alpha)\sum_{i=1}^{n} p_i \log p_i - \alpha \sum_{i=1}^{n} \overline{p}_i \log \overline{p}_i.$$

which shows that uncertainty embedded in $\tilde{P}$ is greater than the convex combination of the uncertainties embedded in P and $\overline{P}$.
(b) $g(x) = \log\frac{1}{x} \Rightarrow g^{'}(x) = x.\left(\frac{-1}{x^2}\right)$. Again (2) gives $\sum_{i=1}^{n} \log\frac{1}{\tilde{p}_i} \leq \sum_{i=1}^{n} \log\frac{1}{p_i}$

$$\Rightarrow -\sum_{i=1}^{n} \log \tilde{p}_i \leq -\sum_{i=1}^{n} \log p_i.$$

Here the function $-\log\alpha$ represents the information content of an event occurring with probability $\alpha(0 \leq \alpha \leq 1)$, and is generally known as the self information of an event with probability α. The above inequality shows that the information embedded in RR model about the occurrence of events is less than the information contained in the original probability distribution. This is due to the fact that RR model contains probabilities that are retained and redistributed. Also from (3), we obtain

$$-\sum_{i=1}^{n}\log\tilde{p}_i \leq -(1-\alpha)\sum_{i=1}^{n}\log p_i - \alpha\sum_{i=1}^{n}\log\overline{p}_i.$$

(c) $f(x) = x\log nx \Rightarrow g^{''}(x) = \frac{1}{nx} > 0$
From (2) we obtain

$$\sum_{i=1}^{n}\tilde{p}_i\log\frac{\tilde{p}_i}{1/n} \leq \sum_{i=1}^{n}p_i\log\frac{p_i}{1/n}$$

$$\Rightarrow \sum_{i=1}^{n}\tilde{p}_i\log\tilde{p}_i + \log n \leq \sum_{i=1}^{n}p_i\log p_i) + \log n$$

$$\Rightarrow \log n - \left(-\sum_{i=1}^{n}\tilde{p}_i\log\tilde{p}_i\right) \leq \log n - \left(-\sum_{i=1}^{n}p_i\log p_i\right).$$

i.e. dissimilarity between $\left\{\frac{1}{n},\frac{1}{n},...,\frac{1}{n}\right\}$ and $\{p_1,p_2,...,p_n\}$ is greater than the dissimilarity between $\left\{\frac{1}{n},\frac{1}{n},...,\frac{1}{n}\right\}$ and $\{\tilde{p}_1,\tilde{p}_2,...,\tilde{p}_n\}$. The uniform distribution is generally referred as the maximum entropy distribution since all the outcomes are equally probable and any deviation from it can be viewed as biasness which occurs due to decrease in uncertainty. Therefore , the distribution $\tilde{P}$ is less biased than the actual probability distribution P. Also from (3), we obtain

$$\log n - (-\textstyle\sum_{i=1}^{n}\tilde{p}_i\log\tilde{p}_i) \leq \log n - ((1-\alpha)p_i\log p_i + \alpha p_i\log p_i)$$

which again shows that the distribution $\tilde{P}$ is less biased than the convex combination of P and $\overline{P}$. Table 1 gives the uncertainty values for $P(3) = \{p_1,p_2,p_3\}$, $\overline{P}(3) = \{\overline{p}_1,\overline{p}_2,\overline{p}_3\}$ and $\tilde{P}(3) = \{\tilde{p}_1,\tilde{p}_2,\tilde{p}_3\}$. For simplicity, we have used the uncertainty function $H(P(n)) = 1 - \sum_{i=1}^{n}{p_i}^2$ used by Yager[6]. Table 2 gives the uncertainty values for $\tilde{P}(3) = \{\tilde{p}_1,\tilde{p}_2,\tilde{p}_3\}$ for different values of α. Evaluated values clearly show that uncertainty embedded in $\tilde{P}$ is greater than the uncertainty embedded in $\overline{P}$ and P. Also for $\alpha = 0.5$, uncertainty in $\tilde{P}$ is maximum(as expected) because the retained and redistributed amounts are equal in this case.

$P(3)$	$\overline{P}(3)$	$\tilde{P}(3); \alpha = 0.5$	$H(P(3)) = 1 - \sum {p_i}^2$	$H(\overline{P(3)}) = 1 - \sum_{i=1}^{n}{\overline{p}_i}^2$	$H_{0.5}(\tilde{P(3)}) = 1 - \sum_{i=1}^{n}\tilde{p}_i^2$
$(1,0,0)$	$(0,0.5,0.5)$	$(0.5,0.25,0.25)$	0	0.5	0.62
$(0.8,0.1,0.1))$	$(0.1,0.45,0.45)$	$(0.45,0.28,0.28)$	0.34	0.58	0.65
$(0.6,0.3,0.1)$	$(0.2,0.35,0.45)$	$(0.4,0.32,0.28)$	0.54	0.63	0.66
$(0.5,0.25,0.25)$	$(0.25,0.38,0.38)$	$(0.37,0.31,0.31)$	0.625	0.66	0.66
$(0.45,0.3,0.25)$	$(0.28,0.35,0.38)$	$(0.36,0.32,0.31)$	0.64	0.66	0.66
$(0.33,0.33,0.33)$	$(0.33,0.33,0.33)$	$(0.33,0.33,0.33)$	0.67	0.66	0.67

Table 1: Uncertainty embedded in $P(3) = \{p_1,p_2,p_3\}$, $\overline{P}(3) = \{\overline{p}_1,\overline{p}_2,\overline{p}_3\}$ and $\tilde{P}(3) = \{\tilde{p}_1,\tilde{p}_2,\tilde{p}_3\}$

3.2 Expectation

The Expectation of a discrete random variable X assuming values $(x_1,x_2,\ldots,x_n)$ with probabilities $P(n) = \{p_1,p_2,\ldots,p_n\}$ is defined as
$E_{\tilde{p}}(X) = \sum_{i=1}^{n}\tilde{p}_i x_i = x_1\tilde{p}_1 + x_2\tilde{p}_2 + ... + x_n\tilde{p}_n$

$P(3)$	$H_{0.25}(\tilde{P})(3)$	$H_{0.5}(\tilde{P})(3)$	$H_{0.75}(\tilde{P})(3)$	$H_1(\tilde{P})(3)$
$(1,0,0)$	0.41	0.62	0.66	0.5
(0.8,0.1,0.1)	0.54	0.65	0.66	0.59
(0.6,0.3,0.1)	0.62	0.66	0.66	0.63
(0.5,0.25,0.25)	0.65	0.66	0.67	0.66
(0.45,0.3,0.25)	0.66	0.67	0.67	0.66
(0.33,0.33,0.33)	0.67	0.67	0.67	0.67

Table 2: Uncertainty embedded in $\tilde{P}(3) = \{\tilde{p}_1, \tilde{p}_2, \tilde{p}_3\}$ for different values of α

$$= x_1 \left((1-\alpha)p_1 + \frac{\alpha}{n-1}p_2 + \frac{\alpha}{n-1}p_3 + ... + \frac{\alpha}{n-1}p_n \right) +$$

$$x_2 \left(\frac{\alpha}{n-1}p_1 + (1-\alpha)p_2 + + \frac{\alpha}{n-1}p_3 + ... + \frac{\alpha}{n-1}p_n \right) +$$

$$... + x_n \left(\frac{\alpha}{n-1}p_1 + \frac{\alpha}{n-1}p_2 + ... + (1-\alpha)p_n \right)$$

$= (1-\alpha)\sum_{i=1}^n p_i x_i + \frac{\alpha}{n-1}\sum_{i=1}^n x_i - \frac{\alpha}{n-1}\sum_{i=1}^n p_i x_i$
$= \sum_{i=1}^n p_i x_i \left((1-\alpha) - \frac{\alpha}{n-1} \right) + \frac{\alpha n}{n-1}\frac{\sum_{i=1}^n x_i}{n}$
$= \sum_{i=1}^n p_i x_i - \frac{\alpha n}{n-1}\sum_{i=1}^n p_i x_i + \frac{\alpha n}{n-1}\frac{\sum_{i=1}^n x_i}{n}$
$= \sum_{i=1}^n p_i x_i$.
Here the probability distribution $P(n)$ is a unimodal symmetric distribution in which the mean, mode and median all fall at the same point. Similarly $\tilde{P}(n)$ is again unimodal and symmetric with mean, mode and median all falling at the same point. It is clear that $\tilde{P}(n)$ has preserved the symmetry since the probabilities are retained and redistributed among all the alternatives without any bias.

4 Conclusion

In the present work, we have developed a mathematical framework which characterizes the retainment and redistribution(RR) of probabilities and which can be viewed as an extension of negation transformation proposed by Yager[7]. However, Yager's negation and the RR Model have various similarities/dissimilarities as far as their mathematical structure is concerned. We list them in the following table.

Work on further extensions and generalizations of the proposed work is in progress and will be communicated elsewhere.

Yager's Negation	**RR model**
The vector $\overline{P}(n)$ is a probability distribution with $\overline{p}_i \in [0,1]$ for $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} \overline{p}_i = 1$.	The vector $\tilde{P}(n)$ is a probability distribution with $\tilde{p}_i \in [0,1]$ for $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} \tilde{p}_i = 1$.
The probabilities $\overline{p}_1, \overline{p}_2, ..., \overline{p}_n$ are the convex combinations of $(n-1)$ elements of the distribution $\{p_1, p_2, ..., p_n\}$ with equal weights(Retainment not allowed).	The probabilities $\tilde{p}_1, \tilde{p}_2, ..., \tilde{p}_n$ are the convex combinations of n elements of the distribution $\{p_1, p_2, ..., p_n\}$ with unequal weights(Retainment allowed).
The matrix $\overline{P}(n)$ is doubly stochastic(row and column sum one) with diagonal entries zero and all other entries equal.	The matrix $\tilde{P}(n)$ is doubly stochastic(row and column sum one) with identical diagonal entries and all other entries also identical but different from diagonal entries.
The model is unbiased in sense that the probabilities are reallocated without any bias.	The model is unbiased in sense that the retained and redistributed components are exactly identical for all the components.

Table 3: Examples - Yager's Negation vs RR Model

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