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HALTING PROBLEM IN FEYNMAN GRAPHON PROCESSES DERIVED FROM THE RENORMALIZATION HOPF ALGEBRA

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Abstract

Thanks to the theory of graphons and random graphs, Feynman graphons are new analytic tools for the study of infinities in (strongly coupled) gauge field theories. We formulate the Halting problem in Feynman graphon processes to build a new theory of computation in dealing with solutions of combinatorial Dyson–Schwinger equations in the context of the Turing machines and the Manin's renormalization Hopf algebra.

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1 Introduction

The theory of graph functions (or graphons) associated to the space of dense or sparse graphs has been developed in Infinite Combinatorics [4, 5, 6, 7, 8, 9, 10, 15] and Computer Science [11, 12, 18, 29]. Recently, some new applications of graphons to Quantum Field Theory are investigated [24, 25] where Feynman graphons, as analytic generalizations of Feynman diagrams, have been built to provide a new analytic platform for the study of the non-perturbative behavior of equations of motions in strongly coupled gauge field theories [26, 27, 28].

Here we are going to build a theory of computation for the study of Feynman graphon processes which contribute to solutions of Dyson–Schwinger equations (i.e. quantum motions). Firstly, we consider the computability of Feynman graphons in terms of the computable analysis. Secondly, we consider the combinatorial Hopf algebra of Feynman graphons of a given gauge field theory where thanks to the Manin's approach to the theory of computation [19, 20, 21], we formulate the Halting problem for partial recursive

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functions defined on the space of labeled Feynman graphons. Thirdly, we introduce a new class of flowcharts which encode labeled Feynman graphon processes. Then we build a new Hopf algebra structure on these flowcharts which enables us to understand the Halting problem of labeled Feynman graphon processes in the context of the BPHZ renormalization program.

1.1 Graphons

Graphons or graph functions are fundamental tools in Infinite Combinatorics for the study of extremely large dense or sparse graphs and complex networks on the basis of the theory of random graph processes together with measure theoretic tools [7, 9, 10, 15]. Certain classes of measurable functions defined on suitable ground measure spaces can be applied to represent graphons. Graphons are useful to formulate graph limits for sequences of finite (weighted) graphs with respect to cut-distance topology where these analytic graphs can topologically complete the space of finite graphs [15]. Graphons determine a class of random graphs which are useful for the approximation of complicated graphs. The theory of graphons has been developed under two different but related settings to deal with dense and sparse graphs. For sequences of dense weighted graphs, one setting focuses on the role of density, homomorphism densities and random graphs [6, 8, 18]. For sequences of sparse weighted graphs, the other setting focuses on the role of rescaling of the ground measure space, renormalization methods and measure theoretic tools to generate non-zero graphons for the description of infinite sparse graphs [4, 5, 7, 9, 10]. It is also possible to study random graphs and homomorphism densities for these non-zero graphons. Several applications of graphons in Combinatorics, Theoretical Computer Science and Quantum Field Theory have been addressed in [7, 9, 11, 12, 18, 24, 25, 26, 27, 28].

1.2 Turing machines and Halting problem

Dealing with the computability of real numbers and real functions is a fundamental topic in Computable Analysis and Theoretical Computer Science. Turing provided the initial steps for the study of computable real numbers. Computable real numbers can be handled in terms of suitable convergent sequences of rational numbers. Recently, real numbers have been studied and classified in terms of weaker versions of computability. In addition, some generalizations of Turing machines have been applied to provide algorithmic structures (on the basis of sequences of rational polygons) for the study of computable real functions. [3, 13, 17, 31, 32]

Turing worked on the existence of a universal machine which encodes a description for any computational program in terms of its corresponding Turing machine [31]. However it has been shown that a certain class of problems such as Halting problem cannot be solved in general by computation. The Halting problem in the study of (universal) Turing machines is one of the fundamental concepts in Theory of Computation [22, 30]. The basic idea of the Halting problem is the ability to determine whether an executed program terminates whenever started from the initial state. Any well-defined computation process which takes some input values and then produces some output values is defined in terms of an algorithm. An algorithm contains a family of programs. Algorithms provide a bridge between programs and computational functions. Programs are represented in terms of a collection of directed graphs together with the compositions of arrows and distinguished loops on vertices namely, flowcharts. The Halting problem concerns the computability of programs in the context of intermediate algorithms. Thanks to the Connes–Kreimer renormalization Hopf algebra of Feynman diagrams, Manin provided a new interpretation of the Halting problem in the context of the Bogoliubov–Parasiuk–Hepp–Zimmermann (BPHZ) perturbative renormalization. He built a graded Hopf algebra of flowcharts decorated by primitive recursive functions and then he applied the BPHZ perturbative renormalization to determine the amount of non-computability of partial recursive functions [17, 19, 20, 21]. The Manin approach has already been developed to Quantum Field Theory where the amount of (non-)computability of intermediate algorithms are studied in terms of the renormalization of solutions of Dyson–Schwinger equations [23].

1.3 Feynman graphons and Dyson–Schwinger equations

The interaction part of the Lagrangian of an interacting gauge field theory Φ contains polynomials with respect to the (running) coupling constants of the physical theory. The strength of the (running) coupling constants govern the behavior of fixed points of Green's functions known as Dyson–Schwinger equations. Solutions of these equations, which provide quantum motions, might have non-perturbative behavior studied in the context of lattice models, large N limits and Hopf algebraic platform. The Connes–Kreimer renormalization Hopf algebra of Feynman diagrams $H_{\mathrm{FG}}(\Phi)$ is topologically enriched with respect to the topology of graphons to formulate the notion of convergence for sequences of Feynman diagrams. For any family of primitive Feynman diagrams $\{\gamma_n\}_{n\geq 1}$ with the corresponding combinatorial Dyson–Schwinger equation

$$X = \mathbb{I} + \sum_{n\geq 1} c(g)\omega_n B^{+}_{\gamma_n}(X^{n+1}) \tag{1}$$

underlying the running coupling constant $c(g)$, its unique solution $X = \sum_{n\geq 0} c(g)^n X_n$ can be interpreted by the Feynman graphon W_X. It is the cut-distance convergent limit of a sequence of Feynman graphon representations W_{Y_m} of the partial sums $Y_m = \sum_{i=0}^{m} c(g)^i X_i$. In other words, W_X is given as an infinite direct sum of W_{X_n}s with the weights $c(g)^n$. This new setting provides a new random graph model for the study of the non-perturbative behavior of quantum equations. [24, 25, 26, 27, 28]

2 From Lebesgue to L^p Feynman graphons

In the theory of graphons, we deal with a family of analytic graphs which have continuum vertex sets. These graphs can be generated in terms of convergent graph limits of sequences of finite weighted dense or sparse graphs. Graphons are useful to construct an important family of finite and infinite random graphs which enable us to recover graph limits and complex networks. In fact, the space of graphons can topologically complete the space of finite graphs. It is possible to build graphons on any arbitrary (σ-finite) measure space (Ω, μ_Ω). Rooted tree representations of Feynman diagrams have been applied to associate a class of graphons to Feynman diagrams and their formal expansions. Rescaled or stretched versions of Lebesgue measure graphons and L^p-graphons are the practical tools to generate non-zero graphons as the convergence limits of sequences of sparse graphs. [25, 27, 28]

Definition 1.
- *A Feynman diagram Γ is a finite oriented decorated graph which obeys the conservation of momenta. It contains a vertex set Γ_0, an edge set Γ_1^{int} of internal edges which have beginning and ending vertices and an edge set Γ_1^{ext} of external edges which have beginning or ending vertex. While Γ might have nested overlapping loops, it does not have any self-loop.*
- *The rooted tree representation t_Γ is a decorated rooted tree such that each vertex in t_Γ is the symbol for a nested loop in Γ.*
- *Consider the Lebesgue measure space $([0,1], m)$ as the ground probability measure space. For a given Feynman diagram Γ, a Feynman graphon W_Γ is a symmetric bounded Lebesgue measurable real valued function on the closed box $[0,1] \times [0,1]$ generated by the adjacency matrix of t_Γ. It is called a labeled Feynman graphon, if it can be presented by*

$$W_\Gamma^\rho(x,y) := W_\Gamma(\rho(x), \rho(y)) \tag{2}$$

in terms of any invertible Lebesgue measure preserving transformation $\rho \in L^1([0,1], m)$ on $[0,1]$.

Remark 1. *For a linear combination $\Gamma = \alpha_1\Gamma_1 + ... + \alpha_n\Gamma_n$ of Feynman diagrams with positive coefficients, its Feynman graphon model W_Γ is given by the normalization of the direct sum $W_{\Gamma_1} + ... + W_{\Gamma_n}$ such that W_{Γ_i} is defined on the subinterval $I_i \times I_i$ with $m(I_i) = \alpha_i$ and $I_i \cap I_j = \emptyset$ for $i \neq j$. In this setting, W_Γ is a stretched labeled Feynman graphon defined on $\sqcup_{i=1}^n I_i \times \sqcup_{i=1}^n I_i$ such that by rescaling methods, we project it as a labeled Feynman graphon defined on $[0,1] \times [0,1]$. [27, 28]*

Pixel picture presentations of the adjacency matrices of decorated rooted trees are elementary examples of labeled Feynman graphons. While $\|W_\Gamma^\rho\|_1 = \|W_\Gamma\|_1$, all relabeled Feynman graphons with respect to Γ are encapsulated by the equivalence class

$$[W_\Gamma] := \{W_\Gamma^\rho,\ \rho : \text{invertible Lebesgue measure preserving}\}. \tag{3}$$

$[W_\Gamma]$ is called an unlabeled Feynman graphon class. In addition, Lebesgue measure preserving transformations on $[0,1]$ generate a larger class of labeled Feynman graphons which are not the same but their random Feynman graph models or their densities are similar.

Definition 2.
- *Feynman diagrams Γ_1, Γ_2 are called weakly isomorphic, if their corresponding Feynman graphon models $W_{\Gamma_1}, W_{\Gamma_2}$ are weakly isomorphic. It means that there exists another Feynman graphon model W and Lebesgue measure preserving transformations ρ_1, ρ_2 on $[0,1]$ such that $W^{\rho_1} = W_{\Gamma_1}$ and $W^{\rho_2} = W_{\Gamma_2}$ almost everywhere.*

- *We extend the equivalence class (3) with respect to the weakly isomorphic relation to define*
$$[W_\Gamma]_\approx := \{W_\Gamma^\rho,\ \rho : \text{Lebesgue measure preserving}\}. \tag{4}$$
$[W_\Gamma]_\approx$ *is called an unlabeled Feynman graphon class up to the weakly isomorphic relation.*

Theorem 1. *For any* $1 \leq p < \infty$*, the space of real graphons* L^p*-completes the space of Feynman diagrams of a given gauge field theory* Φ*.*

Proof. For any Feynman diagram Γ, consider its rooted tree representation t_Γ as a finite weighted graph with the vertices $v_1, ..., v_n$ with the corresponding weights $\alpha_i > 0$ such that $\sum_{i=1}^n \alpha_i = 1$. For any edge $v_i v_j$ in t_Γ, set β_{ij} as its weight. Fix the partition $P : I_1, ..., I_n$ of $[0,1]$ with the corresponding Lebesgue measures $m(I_j) = \alpha_j$, for $j = 1, ..., n$. Now define the labeled Feynman graphon

$$W_\Gamma^P(x,y) := \begin{cases} \beta_{ij}, & \text{if x} \in \mathrm{I_i} \text{ and y} \in \mathrm{I_j} \text{ adjacent} \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

with respect to the partition P. We remove the restriction on the weight and consider t_Γ with the corresponding weight $\alpha_1, ..., \alpha_n$ associated to vertices and edges with the corresponding weights $\beta_{ij}(t_\Gamma)$. Set $\alpha_{t_\Gamma} = \sum_{i=1}^n \alpha_i$. We determine a partition P of $[0,1]$ in terms of measurable subsets $I_1, ..., I_n$ such that $\mu(I_i) = \frac{\alpha_i}{\alpha_{t_\Gamma}}$ such that $([0,1], \mu)$ is a σ-finite measure space. Then a labeled Feynman graphon corresponding to Γ is given by

$$W_\Gamma^P(x,y) = \sum_{i=1}^n \sum_{j=1}^n \chi_{I_i}(x)\chi_{I_j}(y)\beta_{ij}(t_\Gamma). \tag{6}$$

We define a semi-norm structure on the space of labeled Feynman graphons given by

$$||W_\Gamma^\rho||_{\mathrm{cut}} := \sup_{A,B\subset[0,1]} \left| \int_{A\times B} W_\Gamma^\rho(x,y)dxdy \right| \tag{7}$$

such that, up to the weakly isomorphic relation, it is a norm which leads us to the cut-distance metric

$$d_{\mathrm{cut}}([W_{\Gamma_1}]_\approx, [W_{\Gamma_2}]_\approx) := \inf_{\rho_1,\rho_2} \sup_{A,B\subset[0,1]} \left| \int_{A\times B} W_{\Gamma_1}^{\rho_1}(x,y) - W_{\Gamma_2}^{\rho_2}(x,y)dxdy \right|. \tag{8}$$

Thanks to the topology of graphons [15, 18], the space $\mathcal{W}_\Phi([0,1], m)$ of Feynman graphon classes on the Lebesgue measure space $([0,1], m)$ is a compact Hausdorff metric space with respect to the cut-distance topology.

Thanks to [4, 5], we extend our setting to equip the space of Feynman graphons with the L^p-norm. W_Γ is called an L^p-Feynman graphon if $||W_\Gamma||_p < \infty$ such that

$$||W_\Gamma||_p = \left(\int_{[0,1]^2} |W(x,y)|^p \, dxdy \right)^{1/p}. \tag{9}$$

Set

$$d_p(W^{\rho_1}_{\Gamma_1}, W^{\rho_1}_{\Gamma_2}) := ||W^{\rho_1}_{\Gamma_1} - W^{\rho_2}_{\Gamma_2}||_p, \tag{10}$$

then

$$d_{p,\text{cut}}([W_{\Gamma_1}]_{\approx}, [W_{\Gamma_2}]_{\approx}) = \inf_{\rho_1,\rho_2} d_p(W^{\rho_1}_{\Gamma_1}, W^{\rho_2}_{\Gamma_2}). \tag{11}$$

This metric is applied to equip the space of Feynman diagrams with the L^p-norm defined by

$$d_{p,\text{cut}}(\Gamma_1, \Gamma_2) := d_{p,\text{cut}}([W_{\Gamma_1}]_{\approx}, [W_{\Gamma_2}]_{\approx}). \tag{12}$$

We have

$$||\Gamma||_{\text{cut}} \leq ||\Gamma||_{1,\text{cut}} \leq ||\Gamma||_{2,\text{cut}} \leq ||\Gamma||_{\infty,\text{cut}}. \tag{13}$$

Consider a sequence $\{\Gamma_n\}_{n\geq 1}$ of Feynman diagrams with the corresponding sequence $\{t_{\Gamma_n}\}_{n\geq 1}$ of decorated rooted trees (as weighted sparse graphs). When n tends to infinity, $||W^{P_{\Gamma_n}}_{\Gamma_n}||_p$ goes to zero which means that the convergent limit is an infinite graph with almost zero density namely, weakly isomorphic to the 0-graphon. We remove this issue by working on renormalized L^p-Feynman graphons

$$\tilde{W}^{P_{\Gamma_n}}_{\Gamma_n} := \frac{W^{P_{\Gamma_n}}_{\Gamma_n}}{\|W^{P_{\Gamma_n}}_{\Gamma_n}\|_p} \tag{14}$$

for each n. So now $\{\Gamma_n\}_{n\geq 1}$ is L^p-convergent whenever the sequence $\{[\tilde{W}^{P_{\Gamma_n}}_{\Gamma_n}]_{\approx}\}_{n\geq 1}$ of renormalized L^p-Feynman graphons is convergent to a non-zero Feynman graphon with respect to the L^p-norm. The space $\mathcal{W}_{\Phi}([0,1], ||.||_{p,\text{cut}})$ of L^p-Feynman graphon classes is a complete Hausdorff metric space with respect to the L^p-norm. □

Remark 2.

- *It is also possible to consider the stretched version of the canonical graphons where we have* $\tilde{W}^s_{\Gamma_n} := W_{\Gamma_n}(\frac{x}{\|W_{\Gamma_n}\|_p^{1/2}}, \frac{y}{\|W_{\Gamma_n}\|_p^{1/2}})$ *for* $0 \leq x, y \leq \|W_{\Gamma_n}\|_p^{-1/2}$.
- *Thanks to [27, 28], the sequence* $\{\Gamma_n\}_{n\geq 1}$ *with increasing loop numbers is convergent to* W_Γ, *if for each fixed value* k, *the sequence* $\{t_{\Gamma_n}[k]\}_{n\geq 1}$ *of the distributions of random trees is convergent when* n *tends to infinity. In this setting,* $t_{\Gamma_n}[k]$ *is a random tree with* k *vertices given by selecting distinct vertices* $v_1, ..., v_k$ *from* t_{Γ_n} *under a uniformly random process such that with the probability* $W_{\Gamma_n}(v_{i_k}, v_{j_k})$ *(for* $i_k \neq j_k$*), there exists an edge* $v_{i_k}v_{j_k}$ *in* $t_{\Gamma_n}[k]$. *The random tree limit* $t_\Gamma[\infty]$ *of this sequence is determined by the Feynman graphon* W_Γ. *In other words,* $t_\Gamma[\infty]$ *has infinite countable vertices given by selecting distinct vertices* $v_1, v_2, ...$ *from* t_Γ *such that with the probability* $W_\Gamma(v_i, v_j)$, *there exists an edge* v_iv_j *(for* $i \neq j$*) in this infinite random tree.*

One important application of Feynman graphons is their roles for the description of solutions of Dyson–Schwinger equations where we need to deal with the sequences of partial sums and their convergence with respect to the cut-distance topology or L^p-norms.

Corollary 1. *Consider a (strongly coupled) gauge field theory Φ with the corresponding space $\mathcal{W}_\Phi([0,1], ||.||_{p,\mathrm{cut}})$ of L^p-Feynman graphons. The sequence $\{Y_m\}_{m\geq 1}$ of the partial sums of the unique solution X of a combinatorial Dyson–Schwinger equation DSE is L^p-convergent to X.*

Proof. It is a direct result of Theorem 1 and [4, 5, 27, 28]. □

3 A theory of computation for Feynman graphon processes

Turing machines are important examples of abstract models in Theory of Computation. A Turing machine is given by a finite set of states, a finite set of symbols, an input vocabulary collection built from symbols, an initial state, a transition function and a set of final states. The collection of accepted inputs determine the language corresponding to the Turing machine. It is shown by Church–Turing Thesis that there exists a universal Turing machine for each computable problem. All computable functions are computed under a suitable Turing complete system. [30]

In this section, we study the notion of computability of Feynman graphons and Feynman graphon processes in the context of rational polynomials. We explain the structure of a graded Hopf algebra of Feynman graphons which leads us to formulate the Halting problem at the level of Feynman graphons under a renormalization program. We then generalize our Hopf algebraic platform to the level of decorated flowcharts to formulate the Halting problem for Feynman graphon processes which contribute to solutions of combinatorial Dyson–Schwinger equations.

3.1 Computability

The domain and the image of any real graphon are subsets of $[0,1]\times[0,1]$ and $[0,1]$, respectively. In Computable Analysis, real numbers and real functions are classified in terms of their computability level where Turing machines have been applied to determine computable real functions. The computability of graphons are addressed in [1, 2] which are not useful for our platform. Here we plan to understand the computability of (L^p-)Feynman graphons (defined on the ground measure space (Ω, m) such that $\Omega \subseteq \mathbb{R}$) in the context of Computable Analysis and generalized Turing machines to formulate the Halting problem of Feynman graphon processes under the Hopf algebraic setting.

Definition 3.
- *A real number x is called computable, if there exists a sequence $(r_n)_{n\geq 1}$ of rational numbers which effectively converges to x which means that for any n, we have $|r_n - x| \leq 2^{-n}$.*
- *If the effectively convergent sequence $(r_n)_{n\geq 1}$ is increasing, then we call x a left-computable real number while if it is a decreasing effectively convergent sequence, then we call x a right-computable real number.*

A computable real number is left-computable and right-computable. In addition, the sequence $(r_n)_{n\geq 1}$ in Definition 3, which is called the computable name of x, provides some effective approximations of x with effective error estimations.

This notion of "computability" can be formulated for real valued functions where we need to work on some generalized versions of Turing machines. For a given finite alphabet set Σ, the corresponding set Σ^* of finite strings is a countable set. A real function f on Σ^* is computable, if there exists a Turing machine M_f which sends each finite string $s \in \mathrm{Dom}(f)$ to an output $f(s)$ in a finite number of steps. Some generalizations of Turing machines have been considered to study computable functions on uncountable sets such as the set Σ^ω of all infinite strings over Σ. In Computable Analysis, the main task is to classify real numbers and real functions in terms of their level of structural computability. [3, 17]

Definition 4. *For a given finite alphabet set Σ, a type-2 Turing machine M_ω is a Turing machine such that*

- *It is the same as the classical Turing machine on Σ^*,*
- *It can accept inputs from Σ^ω to generate infinite sequences as outputs.*
- *If it halts in a finite number of steps or cannot generate an infinite sequence as an output, then the machine diverges.*

Definition 5. *A real function f defined on $[a,b]$ is computable, if there exists a type-2 Turing machine M_f which computes it. It means that M_f sends any rational sequence $(r_n)_{n\geq 1}$ in $[a,b]$ which effectively converges to a computable real number $x \in [a,b]$ to a rational sequence $(s_n)_{n\geq 1}$ in $\mathrm{Im}(f)$ which effectively converges to the corresponding computable real number $f(x)$.*

Computable real functions have been approximated in terms of sequences of polynomials. A rational polygon is a continuous piecewise linear function which connects a finite set of rational turning points on a closed interval. We replace a sequence of rational polygons with a sequence of rational polynomials. Thanks to Weierstrass approximation Theorem [13, 32], it is shown that a computable (continuous) real function f on a closed interval can be described as the uniformly effectively convergent limit of a computable sequence $(\mathrm{pg}_n)_{n\geq 1}$ of rational polygons. It means that for any fixed $x \in \mathrm{Dom}(f)$ and any $n \geq 1$, we have $|\mathrm{pg}_n(x) - f(x)| \leq 2^{-n}$. [3]

Now we are going to modify this platform for the study of computable (L^p-) Feynman graphons for $1 \leq p < \infty$.

Definition 6. *A labeled Feynman graphon $W_\Gamma^\rho : [0,1] \times [0,1] \to [0,1]$ is called computable, if there exists a type-2 Turing machine $M_{W_\Gamma^\rho}$ such that it sends any sequence $\{(r_n, s_n)\}_{n\geq 1}$ of pairs of rational numbers which effectively converges to a pair (x,y) of computable real numbers to a rational sequence $(t_n)_{n\geq 1}$ which effectively converges to the corresponding computable real number $W_\Gamma^\rho(x,y)$.*

Remark 3. *For a given computable Feynman graphon W_Γ, any labeled Feynman graphon W_Γ^ρ and any other graphon model which is weakly isomorphic to W_Γ is also computable.*

For any labeled Feynman graphon $W_\Gamma^\rho : [0,1] \times [0,1] \to [0,1]$, we project its domain onto the vertical axis ($x = x_0$) and the horizontal axis ($y = y_0$) to obtain real functions

$$W^\rho_{\Gamma, x=x_0} : [0,1] \to [0,1], \; W^\rho_{\Gamma, x=x_0}(y) := W^\rho_\Gamma(x_0, y),$$

$$W^{\rho}_{\Gamma,y=y_0} : [0,1] \to [0,1],\ W^{\rho}_{\Gamma,y=y_0}(x) := W^{\rho}_{\Gamma}(x,y_0). \tag{15}$$

Corollary 2. *For a given computable labeled Feynman graphon W^{ρ}_{Γ}, real functions $W^{\rho}_{\Gamma,x=x_0}, W^{\rho}_{\Gamma,y=y_0}$ are also computable.*

Proof. We have a Turing machine $M_{W^{\rho}_{\Gamma}}$ which encodes the computation process of W^{ρ}_{Γ}. For a fixed computable number $x_0 \in [0,1]$, let $(r^0_n)_{n\geq 1}$ be a sequence which is effectively convergent to x_0. For any computable number $y \in [0,1]$, $M_{W^{\rho}_{\Gamma}}$ sends any rational sequence $\{(r^0_n, s_n)\}_{n\geq 1}$ which is effectively convergent to (x_0, y) to a rational sequence $(u^0_n)_{n\geq 1}$ which effectively converges to $W^{\rho}_{\Gamma}(x_0,y) = W^{\rho}_{\Gamma,x=x_0}(y)$. In addition, for a fixed computable number $y_0 \in [0,1]$, let $(s^0_n)_{n\geq 1}$ be a sequence which is effectively convergent to y_0. For any computable number $x \in [0,1]$, $M_{W^{\rho}_{\Gamma}}$ sends any rational sequence $\{(r_n, s^0_n)\}_{n\geq 1}$ which is effectively convergent to (x, y_0) to a rational sequence $(v^0_n)_{n\geq 1}$ which effectively converges to $W^{\rho}_{\Gamma}(x,y_0) = W^{\rho}_{\Gamma,y=y_0}(x)$. □

Corollary 3. *For a given labeled Feynman graphon W^{ρ}_{Γ}, let real functions $W^{\rho}_{\Gamma,x=x_0}$, $W^{\rho}_{\Gamma,y=y_0}$ are computable on $A_y, A_x \subseteq [0,1]$, respectively. Then real functions $W^{\rho}_{\Gamma,x=x_0} \pm W^{\rho}_{\Gamma,y=y_0}$ and $W^{\rho}_{\Gamma,x=x_0} W^{\rho}_{\Gamma,y=y_0}$ are also computable.*

Proof. Thanks to Definition 5, for computable functions $W^{\rho}_{\Gamma,x=x_0}, W^{\rho}_{\Gamma,y=y_0}$, there exist type-2 Turing machines M_y, M_x such that
- The type-2 Turing machine M_x sends each computable real number $x \in A_x \subseteq \mathrm{Dom}(W^{\rho}_{\Gamma,y=y_0})$ to a computable real number $W^{\rho}_{\Gamma,y=y_0}(x)$ in the image.
- The type-2 Turing machine M_y sends each computable real number $y \in A_y \subseteq \mathrm{Dom}(W^{\rho}_{\Gamma,x=x_0})$ to a computable real number $W^{\rho}_{\Gamma,x=x_0}(y)$ in the image.

Define a new type-2 Turing machine $M_x \pm M_y$ which sends any sequence $\{(p_n, q_n)\}_{n\geq 1}$ of rational pairs in $A_x \times A_y$ which effectively converges to a pair (x, y) of computable numbers to a rational sequence $(r_{p_n q_n})_{n\geq 1}$ in $[0,1]$ which is effectively convergent to $W^{\rho}_{\Gamma}(x,y_0) \pm W^{\rho}_{\Gamma}(x_0,y)$.

In addition, define a new type-2 Turing machine $M_x \times M_y$ which sends any sequence $\{(p_n, q_n)\}_{n\geq 1}$ of rational pairs in $A_x \times A_y$ which effectively converges to a pair (x, y) of computable numbers to a rational sequence $(s_{p_n q_n})_{n\geq 1}$ in $[0,1]$ which is effectively convergent to $W^{\rho}_{\Gamma}(x,y_0) W^{\rho}_{\Gamma}(x_0,y)$. □

Corollary 4. *For a given labeled Feynman graphon W^{ρ}_{Γ} which is computable on $A_x \times A_y \subseteq [0,1] \times [0,1]$, the collection*

$$G_{W^{\rho}_{\Gamma}} := \{W^{\rho}_{\Gamma,x=x_0}, W^{\rho}_{\Gamma,y=y_0} :\ x_0 \in A_x, y_0 \in A_y\} \cup \{W_{\mathbb{I}}\} \tag{16}$$

is an abelian group such that $W_{\mathbb{I}}$ is the 0-graphon with respect to the empty graph.

The space of all computable labeled Feynman graphons on the Lebesgue measure space $([a,b], m)$ or the L^p space $([a,b], ||.||_p)$ is an abelian group.

Theorem 2. *Computable labeled Feynman graphons can be approximated by rational polynomials.*

Proof. It is enough to show that for a given computable labeled Feynman graphon W_Γ^ρ, there exists a computable sequence of rational two variables polynomials which converges uniformly effectively to W_Γ^ρ.

The Turing machine $M_{W_\Gamma^\rho}$ sends any rational sequence of pairs $\{(r_n, s_n)\}_{n\geq 1}$ which converges effectively to the computable pair $(x,y) \in \mathrm{Dom}(W_\Gamma^\rho)$ to an increasing rational sequence $(t_n)_{n\geq 1}$ which is effectively convergent to the corresponding computable real number $W_\Gamma^\rho(x,y)$.

Let $x, y \in [0,1]$ be rational numbers such that we set $(r_j, s_j) = (x,y)$ for all $j \geq 1$ where $M_{W_\Gamma^\rho}$ generates computable numbers $c_1, ..., c_j$ in finite steps via reading a finite number k_j of arrays of the input $\{(r_j, s_j)\}_{j\geq 1}$. Define an open box

$$I_x \times I_y := (x - 2^{-k_j}, x + 2^{-k_j}) \times (y - 2^{-k_j}, y + 2^{-k_j}). \tag{17}$$

For any arbitrary pair $(z_1, z_2) \in I_x \times I_y$ of computable real numbers with the corresponding sequence of pairs $\{(r_n^{z_1}, s_n^{z_2})\}_{n\geq 1}$ which converges effectively to (z_1, z_2), define a new sequence $\{(u_j, v_j)\}_{j\geq 1}$ such that for each $j \geq 1$,

$$(u_j, v_j) = \{^{(x,y),\ j\leq k_j}_{(r_n^{z_1}, s_n^{z_2}),\ j>k_j}. \tag{18}$$

This sequence is effectively convergent to (z_1, z_2) and therefore $M_{W_\Gamma^\rho}$ sends this new sequence to an increasing rational sequence with the initial arrays $c_1, ..., c_j$ which effectively converges to $W_\Gamma^\rho(z_1, z_2)$. For any $(w_1, w_2) \in I_x \times I_y$, $c_j \leq W_\Gamma^\rho(w_1, w_2)$.

The collection $\mathcal{I} := \{I_x \times I_y : x, y \in \mathbb{Q} \cap [0,1]\}$ is an open cover for the compact box $[0,1] \times [0,1]$. There exists a finite sub-cover $\{I_{x_1} \times I_{y_1}, ..., I_{x_l} \times I_{y_l}\}$ for $\mathcal{I}$ such that for each $i = 1, ..., l$, there exist some rational numbers c_j^i where $c_j^i \leq W_\Gamma^\rho(w_1, w_2)$ for any $(w_1, w_2) \in I_{x_i} \times I_{y_i}$.

Therefore for all $i = 1, ..., l$ and for any $(w_1, w_2) \in I_{x_i} \times I_{y_i}$, we can determine a rational two variables polygon pg_j^i such that

$$c_j^i \leq \mathrm{pg}_j^i(w_1, w_2) \leq W_\Gamma^\rho(w_1, w_2). \tag{19}$$

This gives us an increasing sequence of rational two variables polygons such that whenever j goes to infinity, it converges to W_Γ^ρ. □

3.2 Hopf algebra of Feynman graphons and Halting problem

Finding a suitable Turing machine with respect to a given problem is a decision problem. The Halting problem, which concerns whether a machine halt on a given input or not, is undecidable [22, 29]. In this part we equip the space of Feynman graphons associated to Feynman diagrams of a given gauge field theory with a graded Hopf algebra structure. Then we modify the Manin's renormalization program to folrmulate the Halting problem for Feynman graphons.

Theorem 3. *There exists a topological Hopf algebra structure on the space of graphons defined on the ground measure space* $([0,1], m)$ *which encodes the convergence of sequences of finite graphs with increasing vertex numbers.*

Proof. Consider the graded commutative unital algebra generated by the space of finite graphs where the number of vertices provides the graduation parameter and the disjoint union of graphs is the multiplication for this algebra. We equip this algebra with a bialgebra structure in terms of the coproduct

$$\Delta(G) = \sum_V H_V \otimes H_{V_G \setminus V} \tag{20}$$

such that H_V is a subgraph of G with the vertex set V. The grading structure is useful to formulate an antipode recursively. Therefore we obtain a finite type graded commutative non-cocommutative unital co-unital Hopf algebra on the space of finite graphs. [16]

The coproduct (20) determines a graded Hopf algebra structure on the space $\mathcal{W}_{[0,1]}$. Consider the commutative algebra $H_{\mathcal{W}_{[0,1]}}$ free generated by labeled graphons W^ρ in $\mathcal{W}_{[0,1]}$ with respect to all Lebesgue measure preserving transformations on $[0,1]$ such that the 0-graphon $W_{\mathbb{I}}$ with respect to the empty graph is its unit. Labeled graphons corresponding to graphs without any edge belong to $[W_{\mathbb{I}}]_\approx$. However in this algebra we consider them as separate generators. Thanks to Theorem 1, for a given finite graph G with the associated labeled graphons W_G^ρ, define

$$\Delta(W_G^\rho) = \sum_H W_H^\rho \otimes W_{G/H}^\rho \tag{21}$$

as the coproduct of W_G^ρ such that the sum is controlled in terms of subgraphs of the graph G. In addition, any arbitrary labeled graphon $W^\rho \in [W]_\approx$ is identified in terms of the graph limit of a sequence $\{G_n\}_{n\geq 1}$ of finite (simple) graphs [15, 18]. It means that the sequence $\{[W_{G_n}]_\approx\}_{n\geq 1}$ is cut-distance convergent to $[W]_\approx$. The coproduct (21) is linear and bounded which means that it is continuous with respect to the cut-distance topology. Therefore $\Delta(W^\rho)$ is identified in terms of the cut-distance convergent limit of the sequence $\{\Delta([W_{G_n}]_\approx)\}_{n\geq 1}$. In other words, $\Delta(W^\rho)$ is the limit of the sequence $\{\sum_{H_n} W_{H_n}^\rho \otimes W_{G_n/H_n}^\rho\}_{n\geq 1}$ with respect to the cut-distance topology.

The counit on $H_{\mathcal{W}_{[0,1]}}$ is defined by

$$\varepsilon(W^\rho) = 0 \text{ for } W^\rho \neq W_{\mathbb{I}}, \;\; \varepsilon(W_{\mathbb{I}}) = 1 \; . \tag{22}$$

Therefore $H_{\mathcal{W}_{[0,1]}}$ is now equipped with a bialgebra structure. The vertex number of graphs makes a graduation parameter on the bialgebra of labeled graphons. In other words, we have $H_{\mathcal{W}_{[0,1]}} = \bigoplus_{n=0}^\infty H_{\mathcal{W}_{[0,1]}}^{(n)}$ such that for each n, $H_{\mathcal{W}_{[0,1]}}^{(n)}$ is the $\mathbb{K}$-submodule spanned by labeled graphons W_G^ρ such that $|G| = n$. This allows us to define an antipode on $H_{\mathcal{W}_{[0,1]}}$ under a recursive process where for a given finite graph G, we have

$$S(W_G^\rho) = -W_G^\rho - \sum_H S(W_H^\rho) W_{G/H}^\rho \; . \tag{23}$$

In addition, for any arbitrary labeled graphon $W^\rho \in [W]$, we identify $S(W^\rho)$ in terms of the cut-distance convergent limit of the sequence $\{S([W_{G_n}]_\approx)\}_{n\geq 1}$. In other words, $S(W^\rho)$ is the limit of the sequence $\{S(W_{G_n}^\rho)\}_{n\geq 1}$ with respect to the cut-distance topology.

As the result, $H_{\mathcal{W}_{[0,1]}}$ is a graded free commutative non-cocommutative Hopf algebra which is completed with respect to the cut-distance topology. We use the notation $H^{\text{cut}}_{\mathcal{W}_{[0,1]}}$ for the resulting topological Hopf algebra. □

Corollary 5. *The space of (L^p-)Feynman graphons corresponding to Feynman diagrams of a given gauge field theory Φ can be equipped with a topological Hopf algebra.*

Proof. Set $H^{\text{cut}}_{\mathcal{W}_{\Phi}([0,1])}$ as the free commutative algebra generated by Feynman graphon classes $[W_\Gamma]_{\approx}$. The loop number determines the graduation parameter on $H^{\text{cut}}_{\mathcal{W}_{\Phi}([0,1])}$ such that for each $n \geq 1$, $H^{\text{cut},(n)}_{\mathcal{W}_{\Phi}([0,1])}$ is the vector space generated by $[W_\Gamma]_{\approx}$ corresponding to 1PI Feynman diagrams Γ with the loop number n or products of Feynman graphons $[W_{\Gamma_i}]_{\approx}$ corresponding to 1PI Feynman diagrams Γ_i with the overall loop number n. The rest of the proof is a direct result of Theorem 3 and [28]. □

In fact, $H^{\text{cut}}_{\mathcal{W}_{\Phi}([0,1])}$ is the topological renormalization Hopf algebra of Feynman graphons which is rich enough to encode the non-perturbative renormalization of solutions of combinatorial Dyson–Schwinger equations [25, 26, 27, 28].

Manin worked on the construction of a new Hopf algebra of enriched programming encoded by decorated flowcharts to formulate a new interpretation of the Halting problem in the context of perturbative renormalization. He applied the Connes–Kreimer approach to the BPHZ perturbative renormalization to determine the non-computability level of programs in terms of the BPHZ counterterms. He provided a deformed version of the Halting problem for partial recursive functions. He showed that checking whether a positive integer number k belongs to $\mathrm{Dom}(f)$ (of a given partial recursive function f on the constructive world $\mathbb{N}$) or not is equivalent to checking a particular analytic function $\Phi(k, f; z)$ with respect to a complex parameter z (determined on the basis of the Kolmogorov complexity) has a pole at $z = 1$ or not. The BPHZ perturbative renormalization is the key tool to extract the non-computability level in terms of the Birkhoff factorization of a character ψ_Φ on the Hopf algebra of the Halting problem. [19, 20]

Example 1. *For a given partial recursive function f on the constructive world $\mathbb{N}$, define $\tilde{f}$*

$$\tilde{f}(n) = f(n),\ n \in \mathrm{Dom}(f),\ f(x) = 0,\ x \notin \mathrm{Dom}(f)\ , \tag{24}$$

and then consider

$$\Phi(k, f; z) := \sum_{n \geq 0} \frac{z^n}{(1 + n\tilde{f}(k))^2}\ . \tag{25}$$

For any k_1 which is not in $\mathrm{Dom}(f)$, we have $\Phi(k_1, f; z) = \frac{1}{1-z}$ while if $k_2 \in \mathrm{Dom}(f)$, then $\Phi(k_2, f; z)$ is the Taylor series of an analytic function at $|z| < 1$ and continuous at the boundary region $|z| = 1$. [21]

Thanks to this platform, now we formulate the Halting problem at the level of labeled Feynman graphons under a new Hopf algebraic setting.

Corollary 6. *The computability of labeled Feynman graphons can be encoded in terms of the BPHZ perturbative renormalization.*

Proof. The BPHZ perturbative renormalization is formulated in terms of Dimensional Regularization together with Minimal Subtraction. This renormalization scheme is encoded by the Rota–Baxter algebra $(A_{\mathrm{dr}}, R_{\mathrm{ms}})$ such that A_{dr} is the algebra of Laurent series with finite pole parts and $R_{\mathrm{ms}} : A_{\mathrm{dr}} \to A_{\mathrm{dr}}$ is the projection map onto the pole parts. We are going to apply this renormalization scheme to formulate a machinery which approximates whether a given labeled Feynman graphon is computable at an input or not.

We work on the Hopf algebra $H^{\mathrm{cut}}_{\mathcal{W}_\Phi([0,1])}$ and its corresponding complex Lie group $\mathbb{G}_{\mathcal{W}_\Phi([0,1])}(A_{\mathrm{dr}}) = \mathrm{Hom}(H^{\mathrm{cut}}_{\mathcal{W}_\Phi([0,1])}, A_{\mathrm{dr}})$. Thanks to Atkinson Theorem [14], each character $\phi \in \mathbb{G}_{\mathcal{W}_\Phi([0,1])}(A_{\mathrm{dr}})$ has the unique Birkhoff factorization (ϕ_-, ϕ_+) in terms of the factorization $A_{\mathrm{dr}} = A_- \oplus A_+$ such that ϕ_-, ϕ_+ are characters which contribute to the equation $\phi = \phi_-^{-1} * \phi_+$ while $\phi^{-1} = \phi \circ S$ with respect to the antipode of the Hopf algebra $H^{\mathrm{cut}}_{\mathcal{W}_\Phi([0,1])}$. The convolution product $*$ is defined in terms of the coproduct (21). The negative component ϕ_- can be computed in terms of the equation $\phi_- = S_{R_{\mathrm{ms}}} * \phi$ such that $S_{R_{\mathrm{ms}}}$ deforms the antipode S (given by the formula (23)) with respect to the Minimal Subtraction map. In other words,

$$S_{R_{\mathrm{ms}}}(W^\rho_\Gamma) = -R_{\mathrm{ms}}(\phi(W^\rho_\Gamma)) - R_{\mathrm{ms}}\Big\{\sum_\gamma S_{R_{\mathrm{ms}}}(W^\rho_\gamma)\phi(W^\rho_{\Gamma/\gamma})\Big\} \tag{26}$$

such that the sum is controlled by disjoint unions of 1PI Feynman subdiagrams of Γ.

For any $W^\rho_\Gamma \in H^{\mathrm{cut}}_{\mathcal{W}_\Phi([0,1])}$, there exists a sequence $\{\Gamma_n\}_{n\geq 1}$ of finite Feynman diagrams which is cut-distance convergent to $W^\rho_\Gamma \in [W_\Gamma]$. For a partial recursive function F defined on a subspace of the space of labeled Feynman graphons such that $\{W^{\rho_n}_{\Gamma_n}\}_{n\geq 1} \subseteq \mathrm{Dom}(F)$, define a new function

$$\tilde{F}(V) = F(V) :\ V \in \mathrm{Dom}(F),\ \tilde{F}(V) = 0 :\ V \notin \mathrm{Dom}(F)\ . \tag{27}$$

Thanks to (25), define the analytic function

$$\Theta(V, F; z) := \sum_{n\geq 0} \frac{z^n}{(1 + n\|\tilde{F}(V)\|_{\mathrm{cut}})^2}\ , \tag{28}$$

which determines a character $\psi_{\Theta(V,F;z)} \in \mathbb{G}_{\mathcal{W}_\Phi([0,1])}(A_{\mathrm{dr}})$. Its negative component $\psi_{\Theta(V,F;z),-}$ with respect to the Birkhoff factorization can be computed in terms of the formula (26).

Thanks to the Manin renormalization Hopf algebra of the Halting problem [19, 21], the values $\psi_{\Theta(V,F;z),-}(W^{\rho_n}_{\Gamma_n})$ for each $n \geq 1$ approximate the non-computability amount of F at W^ρ_Γ in terms of the character $\psi_{\Theta(V,F;z)}$.

Now if we apply the above setting for $F = \mathrm{id}$, then arrays of the sequence $\{\psi_{\Theta(W^\rho_{\Gamma_n}, \mathrm{id}; z),-}\}_{n\geq 1}$ of analytic functions, which are continuous at $|z| = 1$, encode the non-computability of W^ρ_Γ at $\mathrm{Dom}(W^\rho_\Gamma) \subseteq [0,1] \times [0,1]$. □

Remark 4. *If we change the partial recursive function F with another functions in Proof of Corollary 6, then we obtain another approximations for the non-computability level of labeled Feynman graphons.*

Corollary 7. *There exists a shuffle type product on the space of primitive labeled Feynman graphons of the Hopf algebra $H^{\text{cut}}_{\mathcal{W}_\Phi([0,1])}$. It provides a factorization program (in terms of the Euler product) for labeled Feynman graphons which contribute to solutions of combinatorial Dyson–Schwinger equations.*

Proof. Thanks to the Hopf algebra $H^{\text{cut}}_{\mathcal{W}_\Phi([0,1])}$ given by Corollary 5, consider the free algebra generated by finite sequences $J^{\rho_1,...,\rho_n}_{1,...,n} := (W^{\rho_1}_1, ..., W^{\rho_n}_n)$ of primitive labeled Feynman graphons. For any primitive labeled Feynman graphon W^ρ_p, define a linear operator $B^+_{W^\rho_p}$ on the space of these finite sequences such that

$$B^+_{W^\rho_p}(J^{\rho_1,...,\rho_n}_{1,...,n}) := (W^\rho_p, W^{\rho_1}_1, ..., W^{\rho_n}_n). \tag{29}$$

Thanks to the coproduct structure on labeled Feynman graphons, we define recursively a new coproduct on the space of these sequences given by

$$\Delta(B^+_{W^\rho_p}(J^{\rho_1,...,\rho_n}_{1,...,n})) = B^+_{W^\rho_p}(J^{\rho_1,...,\rho_n}_{1,...,n}) \otimes 1 + (\text{id} \otimes B^+_{W^\rho_p})\Delta(J^{\rho_1,...,\rho_n}_{1,...,n}) \tag{30}$$

such that 1 is the empty sequence as the unit of the algebra. The graduation parameter with respect to the length of sequences allows us to formulate an antipode. We use the notation $H_{\text{prim},\mathcal{W}_\Phi([0,1])}$ for the resulting Hopf algebra on sequences $J^{\rho_1,...,\rho_n}_{1,...,n}$s.

Now we define a new shuffle product on the space of sequences of primitive labeled Feynman graphons. For given primitive labeled Feynman graphons $W^{\sigma_1}_{p_1}, W^{\sigma_2}_{p_2}$ define

$$B^+_{W^{\sigma_1}_{p_1}}(J^{\rho_1,...,\rho_n}_{1,...,n}) \ominus B^+_{W^{\sigma_2}_{p_2}}(J^{\rho_1,...,\rho_m}_{1,...,m}) =$$

$$B^+_{W^{\sigma_1}_{p_1}}(J^{\rho_1,...,\rho_n}_{1,...,n} \ominus B^+_{W^{\sigma_2}_{p_2}}(J^{\rho_1,...,\rho_m}_{1,...,m})) + B^+_{W^{\sigma_2}_{p_2}}(B^+_{W^{\sigma_1}_{p_1}}(J^{\rho_1,...,\rho_n}_{1,...,n}) \ominus J^{\rho_1,...,\rho_m}_{1,...,m}). \tag{31}$$

The product $\ominus$ is commutative and associative on the space of sequences $J^{\rho_1,...,\rho_n}_{1,...,n}$s such that the empty sequence is its unit.

For any non-zero real number α, consider combinatorial Dyson–Schwinger equations with the general form

$$X(\alpha) = 1 + \alpha \sum_{W^\rho_p} B^+_{W^\rho_p}(X(\alpha)) \tag{32}$$

in $H_{\text{prim},\mathcal{W}_\Phi([0,1])}[[\alpha]]$ together with characters ϕ_s

$$\phi_s(J^{\rho_1,...,\rho_n}_{1,...,n}) = \frac{1}{n!}\omega(J^{\rho_1,...,\rho_n}_{1,...,n})^{-s} \tag{33}$$

on $H_{\text{prim},\mathcal{W}_\Phi([0,1])}$ (for real numbers $s > 1$) such that $\omega(J^{\rho_1,...,\rho_n}_{1,...,n}) := \|W^{\rho_1}_1\|_{\text{cut}} \cdots \|W^{\rho_n}_n\|_{\text{cut}}$. For any sequence $J^{\rho_1,...,\rho_n}_{1,...,n}$, we apply rescaling methods to replace each $W^{\rho_i}_i : [0,1]^2 \to [0,1]$ with a new labeled Feynman graphon $\tilde{W}^{\rho_i}_i : [0,1]^2 \to [0,a_i]$ such that $\|\tilde{W}^{\rho_i}_i\|_{\text{cut}} = p_i$ is the ith prime number. Then we can check that

$$\lim_{\alpha\to 1}\phi_s(X(\alpha)) = \lim_{\alpha\to 1}\phi_s(\prod_p^{\ominus} \frac{1}{1-\alpha(\tilde{W}^\rho_p)}) = \prod_{p_{j_i}} \frac{1}{1-p_i^{-s}} = \zeta(s) \tag{34}$$

such that

$$\frac{1}{1-\alpha(\tilde{W}^\rho_p)} = 1 + \alpha(\tilde{W}^\rho_p) + \alpha^2(\tilde{W}^\rho_p) \ominus (\tilde{W}^\rho_p) + \alpha^3(\tilde{W}^\rho_p) \ominus (\tilde{W}^\rho_p) \ominus (\tilde{W}^\rho_p) + ... \tag{35}$$

□

3.3 Halting problem of labeled Feynman graphon processes

In Theory of Computation, it is possible to describe operations in terms of sequences of decimal digits which enables us to encode programs in terms of joining together decimal digits. Flowcharts are basic tools to describe programs in terms of some algorithmic processes. Turing machines have been introduced as the fundamental concept for the study of intermediate algorithms and programs. We describe programs in terms of a certain class of functions namely, partial recursive functions. These functions are useful to encode physical processes that take steps in a limited period of time or an infinite time. The computability is about whether a partial recursive function can be computed via a program or not. [17, 30]

Thanks to Corollary 1, the (non-perturbative) solution X of a given combinatorial Dyson–Schwinger equation DSE can be interpreted as the infinite direct sum of Feynman graphons W_{X_i} of weight $c(g)^i$. The Feynman graphon representations of the partial sums Y_m are applied to study the evolution of the equation DSE in terms of the sequence $\{W_{Y_m}\}_{m\geq 1}$. This Feynman graphon process determines the real time dynamics of the equation DSE. Here we study the amount of computability of Feynman graphon processes in the context of the renormalization Hopf algebra of Halting problems of flowcharts.

Lemma 1. *The unique solution X of a given combinatorial Dyson–Schwinger equation DSE is encoded by a random graph process.*

Proof. We show that any labeled Feynman graphon $W_X^\rho \in [W_X]_\approx$ determines a random graph process which converges to an infinite random graph. For any natural number $1 \leq n \leq \infty$, set $[n] := \{i \in \mathbb{N} : i \leq n\}$ and then define a random graph $R(n, W_X^\rho)$ with the vertex set $[n]$ in terms of taking n random real numbers $x_1, ..., x_n$ in $[0,1]$ such that with the probability $W_X^\rho(x_i, x_j)$, there exists an edge between x_i and x_j in $R(n, W_X^\rho)$ for $i, j \in [n]$. We generalize this method to build an infinite random graph $R(\infty, W_X^\rho)$ with the vertex set $[\infty]$ in terms of taking infinite countable random real numbers $x_1, x_2, ...$ in $[0,1]$ such that with the probability $W_X^\rho(x_i, x_j)$, there exists an edge between x_i and x_j in $R(\infty, W_X^\rho)$ for $i, j \in [\infty]$. This class of infinite random graphs are invariant under permutations of vertices.

Consider the sequence $\{Y_m^{\rho_m}\}_{m\geq 1}$ of finite partial sums with the corresponding sequence $\{W_{Y_m}^{\rho_m}\}_{m\geq 1}$ of Feynman graphons which cut-distance converges to W_X^ρ when m tends to infinity. For each m and a fixed k, we build a random graph $R(k, W_{Y_m}^{\rho_m})$ on the vertex set $[k]$ such that with the probability $W_{Y_m}^{\rho_m}(x_i, x_j)$, there exists an edge between x_i, x_j for $i, j \in [k]$. Thanks to [25, 27], for each k, the sequence $\{R(k, W_{Y_m}^{\rho_m})\}_{m\geq 1}$ is convergent to the random graph $R(k, W_X^\rho)$. Therefore the distributions of the random graphs $R(k, W_{Y_m}^{\rho_m})$ converge when m tends to infinity. □

Remark 5.

- *If labeled Feynman graphons $W_{\mathrm{DSE}_1}^{\rho_1}$ and $W_{\mathrm{DSE}_2}^{\rho_2}$ are weakly isomorphic, then their corresponding infinite random graphs $R(\infty, W_{\mathrm{DSE}_1}^{\rho_1})$ and $R(\infty, W_{\mathrm{DSE}_2}^{\rho_2})$ have the same distribution.*

- *For a given labeled Feynman graphon W_X^ρ and any Feynman diagram Γ, the homomorphism density $t(\Gamma, W_X^\rho)$ determines the probability that Γ can be interpreted* *as a subgraph of the infinite random graph* $R(\infty, W_X^\rho)$.

Definition 7. *Let* $R(\infty, W_X^\rho)$ *be an infinite random graph generated by the labeled Feynman graphon* W_X^ρ.

- *A labeled Feynman graphon process initiated from* $R(\infty, W_X^\rho)$ *is a sequence* $\{W_{\Gamma_n}^{\rho_n}\}_{n\geq 1}$ *of labeled Feynman graphons corresponding to (finite formal expansions of) Feynman diagrams* Γ_n *which is cut-distance convergent to* W_X^ρ. *It is called a labeled* L^p*-Feynman graphon process, if* $\{W_{\Gamma_n}^{\rho_n}\}_{n\geq 1}$ *is* L^p*-convergent to* W_X^ρ.
- *A computable labeled Feynman graphon process initiated from* $R(\infty, W_X^\rho)$ *is a sequence* $\{W_{\Gamma_n}^{\rho_n}\}_{n\geq 1}$ *of computable labeled Feynman graphons corresponding to (finite formal expansions of) Feynman diagrams* Γ_n *which is cut-distance convergent to* W_X^ρ.

We want to organize labeled Feynman graphon processes into a new Hopf algebra structure which is useful to encode the Halting problem.

Definition 8. *Let* $\{W_{\Gamma_n}^{\rho_n}\}_{n\geq 1}$ *be a labeled Feynman graphon process initiated from an infinite random graph* $R(\infty, W_X^\rho)$. *Let there exist a family* $M := \{\rho_{st} : W_{\Gamma_s}^{\rho_s} \to W_{\Gamma_t}^{\rho_t}\}_{s,t}$ *of Lebesgue measurable maps. A proper admissible partition* c *for this sequence is a cut which divides the sequence into a disjoint union of two non-empty sub-sequences* $\{U_i^c\}_{i\geq 1}$ *and* $\{V_j^c\}_{j\geq 1}$ *such that*

- *All labeled Feynman graphons in* $\{W_{\Gamma_n}^{\rho_n}\}_{n\geq 1}$ *belong either to* $\{U_i^c\}_{i\geq 1}$ *or* $\{V_j^c\}_{j\geq 1}$.
- *Any map* ρ_{ij} *in* M *describes the information flow from* $W_{\Gamma_i}^{\rho_i}$ *to* $W_{\Gamma_j}^{\rho_j}$.
- *A cut* c *is called trivial, if one of the sub-sequences* $\{U_i^c\}_{i\geq 1}$ *or* $\{V_j^c\}_{j\geq 1}$ *are empty.*

Definition 9. *Consider the collection of all labeled Feynman graphon processes such as* $\{W_{\Gamma_n}^{\rho_n}\}_{n\geq 1}$ *together with a family* $M := \{\rho_{st} : W_{\Gamma_s}^{\rho_s} \to W_{\Gamma_t}^{\rho_t}\}_{s,t}$ *of Lebesgue measurable maps. A sub-collection* $\mathrm{Fl}_{\mathrm{graphon}}$ *of labeled Feynman graphon processes is called admissible, if*

- *Each sub-sequence* $\{U_i\}_{i\geq 1}$ *together with the family* $M|_{\{U_i\}_{i\geq 1}}$ *of any given Feynman graphon process* $\{W_{\Gamma_n}^{\rho_n}\}_{n\geq 1}$ *in* $\mathrm{Fl}_{\mathrm{graphon}}$ *belongs to* $\mathrm{Fl}_{\mathrm{graphon}}$.
- *Each disjoint union of sequences of labeled Feynman graphons in* $\mathrm{Fl}_{\mathrm{graphon}}$ *belongs to* $\mathrm{Fl}_{\mathrm{graphon}}$.
- *The* 0*-graphon belongs to* $\mathrm{Fl}_{\mathrm{graphon}}$.
- *For any labeled Feynman graphon process* $\{W_{\Gamma_n}^{\rho_n}\}_{n\geq 1}$ *in* $\mathrm{Fl}_{\mathrm{graphon}}$ *and any proper cut* c*, the corresponding sub-sequences* $\{U_i^c\}_{i\geq 1}$ *and* $\{V_j^c\}_{j\geq 1}$ *belong to* $\mathrm{Fl}_{\mathrm{graphon}}$. *Objects of the sub-collection* $\mathrm{Fl}_{\mathrm{graphon}}$ *are called flowcharts decorated by labeled Feynman graphon processes.*

Now we modify the Manin renormalization Hopf algebra of the Halting problem ([19, 20, 21]) to flowcharts decorated by labeled Feynman graphon processes.

Theorem 4. *Admissible cuts given by Definition 8 determine a graded Hopf algebra structure on the space of flowcharts decorated by labeled Feynman graphon processes.*

Proof. For a given admissible sub-collection $\mathrm{Fl}_{\mathrm{graphon}}$, set $H_{\mathrm{Fl}_{\mathrm{graphon}}}$ as the $\mathbb{K}$-linear span of flowcharts in $\mathrm{Fl}_{\mathrm{graphon}}$. The disjoint union gives a multiplication on this space and the admissible cuts provide a coproduct structure on this space given by

$$\Delta(\{W_{G_n}^{\rho_n}\}_{n\geq 1}) = \sum_{c} \{U_i^c\}_{i\geq 1} \otimes \{V_j^c\}_{j\geq 1}. \tag{36}$$

The cut-distance topology guarantees that this infinite formal sum is well-defined.

We consider $H_{\mathrm{Fl}_{\mathrm{graphon}}} = \oplus_{j=1}^{\infty} H^{j}_{\mathrm{Fl}_{\mathrm{graphon}}}$ as a $\mathbb{N}^{\infty}$-multigraded bialgebra such that $j = (j_1, j_2, ...)$. For each j, $H^{j}_{\mathrm{Fl}_{\mathrm{graphon}}}$ is the $\mathbb{K}$-submodule generated by those labeled Feynman graphon processes $\{W_{G_n}^{\rho_n}\}_{n\geq 1}$ such that for each n, $W_{G_n}^{\rho_n} \in H^{(j_n)}_{\mathbb{W}_{[0,1]}}$ (according to the Proof of Theorem 3).

Thanks to this multigraded graduation parameter and the coproduct (36), we formulate the antipode recursively to obtain our promising Hopf algebra. □

Corollary 8. *The Halting problem in the construction of infinite random graphs generated by labeled Feynman graphon processes which contribute to solutions of combinatorial Dyson–Schwinger equations can be encoded in terms of the Hopf algebra* $H_{\mathrm{Fl}_{\mathrm{graphon}}}$.

Proof. Thanks to Lemma 1, Remark 5 and Definition 7, we only need to update the Proof of Corollary 6 for the Hopf algebra $H_{\mathrm{Fl}_{\mathrm{graphon}}}$ and its corresponding complex Lie group $\mathbb{G}_{\mathrm{Fl}_{\mathrm{graphon}}}(A_{\mathrm{dr}})$. □

4 Conclusion

We studied the computability of Feynman graphons and Feynman graphon processes in the context of the computable analysis, the generalized Turing machines and the renormalization Hopf algebra of the Halting problem. In this direction, we equipped the space of flowcharts decorated by Feynman graphon processes with a new graded Hopf algebra structure which led us to interpret the Halting problem of algorithms associated to Feynman graphon processes in terms of the BPHZ renormalization program.

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