Bulletin of the *Transilvania* University of Braşov
Series III: Mathematics and Computer Science, Vol. 2(64), No. 1 - 2022, 159-176
https://doi.org/10.31926/but.mif.2022.2.64.1.11

AN ANALYSIS OF (0,1,2;0) POLYNOMIAL INTERPOLATION INCLUDING INTERPOLATION ON BOUNDARY POINTS OF INTERVAL $[-1, 1]$

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Abstract

In this paper, we survey an interpolation on polynomials with Hermite conditions on the zeros of ultraspherical polynomials at interval [-1,1]. Our aim is to demonstrate the existence, uniqueness, explicit representation and convergence theorem of the interpolatory polynomials, which are the zeros of the polynomials $P_n^{(k)}(x)$ and $P_{n-1}^{(k+1)}(x)$ respectively, where $P_n^{(k)}(x)$ is the ultraspherical polynomial of degree n.

2000 *Mathematics Subject Classification:* 41A05, 33C45
Key words: Lagrange interpolation, ultraspherical polynomials, fundamental polynomials, explicit form, order of convergence.

1 Introduction:

In 1959, R.B. Sexna [6] modified the (0,2) interpolation problem of [1] including the conditions $R_n'''(+1)$ and $R_n'''(-1)$ on knots [6](2.1). She considered a interpolatory polynomial $f_{2n+1}(x)$ of degree at most 2n+1, which satisfied the conditions [6](2.2) on the zeros of the polynomial $\Pi(x)$=$(1-x^2)P_{n-1}'(x)$. Later, L.Szili [10] studied the above problem of (0,2) interpolation on the roots of all classical orthogonal polynomials with respect to its existence, uniqueness and explicit form. His problem [10] was studied by M. Lenard [5] including additional interpolatory conditions.
Further, R. Srivastava and Y. Singh [9] investigated the problem of (0;1) interpolation on the knots [9](1) with interpolatory conditions [9](2)-(5) on the zeros of polynomials $P_{n-1}^{(k+1)'}(x)$ and $P_{n-1}^{(k+1)}(x)$ respectively. Later, the authors [8], [7] extended the above problem at the zeros of ultraspherical polynomials

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including the hermite conditions.
The convergence of this interpolation process was studied by Xie [12], if $f \in C^r[-1,1]$ for $x \in [-1,1]$, then

$$|f(x) - \mathcal{R}_{2n+1}(x;f)| = O\left(n^{-r+1}\right)\omega\left(f^{(r)};\frac{1}{n}\right). \tag{1.1}$$

For $k \geq 1$ Lenard [3] proved that if $f \in C^r[-1,1]$ for $x \in [-1,1]$,then

$$|f(x) - \mathcal{R}_m(x;f)| = O\left(n^{k-r+\frac{1}{2}}\right)\omega\left(f^{(r)};\frac{1}{n}\right). \tag{1.2}$$

For $k \geq 0$ Lenard [4] proved that if $f \in C^r[-1,1]$ for $x \in [-1,1]$, then

$$|f'(x) - \mathcal{R}'_m(x;f)| = O\left(n^{k-r+\frac{5}{2}}\right)\omega\left(f^{(r)};\frac{1}{n}\right), \tag{1.3}$$

that is, if $f \in C^{k+2}[-1,1]$, $f^{k+2} \in Lip\alpha$, $\alpha > \dfrac{1}{2}$, then $\mathcal{R}_m(x;f)$ and $\mathcal{R}'_m(x;f)$ uniformly converges to $f(x)$ and $f'(x)$ respectively on [-1,1] , where $\omega(f^{(r)},.)$ denotes the modulus of continuity of the r^{th} derivative of the function $f(x)$.

The aim of this paper is to extend the study of problem [7] to the case of (0,1,2;0) interpolation with Hermite-type boundary conditions at interval $[-1,1]$. The following problem is given.

Problem:

Let the set of knots be given by

$$-1 = x_n^* < x_n < x_{n-1}^* < x_{n-1} < ..... < x_1^* < x_1 < x_0^* = 1, \quad n \geq 2, \tag{1.4}$$

where $\{x_i\}_{i=1}^n$ and $\{x_i^*\}_{i=1}^{n-1}$ are the roots of ultraspherical polynomials $P_n^{(k)}(x)$ and $P_{n-1}^{(k+1)}(x)$ respectively, then on the nodal points (1.4) there exists a unique polynomial $\mathcal{R}_m(x)$ of degree at most m=$4n + 2k + 1$ satisfying the following interpolation conditions i.e

$$\mathcal{R}_m(x_i) = y_i \quad (i = 1,2,...,n), \tag{1.5}$$

$$\mathcal{R}'_m(x_i) = y_i{}' \quad (i = 1,2,...,n), \tag{1.6}$$

$$\mathcal{R}''_m(x_i) = y_i{}'' \quad (i = 1,2,...,n), \tag{1.7}$$

$$\mathcal{R}_m(x_i^*) = y_i^* \quad (i = 1,2,...,n-1), \tag{1.8}$$

with (Hermite) boundary conditions

$$\mathcal{R}_m^{(l)}(1) = \alpha_l \quad (l = 0,1,...,k), \tag{1.9}$$

$$\mathcal{R}_m^{(l)}(-1) = \beta_l \quad (l = 0, 1, ...k + 1), \tag{1.10}$$

where y_i, y_i', y_i'', y_i^*, α_l and β_l are arbitrary real numbers and k is a fixed non-negative integer. In section 2 and section 3, we give some preliminaries and prove explicit representation respectively. Existence and uniqueness of the interpolational polynomial are proved in section 4. Furthermore, the estimation of the fundamental polynomials and convergence theorem are proved in section 5.

2 Preliminaries:

Let $P_n^{(k)}(x)$=$P_n^{(k,k)}(x)$ $(k > -1, n \geq 0)$ denote the ultraspherical polynomial of degree n. We refer to [11] (4.2.1).

$$(1 - x^2)P_n^{(k)''}(x) - 2x(k + 1)P_n^{(k)'}(x) + n(n + 2k + 1)P_n^{(k)}(x) = 0, \tag{2.1}$$

$$P_n^{(k)'}(x) = \frac{(n + 2k + 1)}{2} P_{n-1}^{(k+1)}(x), \tag{2.2}$$

$$|P_n^{(k)}(x)| = O(n^k), \quad x \in [-1, 1], \tag{2.3}$$

$$(1 - x^2)^{\frac{k}{2}+\frac{1}{4}}|P_n^{(k)}(x)| = O(\frac{1}{\sqrt{n}}), \tag{2.4}$$

The fundamental polynomials of Lagrange interpolation are given by:

$$l_j^*(x) = \frac{P_{n-1}^{(k+1)}(x)}{P_{n-1}^{(k+1)'}(x_j^*)(x - x_j^*)} \tag{2.5}$$

and

$$l_j(x) = \frac{P_n^{(k)}(x)}{P_n^{(k)'}(x_j)(x - x_j)} = \frac{\tilde{h}_n^{(k)}}{(1 - x_j^2)\{P_n^{(k)'}(x_j)\}^2} \sum_{\nu=0}^{n-1} \frac{1}{h_\nu^{(k)}} P_\nu^{(k)}(x_j)P_\nu^{(k)}(x), \tag{2.6}$$

where

$$\tilde{h}_n^{(k)} = \frac{2^{2k}\Gamma^2\left(n + k + 1\right)}{\Gamma\left(n + 1\right)\Gamma\left(n + 2k + 1\right)} \sim \omega_1, \tag{2.7}$$

$$h_\nu^{(k)} = \frac{2^{2k+1}}{(2\nu + 2k + 1)} \frac{\Gamma^2\left(\nu + k + 1\right)}{\Gamma\left(\nu + 1\right)\Gamma\left(\nu + 2k + 1\right)} \begin{cases} \sim \frac{1}{\nu} & (\nu > 0), \\ = \omega_2 & (\nu = 0), \end{cases} \tag{2.8}$$

where the constants ω_1 and ω_2 depend on k.

$$l_j(x_i) = \delta_{ij} = \begin{cases} 0 & \text{if} \quad i \neq j \\ 1 & \text{if} \quad i = j \end{cases}, \tag{2.9}$$

$$l_j^*(x_i^*) = \delta_{ij} = \begin{cases} 0 & \text{if} \quad i \neq j \\ 1 & \text{if} \quad i = j \end{cases}, \tag{2.10}$$

$$l_j{}'(x_i) = \begin{cases} \frac{P_n^{(k)'}(x_i)}{P_n^{(k)'}(x_j)(x_i - x_j)}, \text{ when } & i \neq j \\ \frac{x_j(k+1)}{(1-x_j^2)}, \text{ when } & i = j, \end{cases} \tag{2.11}$$

$$l_j{}''(x_j) = \frac{4x_j^2(k+1)(k+2)}{3(1-x_j{}^2)^2} - \left\{ \frac{n(n+2k+1) - 2(k+1)}{3(1-x_j{}^2)} \right\}. \tag{2.12}$$

If $x_1 > x_2 > ..... > x_n$ are the roots of $P_n^{(k)}(x)$, then the following relations hold [8].

$$(1 - x_j^2) \sim \begin{cases} \frac{j^2}{n^2} & (x_j \geq 0), \\ \frac{(n-j)^2}{n^2} & (x_j < 0), \end{cases} \tag{2.13}$$

$$|P_n^{(k)'}(x_j)| \sim \begin{cases} \frac{n^{k+2}}{j^{k+\frac{3}{2}}} & (x_j \geq 0), \\ \frac{n^{k+2}}{(n-j)^{k+\frac{3}{2}}} & (x_j < 0). \end{cases} \tag{2.14}$$

3 Explicit representation of interpolatory polynomials:

The polynomial $\mathcal{R}_m(x)$ can be explicitly written as

$$\begin{aligned} \mathcal{R}_m(x) &= \sum_{j=1}^{n} \breve{\mathcal{A}}_j(x) y_j + \sum_{j=1}^{n} \breve{\mathcal{B}}_j(x) y_j{}' + \sum_{j=1}^{n} \breve{\mathcal{C}}_j(x) y_j{}'' + \sum_{j=1}^{n-1} \breve{\mathcal{D}}_j(x) y_j^* \\ &\quad + \sum_{j=0}^{k} \breve{\mathcal{E}}_j(x) \alpha_j + \sum_{j=0}^{k+1} \breve{\mathcal{F}}_j(x) \beta_j, \end{aligned} \tag{3.1}$$

which satisfies the conditions (1.5) - (1.10), where $\breve{\mathcal{A}}_j(x)$ and $\breve{\mathcal{D}}_j(x)$ are the fundamental polynomials of first kind, $\breve{\mathcal{B}}_j(x)$ and $\breve{\mathcal{C}}_j(x)$ are the fundamental polynomials of second kind and third kind respectively. $\breve{\mathcal{E}}_j(x)$ and $\breve{\mathcal{F}}_j(x)$ are the fundamental polynomials which correspond to the boundary conditions each of degree $\leq 4n + 2k + 1$, uniquely determined by the following conditions, which shown in following tables.

Table 1: **Interpolatory conditions for first kind of fundamental polynomials** $\breve{\mathcal{A}}_j(x)$

For j=1,2...,n	
$\breve{\mathcal{A}}_j(x_i) = \delta_{ji}$ $\breve{\mathcal{A}}'_j(x_i) = 0$ $\breve{\mathcal{A}}''_j(x_i) = 0$	$i = 1, 2..., n$
$\breve{\mathcal{A}}_j(x_i^*) = 0$	$i = 1, 2..., n-1$
$\breve{\mathcal{A}}_j^{(l)}(1) = 0$	$l = 0, 1..., k$
$\breve{\mathcal{A}}_j^{(l)}(-1) = 0$	$l = 0, 1..., k+1$

Table 2: **Interpolatory conditions for second kind of fundamental polynomials** $\breve{\mathcal{B}}_j(x)$

For j=1,2...,n	
$\breve{\mathcal{B}}_j(x_i) = 0$ $\breve{\mathcal{B}}'_j(x_i) = \delta_{ji}$ $\breve{\mathcal{B}}''_j(x_i) = 0$	$i = 1, 2..., n$
$\breve{\mathcal{B}}_j(x_i^*) = 0$	$i = 1, 2..., n-1$
$\breve{\mathcal{B}}_j^{(l)}(1) = 0$	$l = 0, 1..., k$
$\breve{\mathcal{B}}_j^{(l)}(-1) = 0$	$l = 0, 1..., k+1$

Table 3: **Interpolatory conditions for third kind of fundamental polynomials** $\breve{\mathcal{C}}_j(x)$

For j=1,2...,n	
$\breve{\mathcal{C}}_j(x_i) = 0$ $\breve{\mathcal{C}}'_j(x_i) = 0$ $\breve{\mathcal{C}}''_j(x_i) = \delta_{ji}$	$i = 1, 2..., n$
$\breve{\mathcal{C}}_j(x_i^*) = 0$	$i = 1, 2..., n-1$
$\breve{\mathcal{C}}_j^{(l)}(1) = 0$	$l = 0, 1..., k$
$\breve{\mathcal{C}}_j^{(l)}(-1) = 0$	$l = 0, 1..., k+1$

Table 4: **Interpolatory conditions for first kind of fundamental polynomials $\breve{\mathfrak{D}}_j(x)$**

For j=1,2...,n-1	
$\breve{\mathfrak{D}}_j(x_i)=0$ $\breve{\mathfrak{D}}'_j(x_i)=0$ $\breve{\mathfrak{D}}''_j(x_i)=0$	$i=1,2...,n$
$\breve{\mathfrak{D}}_j(x_i^*)=\delta_{ji}$	$i=1,2...,n-1$
$\breve{\mathfrak{D}}_j^{(l)}(1)=0$	$l=0,1...,k$
$\breve{\mathfrak{D}}_j^{(l)}(-1)=0$	$l=0,1...,k+1$

Table 5: **Interpolatory conditions for the fundamental polynomials $\breve{\mathfrak{E}}_j(x)$ which correspond to boundary conditions**

For j=0,1...,k	
$\breve{\mathfrak{E}}_j(x_i)=0$ $\breve{\mathfrak{E}}'_j(x_i)=0$ $\breve{\mathfrak{E}}''_j(x_i)=0$	$i=1,2...,n$
$\breve{\mathfrak{E}}_j(x_i^*)=0$	$i=1,2...,n-1$
$\breve{\mathfrak{E}}_j^{(l)}(1)=\delta_{jl}$	$l=0,1...,k$
$\breve{\mathfrak{E}}_j^{(l)}(-1)=0$	$l=0,1...,k+1$

Table 6: **Interpolatory conditions for the fundamental polynomials $\breve{\mathfrak{F}}_j(x)$ which correspond to boundary conditions**

For j=0,1...,k+1	
$\breve{\mathfrak{F}}_j(x_i)=0$ $\breve{\mathfrak{F}}'_j(x_i)=0$ $\breve{\mathfrak{F}}''_j(x_i)=0$	$i=1,2...,n$
$\breve{\mathfrak{F}}_j(x_i^*)=0$	$i=1,2...,n-1$
$\breve{\mathfrak{F}}_j^{(l)}(1)=0$	$l=0,1...,k$
$\breve{\mathfrak{F}}_j^{(l)}(-1)=\delta_{jl}$	$l=0,1...,k+1$

The explicit forms are given in the following Lemmas.

Lemma 3.1. The fundamental polynomials $\breve{\mathcal{A}}_j(x)$, for j=$1, 2, ..., n$ satisfying the interpolatory conditions (Table 1.) are given by:

$$\breve{\mathcal{A}}_j(x) = \{\tilde{C_1} + \tilde{C_2}(x - x_j)\}(1 - x^2)^{k+2}\{l_j(x)\}^3 P_{n-1}^{(k+1)}(x) + \tilde{C_3}\breve{\mathcal{B}}_j(x), \tag{3.2}$$

where

$$\tilde{C_1} = \frac{1}{(1 - {x_j}^2)^{k+2} P_{n-1}^{(k+1)}(x_j)}, \tag{3.3}$$

$$\tilde{C_2} = \frac{\tilde{C_1}}{x_j(3k+1)} \left\{(n^2 + 2nk + n - k) - \frac{{x_j}^2(k+1)(5k+1)}{(1 - {x_j}^2)}\right\}, \tag{3.4}$$

$$\tilde{C_3} = -\frac{1}{(3k+1)} \left\{\frac{(n^2 + 2nk + n - k)}{x_j} + \frac{4x_j k^2}{(1 - {x_j}^2)}\right\} \tag{3.5}$$

and $\breve{\mathcal{B}}_j(x)$ are given in Lemma 3.2.

Lemma 3.2. The fundamental polynomials $\breve{\mathcal{B}}_j(x)$, for j=$1, 2, ..., n$ satisfying the interpolatory conditions (Table 2.) are given by:

$$\begin{aligned}\breve{\mathcal{B}}_j(x) =& \tilde{C_4}(1 + x)(1 - x^2)^{k+1} P_n^{(k)}(x) P_{n-1}^{(k+1)}(x)\{l_j(x)\}^2 \\ &+ \tilde{C_5}\breve{\mathcal{C}}_j(x),\end{aligned} \tag{3.6}$$

where

$$\tilde{C_4} = \frac{(n + 2k + 1)}{2(1 + x_j)(1 - {x_j}^2)^{k+1}\{P_n^{(k)'}(x_j)\}^2}, \tag{3.7}$$

$$\tilde{C_5} = -2\left\{\frac{(1 - x_j) + 3x_j(k+1)}{(1 - {x_j}^2)}\right\} \tag{3.8}$$

and $\breve{\mathcal{C}}_j(x)$ are given in Lemma 3.3.

Lemma 3.3. The fundamental polynomials $\breve{\mathcal{C}}_j(x)$, for j=$1, 2, ..., n$ satisfying the interpolatory conditions (Table 3.) are given by:

$$\breve{\mathcal{C}}_j(x) = \frac{(n + 2k + 1)(1 + x)(1 - x^2)^{k+1}\{P_n^{(k)}(x)\}^2 P_{n-1}^{(k+1)}(x) l_j(x)}{4(1 + x_j)(1 - {x_j}^2)^{k+1}\{P_n^{(k)'}(x_j)\}^3}. \tag{3.9}$$

Lemma 3.4. The fundamental polynomials $\breve{\mathcal{D}}_j(x)$, for j=$1, 2, ..., n-1$ satisfying the interpolatory conditions (Table 4.) are given by:

$$\breve{\mathcal{D}}_j(x) = \frac{(1 + x)(1 - x^2)^{k+1}\{P_n^{(k)}(x)\}^3 l_j^*(x)}{(1 + x_j^*)(1 - x_j^{*2})^{k+1}\{P_n^{(k)}(x_j^*)\}^3}. \tag{3.10}$$

Lemma 3.5. The fundamental polynomials $\breve{\mathcal{E}}_j(x)$, $for\ j{=}0,1,...,k$ which correspond to the boundary conditions, satisfying the interpolatory conditions (Table 5.) are given by:

$$\breve{\mathcal{E}}_j(x) = (1-x)^j(1+x)^{k+2}\{P_n^{(k)}(x)\}^3 P_{n-1}^{(k+1)}(x)\tilde{p}_j(x), \tag{3.11}$$

where $\tilde{p}_j(x)$ is a uniquely determined polynomial of degree $\leq k-j$.

Lemma 3.6. The fundamental polynomials $\breve{\mathcal{F}}_j(x)$, $for\ j{=}0,1,...,k+1$ which correspond to the boundary conditions, satisfying the interpolatory conditions (Table 6.) are given by:

$For\ j{=}0,1...,k$

$$\breve{\mathcal{F}}_j(x) = (1+x)^j(1-x)^{k+1}\{P_n^{(k)}(x)\}^3 P_{n-1}^{(k+1)}(x)\tilde{q}_j(x), \tag{3.12}$$

where $\tilde{q}_j(x)$ is a uniquely determined polynomial of degree $\leq k+1-j$,
$for\ j{=}k+1$

$$\breve{\mathcal{F}}_{k+1}(x) = \frac{(1-x^2)^{k+1}\{P_n^{(k)}(x)\}^3 P_{n-1}^{(k+1)}(x)}{2^{k+1}(k+1)!\{P_n^{(k)}(-1)\}^3 P_{n-1}^{(k+1)}(-1)}. \tag{3.13}$$

Proof of Lemma 3.1. - Lemma 3.6.

Proof. We consider $\breve{\mathcal{D}}_j(x) = \tilde{C}_6(1+x)(1-x^2)^{k+1}\{P_n^{(k)}(x)\}^3 l_j^*(x)$ which satisfies the conditions (Table 4.). Now, we can see that when $i \neq j$, then $\breve{\mathcal{D}}_j(x_i^*) = 0$, when $i = j$, then

$$\breve{\mathcal{D}}_j(x_j^*) = \tilde{C}_6(1+x_j^*)(1-x_j^{*2})^{k+1}\{P_n^{(k)}(x_j^*)\}^3 l_j^*(x_j^*) = 1,$$

by using (2.10), we find that

$$\tilde{C}_6 = \frac{1}{(1+x_j^*)(1-x_j^{*2})^{k+1}\{P_n^{(k)}(x_j^*)\}^3},$$

then we get Lemma 3.4. In similar way, we can also prove Lemma 3.3, which satisfies the conditions (Table 3). Next, we assume that

$$\begin{aligned}\breve{\mathcal{B}}_j(x) =& \tilde{C}_4(1+x)(1-x^2)^{k+1}P_n^{(k)}(x)P_{n-1}^{(k+1)}(x)\{l_j(x)\}^2 \\ &+ \tilde{C}_5\breve{\mathcal{C}}_j(x),\end{aligned}$$

from Table 2, we see that $\breve{\mathcal{B}}_j'(x)|_{x=x_i} = 0$, when $i \neq j$.
If i=j and by using (2.9), we have

$$\breve{\mathcal{B}}_j'(x)|_{x=x_j} = \tilde{C}_4(1+x_j)(1-{x_j}^2)^{k+1}{P_n^{(k)}}'(x_j)P_{n-1}^{(k+1)}(x_j) = 1, \tag{3.14}$$

then we get the value of $\tilde{C}_4$ (3.7). Furthermore, if $i \neq j$ then $\breve{\mathcal{B}}''_j(x_i) = 0$ and if i=j, we find it

$$\breve{\mathcal{B}}''_j(x_j) = \tilde{C}_4(1 - {x_j}^2)^{k+1}\{2{P_n^{(k)}}'(x_j)P_{n-1}^{(k+1)}(x_j) + 2(1 + x_j){P_n^{(k)}}'(x_j){P_{n-1}^{(k+1)}}'(x_j) + (1 + x_j)P_{n-1}^{(k+1)}(x_j){P_n^{(k)}}''(x_j)\} + \tilde{C}_5 = 0, \tag{3.15}$$

by using (2.9), (2.11) and table 3.($\breve{\mathcal{C}}''_j(x_j)$=1).
Now, from (2.1) and (2.2), we have

$${P_n^{(k)}}''(x_j) = \frac{2x_j(k + 1){P_n^{(k)}}'(x_j)}{(1 - {x_j}^2)} \tag{3.16}$$

and

$${P_{n-1}^{(k+1)}}'(x_j) = \frac{2x_j(k + 1)P_{n-1}^{(k+1)}(x_j)}{(1 - {x_j}^2)}. \tag{3.17}$$

Substituting the value of C_4, value of (3.16) and (3.17) in equation (3.15), then we obtain the value of $\tilde{C}_5$ (3.8). Hence, finally we obtain Lemma 3.2. As using the above process, we can also find Lemma 3.1. Next, from Table 5. $\breve{\mathcal{E}}_j(x)$ satisfies the interpolatory conditions. It is clear that $\breve{\mathcal{E}}_j^{(l)}(1)$=0 for $l = 0, ...j - 1$. Now, we write the polynomial $\tilde{p}_j(x)$ from (3.11) in the form

$$\tilde{p}_j(x) = \tilde{a}_0^{(j)} + \tilde{a}_1^{(j)}(1 - x) + ... + \tilde{a}_{k-j}^{(j)}(1 - x)^{k-j}.$$

The coefficients of the polynomial $\tilde{p}_j(x)$ are determined by the system

$$\begin{aligned} \breve{\mathcal{E}}_j^{(l)}(1) &= \frac{d^l}{dx^l}[(1 - x)^j(1 + x)^{k+2}\{P_n^{(k)}(x)\}^3P_{n-1}^{(k+1)}(x)\tilde{p}_j(x)]_{x=1} \\ &= \delta_{jl} \quad (l = j, j + 1, ..., k). \end{aligned}$$

Similar to above process, we can also prove Lemma 3.6. □

4 Existence and uniqueness

Theorem 4.1. If $\{y_i\}_{i=1}^n$, $\{{y_i}'\}_{i=1}^n$, $\{{y_i}''\}_{i=1}^n$, $\{y_i^*\}_{i=1}^{n-1}$, $\{\alpha_l\}_{l=0}^k$ and $\{\beta_l\}_{l=0}^{k+1}$ are given real numbers, $k \geq 0$, $n \geq 2$ are arbitrary fixed integers, then on the nodal points (1.4) there exists a unique polynomial $\mathcal{R}_m(x)$ (can see in (3.1)) of degree at most $4n + 2k + 1$ satisfying the conditions (1.5) - (1.10).

Proof. By Lemma 3.1 to Lemma 3.6, we can see that polynomial $\mathcal{R}_m(x)$ in (3.1) satisfies the conditions (1.5) - (1.10), which proves the existence of interpolational polynomial $\mathcal{R}_m(x)$. For the uniqueness we assume that there is another polynomial $\mathcal{R}_m^*(x)$ of degree at most m=4n+2k+1 which also satisfies the conditions. Then the polynomial $\mathcal{Q}_m(x) = \mathcal{R}_m(x) - \mathcal{R}_m^*(x)$ satisfies the equations

$$\mathcal{Q}_m(x_i) = 0,\ \mathcal{Q}'_m(x_i) = 0,\ \mathcal{Q}''_m(x_i) = 0 \quad (i = 1, 2..., n), \tag{4.1}$$

$$\mathcal{Q}_m(x_i^*) = 0 \quad (i = 1, 2..., n-1), \tag{4.2}$$

$$\mathcal{Q}_m^{(l)}(1) = 0 \quad (l = 0, 1..., k) \tag{4.3}$$

and

$$\mathcal{Q}_m^{(l)}(-1) = 0 \quad (l = 0, 1, ...k+1), \tag{4.4}$$

so it can be written in the form

$$\mathcal{Q}_m(x) = (1-x^2)^{k+1} P_n^{(k)}(x) P_{n-1}^{(k+1)}(x) g_{2n}(x), \tag{4.5}$$

where $g_{2n}(x)$ is a polynomial of degree at most 2n. Now, we can see that the conditions $\mathcal{Q}_m(x_i) = 0$ and $\mathcal{Q}_m(x_i^*) = 0$ satisfy in (4.5). Furthermore,

$$\mathcal{Q}_m'(x_i) = (1-{x_i}^2)^{k+1} {P_n^{(k)}}'(x_i) P_{n-1}^{(k+1)}(x_i) g_{2n}(x_i) = 0, \quad i.e \;\; g_{2n}(x_i) = 0 \tag{4.6}$$

and

$$\mathcal{Q}_m''(x_i) = 2(1-{x_i}^2)^{k+1} {P_n^{(k)}}'(x_i) P_{n-1}^{(k+1)}(x_i) {g_{2n}}'(x_i) = 0 \tag{4.7}$$

that is

$${g_{2n}}'(x_i) = 0, \;\; for \;\; i = 1, 2..., n. \tag{4.8}$$

It is possible in that case when ${g_{2n}}'(x) \equiv 0$, so $g_{2n}(x) \equiv a$, hence

$$\mathcal{Q}_m(x) = a(1-x^2)^{k+1} P_n^{(k)}(x) P_{n-1}^{(k+1)}(x), \tag{4.9}$$

from equation (4.4) and (4.9), we find it

$$\begin{aligned}\mathcal{Q}_m^{(k+1)}(-1) = a2^{k+1}(k+1)! P_n^{(k)}(-1) P_{n-1}^{(k+1)}(-1) = 0 \\ \Longrightarrow a = 0,\end{aligned}$$

therefore $\mathcal{Q}_m(x) \equiv 0$, which proves the uniqueness. □

5 Estimation of the fundamental polynomials.

Theorem 5.1. If $k > 0$, $n \geq 2$, for the first derivative of the first kind fundamental polynomials $\{\breve{\mathfrak{D}}_j(x)\}_{j=1}^{n-1}$ on [-1,1] holds

$$\sum_{j=1}^{n-1} (1-{x_j^*}^2) |\breve{\mathfrak{D}}_j'(x)| = O(n^{2k+4}). \tag{5.1}$$

Proof. From equation (2.1) and (2.2), we have

$$P_n^{(k)}(x_j^*) = -\frac{(1-{x_j^*}^2) {P_{n-1}^{(k+1)}}'(x_j^*)}{2n}. \tag{5.2}$$

Differentiating (3.10), we have

$$\sum_{j=1}^{n-1}(1-x_j^{*2})|\breve{\mathfrak{D}}_j'(x)| = \eta_1+\eta_2+\eta_3,$$

where we use the decomposition (2.5) in η_1 for $l_j^*(x)$ and from (5.2), we get

$$\eta_1 \le \sum_{j=1}^{n-1}\frac{8n^3(1-x_j^*)(1+x)\{1-x+2x(k+1)\}(1-x^2)^{\frac{k}{2}+\frac{1}{4}}|P_n^{(k)}(x)|^3\times \tilde{h}_{n-1}^{(k+1)}}{(1-x_j^{*2})^{\frac{3k}{2}+\frac{23}{4}}|P_{n-1}^{(k+1)'}(x_j^*)|^5}$$
$$\times\left\{\gamma_1+\sum_{\nu=1}^{n-2}\frac{1}{h_\nu^{(k+1)}}(1-x_j^{*2})^{\frac{k}{2}+\frac{3}{4}}|P_\nu^{(k+1)}(x_j^*)|(1-x^2)^{\frac{k}{2}+\frac{3}{4}}|P_\nu^{(k+1)}(x)|\right\},$$

where γ_1 is a constant, independent of n, by (2.3), (2.4), (2.8), (2.13) and (2.14), we have

$$\eta_1 = O(n^{2k+2}).$$

Again use the decomposition (2.5) in η_2 for $l_j^*(x)$, we get

$$\eta_2 \le \sum_{j=1}^{n-1}\frac{12n^3(n+2k+1)(1-x_j^*)(1+x)(1-x^2)^{\frac{k}{2}+\frac{1}{4}}|P_n^{(k)}(x)|^2|P_{n-1}^{(k+1)}(x)|\tilde{h}_{n-1}^{(k+1)}}{(1-x_j^{*2})^{\frac{3k}{2}+\frac{23}{4}}|P_{n-1}^{(k+1)'}(x_j^*)|^5}$$
$$\times\left\{\gamma_2+\sum_{\nu=1}^{n-2}\frac{(1-x_j^{*2})^{\frac{k}{2}+\frac{3}{4}}}{h_\nu^{(k+1)}}|P_\nu^{(k+1)}(x_j^*)|(1-x^2)^{\frac{k}{2}+\frac{3}{4}}|P_\nu^{(k+1)}(x)|\right\},$$

where γ_2 is a constant, independent of n, by (2.3), (2.4), (2.8), (2.13) and (2.14), we find

$$\eta_2 = O(n^{2k+4}).$$

As such, to use the decomposition (2.5) in η_3 for $l_j^*(x)$, we obtain

$$\eta_3 = O(n^{2k+3}).$$

Hence, the theorem is proved. □

Theorem 5.2. If $k>0$, $n\ge 2$, for the first derivative of the third kind fundamental polynomials $\{\breve{\mathfrak{C}}_j(x)\}_{j=1}^n$ on [-1,1] holds

$$\sum_{j=1}^{n}|\breve{\mathfrak{C}}_j'(x)| = O(n^{2k+3}). \tag{5.3}$$

Proof. Taking derivative and applying the sum (for $j=1$ to $j=n$) in (3.9), we have

$$\sum_{j=1}^{n}|\breve{\mathfrak{C}}_j'(x)| = \zeta_1+\zeta_2+\zeta_3+\zeta_4,$$

where we use the decomposition (2.6) in ζ_1 for $l_j(x)$, we have

$$\zeta_1 \leq \sum_{j=1}^{n} \frac{(n+2k+1)(1-x_j)(1+x)\{1-x+2x(k+1)\}(1-x^2)^{\frac{k}{2}+\frac{1}{4}}|P_n^{(k)}(x)|^2}{4(1-x_j^2)^{\frac{3k}{2}+\frac{13}{4}}|P_n^{(k)'}(x_j)|^5}$$
$$\times (1-x^2)^{\frac{k}{2}+\frac{3}{4}}|P_{n-1}^{(k+1)}(x)|\tilde{h}_n^{(k)} \left\{\gamma_3 + \sum_{\nu=1}^{n-1} \frac{1}{h_\nu^{(k)}}(1-x_j^2)^{\frac{k}{2}+\frac{1}{4}}|P_\nu^{(k)}(x_j)|P_\nu^{(k)}(x)|\right\},$$

where γ_3 is a constant, independent of n, by (2.3), (2.4), (2.8), (2.13) and (2.14), we obtain

$$\zeta_1 = O(n^{2k}).$$

Further use the decomposition (2.6) in ζ_2 for $l_j(x)$, we have

$$\zeta_2 \leq$$
$$\leq \sum_{j=1}^{n} \frac{(n+2k+1)^2(1-x_j)(1+x)(1-x^2)^{\frac{k}{2}+\frac{1}{4}}|P_n^{(k)}(x)|(1-x^2)^{\frac{k}{2}+\frac{3}{4}}|P_{n-1}^{(k+1)}(x)|^2\tilde{h}_n^{(k)}}{4(1-x_j^2)^{\frac{3k}{2}+\frac{13}{4}}|P_n^{(k)'}(x_j)|^5}$$
$$\times \left\{\gamma_4 + \sum_{\nu=1}^{n-1} \frac{1}{h_\nu^{(k)}}(1-x_j^2)^{\frac{k}{2}+\frac{1}{4}}|P_\nu^{(k)}(x_j)||P_\nu^{(k)}(x)|\right\},$$

where γ_4 is a constant, independent of n, by using (2.3), (2.4), (2.8), (2.13) and (2.14), we obtain

$$\zeta_2 = O(n^{2k+2}),$$

similarly using the decomposition on ζ_3 and ζ_4, we have

$$\zeta_3 = O(n^{2k+3}) \ \ and \ \ \zeta_4 = O(n^{2k+2}).$$

Hence, the theorem is proved. □

Theorem 5.3. If $k > 0$, $n \geq 2$, for the first derivative of the second kind fundamental polynomials $\{\breve{\mathcal{B}}_j(x)\}_{j=1}^n$ on [-1,1] holds

$$\sum_{j=1}^{n} |\breve{\mathcal{B}}_j'(x)| = O(n^{2k+4}). \tag{5.4}$$

Proof. Differentiating (3.6), using (3.7) and (3.8), we have

$$\sum_{j=1}^{n} |\breve{\mathcal{B}}_j'(x)| = \xi_1 + \xi_2 + \xi_3 + \xi_4 + \xi_5,$$

where ξ_1 is used in decomposition (2.6) for $l_j(x)$, we have

$$\xi_1 \leq \sum_{j=1}^{n} \frac{1}{2\{(1-x_j^2)^{\frac{k}{3}+\frac{3}{4}}|P_n^{(k)'}(x_j)|\}^6} \cdot (n+2k+1)(1-x_j)(1+x)$$
$$\times \{1-x+2x(k+1)\}(1-x^2)^{\frac{k}{2}+\frac{1}{4}}|P_n^{(k)}(x)|(1-x^2)^{\frac{k}{2}+\frac{3}{4}}|P_{n-1}^{(k+1)}(x)|\{\tilde{h}_n^{(k)}\}^2$$
$$\times \left\{\gamma_5 + \sum_{\nu=1}^{n-1}\sum_{\nu=1}^{n-1} \frac{1}{\{h_\nu^{(k)}\}^2}\{(1-x_j^2)^{\frac{k}{2}+\frac{1}{4}}|P_\nu^{(k)}(x_j)|\}^2|P_\nu^{(k)}(x)|^2\right\},$$

where γ_5 is a constant, independent of n,

by using (2.3), (2.4), (2.8), (2.13) and (2.14), we obtain

$$\xi_1 = O(n^{2k+1}).$$

In such a way, ξ_2 is also used in decomposition (2.6), we get

$$\xi_2 \leq \sum_{j=1}^{n} \frac{(n+2k+1)(1-x_j)(1+x)(1-x^2)^{\frac{k}{2}+\frac{3}{4}}|P_{n-1}^{(k+1)}(x)||P_n^{(k)'}(x)|\{\tilde{h}_n^{(k)}\}^2}{2\{(1-x_j^2)^{\frac{k}{3}+\frac{3}{4}}|P_n^{(k)'}(x_j)|\}^6}$$
$$\times \left\{\gamma_6 + \sum_{\nu=1}^{n-1}\sum_{\nu=1}^{n-1} \frac{1}{\{h_\nu^{(k)}\}^2}\{(1-x_j^2)^{\frac{k}{2}+\frac{1}{4}}|P_\nu^{(k)}(x_j)|\}^2(1-x^2)^{\frac{k}{2}+\frac{1}{4}}|P_\nu^{(k)}(x)|^2\right\},$$

where γ_6 is a constant, independent of n, by (2.2), (2.3), (2.4), (2.8), (2.13) and (2.14), we obtain

$$\xi_2 = O(n^{2k+3}),$$

similarly using the decomposition on ξ_3 and ξ_4 for $l_j(x)$, we have

$$\xi_3 = O(n^{2k+4}) \quad , \quad \xi_4 = O(n^{2k+3})$$

and

$$\xi_5 = \sum_{j=1}^{n} |C_5|\breve{\mathfrak{C}}_j'(x)|,$$

from (3.8) and (5.3), we obtain

$$\xi_5 = O(n^{2k+3}).$$

Hence, the theorem is proved. □

Theorem 5.4. If $k > 0$, $n \geq 2$, for the first derivative of the first kind fundamental polynomials $\{\breve{\mathcal{A}}_j(x)\}_{j=1}^n$ on [-1,1] holds

$$\sum_{j=1}^{n}(1-x_j^2)|\breve{\mathcal{A}}_j'(x)| = O(n^{2k+6}). \tag{5.5}$$

Proof. Differentiating (3.2), using (3.3), (3.4) and (3.5), we have

$$\sum_{j=1}^{n}(1-x_j^2)|\breve{\mathcal{A}}_j'(x)| = \varsigma_1+\varsigma_2+\varsigma_3+\varsigma_4,$$

where we use the decomposition (2.6) in ς_1 for $l_j(x)$, we have

$$\varsigma_1 \leq \sum_{j=1}^{n} \frac{b(n+2k+1)(1-x^2)(1-x^2)^{\frac{k}{2}+\frac{3}{4}}|P_{n-1}^{(k+1)}(x)|\{\tilde{h}_n^{(k)}\}^3}{2(1-x_j^2)^{\frac{5k}{2}+\frac{19}{4}}|P_n^{(k)'}(x_j)|^7} \times \left\{\gamma_7 + \sum_{\nu=1}^{n-1}\sum_{\nu=1}^{n-1}\sum_{\nu=1}^{n-1}\frac{1}{\{h_\nu^{(k)}\}^3}\{(1-x_j^2)^{\frac{k}{2}+\frac{1}{4}}|P_\nu^{(k)}(x_j)|\}^3(1-x^2)^{\frac{k}{2}+\frac{1}{4}}|P_\nu^{(k)}(x)|^3\right\},$$

where γ_7 is a constant, independent of n and

$$b = \frac{1}{x_j(3k+1)}\left\{(n^2+2nk+n-k) - \frac{{x_j}^2(k+1)(5k+1)}{(1-{x_j}^2)}\right\}, \tag{5.6}$$

by using (2.3), (2.4), (2.8), (2.13), (2.14) and (5.6), we obtain

$$\varsigma_1 = O(n^{2k+4}).$$

Now, next part of partial sum is

$$\varsigma_2 = \sum_{j=1}^{n}\frac{(n+2k+1)\{1+b(x-x_j)\}}{2(1-x_j^2)^{k+2}|P_n^{(k)'}(x_j)|}\{2x(k+2)|P_{n-1}^{(k+1)}(x)|+(1-x^2)|P_{n-1}^{(k+1)'}(x)|\} \times(1-x^2)^{k+1}|l_j(x)|^3, \tag{5.7}$$

by using the decomposition in ς_2 for $l_j(x)$, we find it

$$\varsigma_2 \leq \sum_{j=1}^{n}\frac{\left\{2x(k+2)(1-x^2)^{\frac{k}{2}+\frac{3}{4}}|P_{n-1}^{(k+1)}(x)|+\frac{1}{2}(n+2k+2)(1-x^2)^{\frac{k}{2}+\frac{7}{4}}|P_{n-2}^{(k+2)}(x)|\right\}}{2(1-x_j^2)^{\frac{5k}{2}+\frac{19}{4}}|P_n^{(k)'}(x_j)|^7} \times(n+2k+1)\{1+b(x-x_j)\}\{\tilde{h}_n^{(k)}\}^3 \times\left\{\gamma_8+\sum_{\nu=1}^{n-1}\sum_{\nu=1}^{n-1}\sum_{\nu=1}^{n-1}\frac{1}{\{h_\nu^{(k)}\}^3}\{(1-x_j^2)^{\frac{k}{2}+\frac{1}{4}}|P_\nu^{(k)}(x_j)|\}^3(1-x^2)^{\frac{k}{2}+\frac{1}{4}}|P_\nu^{(k)}(x)|^3\right\},$$

where γ_8 is a constant, independent of n, by using (2.3), (2.4), (2.8), (2.13), (2.14) and (5.6), we obtain

$$\varsigma_2 = O(n^{2k+5}).$$

Similarly in a way, using the decomposition (2.6) in ς_3, we have

$$\varsigma_3 = O(n^{2k+6})$$

and

$$\varsigma_4 = \sum_{j=1}^{n} |C_3|(1-x_j^2)|\breve{\mathcal{B}}_j'(x)|,$$

by (2.13), (3.5) and (5.4), we obtain

$$\varsigma_4 = O(n^{2k+6}).$$

Hence, the theorem is proved. □

Main Theorem:

Let $k \geq 0$ be a fixed integer, m=4n+2k+1 and let the knots $\{x_i\}_{i=1}^{n}$ and $\{x_i^*\}_{i=1}^{n-1}$ be the roots of the ultraspherical polynomials $P_n^{(k)}(x)$ and $P_{n-1}^{(k+1)}(x)$ respectively. If $f \in C^r[-1,1]$ $(r \geq k+1, 2n \geq 2r-k+2)$, then the interpolational polynomial

$$\mathcal{R}_m(x;f) = \sum_{i=1}^{n} \breve{\mathcal{A}}_i(x) f(x_i) + \sum_{i=1}^{n} \breve{\mathcal{B}}_i(x) f'(x_i) + \sum_{i=1}^{n} \breve{\mathcal{C}}_i(x) f''(x_i) + \sum_{i=1}^{n-1} \breve{\mathcal{D}}_i(x) f(x_i^*) + \sum_{j=0}^{k} \breve{\mathcal{E}}_j(x) f^{(j)}(1) + \sum_{j=0}^{k+1} \breve{\mathcal{F}}_j(x) f^{(j)}(-1), \tag{5.8}$$

satisfies (5.9) for $x \in [-1,1]$

$$|f'(x) - \mathcal{R}_m'(x;f)| = \omega(f^{(r)}; \frac{1}{n}) O(n^{2k+6-r}), \tag{5.9}$$

where the fundamental polynomials $\breve{\mathcal{A}}_i(x)$, $\breve{\mathcal{B}}_i(x)$, $\breve{\mathcal{C}}_i(x)$, $\breve{\mathcal{D}}_i(x)$, $\breve{\mathcal{E}}_i(x)$ and $\breve{\mathcal{F}}_i(x)$ are given in Lemma 3.1 - Lemma 3.6.

Proof

For k=0 we refer to (1.1) , proved by Xie and Zhou [13]. Let $f \in C^r[-1,1]$, by the theorem of Gopengauz [2] for every $m \geq 4r+5$ there exists a polynomial $p_m(x)$ of degree at most m such that for $s = 0, ..., r$

$$|f^{(s)}(x) - p_m^{(s)}(x)| \leq M_{r,s} \left(\frac{\sqrt{1-x^2}}{m} \right)^{r-s} \omega \left(f^{(r)}; \frac{\sqrt{1-x^2}}{m} \right), \tag{5.10}$$

where $\omega(f^{(r)};.)$ denotes the modulus of continuity of the function $f^{(r)}(x)$ and the constants $M_{r,s}$ depend only on r and j. Furthermore,

$$f^{(s)}(\pm 1) = p_m^{(s)}(\pm 1) \quad (s = 0, ..., r).$$

By the uniqueness of the interpolational polynomials $\mathcal{R}_m(x;f)$ it is clear that $\mathcal{R}_m(x;p_m)$=$p_m(x)$. Hence for $x \in [-1,1]$

$$\begin{aligned}|f'(x) - \mathcal{R}'_m(x;f)| &\le |f'(x) - p'_m(x)| + |\mathcal{R}'_m(x;p_m) - \mathcal{R}'_m(x;f)| \\ &\le |f'(x) - p'_m(x)| + \sum_{i=1}^{n} |f(x_i) - p_m(x_i)||\breve{\mathcal{A}}'_i(x)| \\ &+ \sum_{i=1}^{n} |f'(x_i) - p'_m(x_i)||\breve{\mathcal{B}}'_i(x)| + \sum_{i=1}^{n} |f''(x_i) - p''_m(x_i)||\breve{\mathcal{C}}'_i(x)| \\ &+ \sum_{i=1}^{n-1} |f(x_i^*) - p_m(x_i^*)||\breve{\mathcal{D}}'_i(x)|,\end{aligned}$$

by using (5.8) and (5.10). Furthermore, applying the estimates (5.1), (5.3), (5.4), (5.5) and using (2.13), we obtain (5.9) which is statement of the theorem. By Main Theorem and (1.2) we can state the conclusion of the above theorem.

Conclusion

Let $k \ge 0$ be a fixed integer, m=4n+2k+1 , $2n \ge k+4$, let $\{x_i\}_{i=1}^{n}$ and $\{x_i^*\}_{i=1}^{n-1}$ be the roots of the ultraspherical polynomials $P_n^{(k)}(x)$ and $P_{n-1}^{(k+1)}(x)$ respectively. If $f \in C^{2k+6}[-1,1]$, $f^{2k+6} \in Lip\alpha$, $\alpha > \frac{1}{2}$, then $\mathcal{R}_m(x;f)$ and $\mathcal{R}'_m(x;f)$ uniformly converge to $f(x)$ and $f'(x)$, respectively on [-1,1] as $n \to \infty$.

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