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ON THE GEOMETRY OF THE MUS-CHEEGER-GROMOLL METRIC

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Abstract

Let (M,g) be an n-dimensional smooth Riemannian manifold. In the present paper, we introduce a new class of natural metrics denoted by G and called the Mus-Cheeger-Gromoll metric on the tangent bundle TM. We calculate its Levi-Civita connection and Riemannian curvature tensor. Also we study the geometry of (TM,G) .

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1 Introduction

We recall some basic facts about the geometry of the tangent bundle. In the present paper, we denote by $\Gamma(TM)$ the space of all vector fields of a Riemannian manifold (M,g). Let (M,g) be an n-dimensional Riemannian manifold and (TM,π,M) be its tangent bundle. A local chart $(U,x^i)_{i=1\dots n}$ on M induces a local chart $(\pi^{-1}(U),x^i,y^i)_{i=1\dots n}$ on TM. Denote by Γ^k_{ij} the Christoffel symbols of g and by ∇ the Levi-Civita connection of g.

We have two complementary distributions on TM, the vertical distribution $\mathcal{V}$ and the horizontal distribution $\mathcal{H}$, defined by

$$\begin{aligned}\mathcal{V}_{(x,u)} &= \ker(d\pi_{(x,u)})\\ &= \left\{a^i\frac{\partial}{\partial y^i}_{|(x,u)};a^i\in\mathbb{R}\right\}\\ \mathcal{H}_{(x,u)} &= \left\{\frac{\partial}{\partial x^i}|_{(x,u)}-a^iu^j\Gamma^k_{ij}\frac{\partial}{\partial y^k}|_{(x,u)};a^i\in\mathbb{R}\right\}\end{aligned}$$

where $(x,u)\in TM$, such that $T_{(x,u)}TM=\mathcal{H}_{(x,u)}\oplus\mathcal{V}_{(x,u)}$.

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Let $X = X^i \dfrac{\partial}{\partial x^i}$ be a local vector field on M. The vertical and the horizontal lifts of X are defined by

$$X^V = X^i \frac{\partial}{\partial y^i} \tag{1}$$

$$X^H = X^i \frac{\delta}{\delta x^i} = X^i \Big\{ \frac{\partial}{\partial x^i} - y^j \Gamma^k_{ij} \frac{\partial}{\partial y^k} \Big\}. \tag{2}$$

For consequences, we have $\Big(\dfrac{\partial}{\partial x^i}\Big)^H = \dfrac{\delta}{\delta x^i}$ and $\Big(\dfrac{\partial}{\partial x^i}\Big)^V = \dfrac{\partial}{\partial y^i}$, then $\Big(\dfrac{\delta}{\delta x^i}, \dfrac{\partial}{\partial y^i}\Big)_{i=1..n}$ is a local adapted frame in TTM. The tangent bundle TM of a Riemannian manifold (M, g) can be endowed in a natural way with a Riemannian metric g^s, the Sasaki metric, depending only on the Riemannian structure g of the base manifold M. It is uniquely determined by

$$\begin{array}{rcl} g^s(X^H, Y^H) &=& g(X,Y) \circ \pi \\ g^s(X^H, Y^V) &=& 0 \\ g^s(X^V, Y^V) &=& g(X,Y) \circ \pi \end{array} \tag{3}$$

for all vector fields X and Y on M. More intuitively, the metric g^s is constructed in such a way that the vertical and horizontal sub bundles are orthogonal and the bundle map $\pi : (TM, g^s) \longrightarrow (M, g)$ is a Riemannian submersion.
The geometry of the tangent bundle TM equipped with Sasaki metric has been studied by many authors K. Yano and S. Ishihara [17], A. Salimov, A. Gezer and N. Cengiz (see [3], [9], [10], [15]) etc. The rigidity of Sasaki metric has incited some geometers to construct and study other metrics on TM. J. Cheeger and D. Gromoll have introduced the notion of Cheeger-Gromoll [4]. It is uniquely determined by

$$\begin{array}{rcl} g_{CG}(X^H, Y^H) &=& g(X,Y) \circ \pi \\ g_{CG}(X^H, Y^V) &=& 0 \\ g_{CG}(X^V, Y^V) &=& \dfrac{1}{\alpha}\{g(X,Y) + g(X,u)g(Y,u)\} \circ \pi \end{array} \tag{4}$$

Where $X, Y \in \Gamma(TM)$, $(x,u) \in TM$, $\alpha = 1 + g_x(u,u)$.

M. Benyounes, E. Loubeau, and C. M. Wood in [2] introduced the geometry of the tangent bundle equipped with a two-parameter family of Riemannian metrics is called generalized Cheeger-Gromoll metric given by

$$\begin{array}{rcl} G(X^H, Y^H)_{(x,u)} &=& g_x(X,Y) \\ G(X^H, Y^V)_{(x,u)} &=& 0 \\ G(X^V, Y^V)_{(x,u)} &=& \omega^p(g_x(X,Y) + q g_x(X,u) g_x(Y,u)) \end{array} \tag{5}$$

for all vector fields $X, Y \in \Gamma(TM)$, $r^2 = \|u\| = \sqrt{g(u,u)}$, where $\omega = (1 + \|u\|^2)^{-1}$, $p, q \in \mathbb{R}$ and q positive ensure non-degeneracy.

In [1] we have defined a metric on TM called the vertical rescaled generalized Cheeger-Gromoll metric, given by

$$\begin{array}{rcl} G^f(X^H, Y^H)_{(x,u)} & = & g_x(X,Y) \\ G^f(X^H, Y^V)_{(x,u)} & = & 0 \\ G^f(X^V, Y^V)_{(x,u)} & = & f(x)\omega^p(g_x(X,Y) + qg_x(X,u)g_x(Y,u)) \end{array}$$

for all vector fields $X, Y \in \Gamma(TM)$, and $r^2 = \|u\| = \sqrt{g(u,u)}$, where $\omega = (1 + \|u\|^2)^{-1}$, $p, q \in \mathbb{R}$ and q positive ensure non-degeneracy.

Motivated by the above studies, we define a new class of naturally metric on TM called Mus-Cheeger-Gromoll metric, given by

$$\begin{array}{rcl} G(X^H, Y^H)_{(x,u)} & = & g_x(X,Y) \\ G(X^V, Y^H)_{(x,u)} & = & 0 \\ G(X^V, Y^V)_{(x,u)} & = & f(x)\omega(r^2)\big(g_x(X,Y) + \alpha(r^2)g(X,u)g(Y,u)\big) \end{array}$$

for all vector fields $X, Y \in \Gamma(TM)$,$(x,u) \in TM$ where $f : M \to \mathbb{R}_+$ and $\omega, \alpha : \mathbb{R} \to \mathbb{R}_+$, $r^2 = g(u,u)$.

In this paper, we introduce the Mus-Cheeger-Gromoll metric on the tangent bundle TM as a new natural metric non-rigid on TM. First, we investigate the geometry of the Mus-Cheeger-Gromoll metric and we characterize the sectional curvature and the scalar curvature.

2 Mus-Cheeger-Gromoll metric

Definition 1. *Let (M,g) be a Riemannian manifold, and $f : M \to]0, +\infty[$. We define the Mus-Cheeger-Gromoll metric G on the tangent bundle TM by*

$$\begin{array}{rcl} G(X^H, Y^H)_{(x,u)} & = & g_x(X,Y), \\ G(X^V, Y^H)_{(x,u)} & = & 0, \\ G(X^V, Y^V)_{(x,u)} & = & f(x)\omega(r^2)\big(g_x(X,Y) + \alpha(r^2)g(X,u)g(Y,u)\big). \end{array}$$

for all vector fields $X, Y \in \Gamma(TM)$,$(x,u) \in TM$ where $f : M \to \mathbb{R}_+$ and $\omega, \alpha : \mathbb{R} \to \mathbb{R}_+$, $r^2 = g(u,u)$.

Remark 1.

1. *If $f = 1$, $\omega = 1$ and $\alpha = 0$, then G is the Sasaki metric.*
2. *If $f = 1$, $\omega = \frac{1}{1+r^2}$ and $\alpha = 1$, then G is the Cheeger-Gromoll metric.*
3. *If $f = 1$, $\omega = \left(\frac{1}{1+r^2}\right)^p$ and α is constant, then G is the generalized Cheeger-Gromoll metric.*

4. *If* $\omega = \left(\frac{1}{1+r^2}\right)^p$ *and* α *is constant, then* G *is the vertical rescaled generalized Cheeger-Gromoll metric.*

Lemma 1. *Let* (M,g) *be a Riemannian manifold, and* $h : \mathbb{R} \to \mathbb{R}$ *a smooth function.*
For all $X, Y \in \Gamma(TM), p = (x,u) \in TM$ *and* $u = u^i = \frac{\partial}{\partial x^i}$, *we have*

$$\begin{aligned} X^H_{(p,u)}(h(r^2)) &= 0, \\ X^V_{(p,u)}(h(r^2)) &= 2h'(r^2)g(X,u). \end{aligned}$$

Proof. Locally, if $X = X^i \frac{\partial}{\partial x^i}$ then

$$X^V = X^i \frac{\partial}{\partial u^i}, X^H = X^i \frac{\partial}{\partial x^i} + X^i u^j \Gamma^k_{ij} \frac{\partial}{\partial u^k}$$

$$\begin{aligned} X^H(h(r^2)) &= \frac{\partial}{\partial x^i}(h(r^2)) - \Gamma^j_{ik} u^k \frac{\partial}{\partial u^j}(h(r^2)) \\ &= h'(r^2)\left\{\frac{\partial}{\partial x^i}(g(u,u)) - \Gamma^j_{ik} u^k \frac{\partial}{\partial u^j}(g(u,u))\right\} \\ &= h'(r^2)\left\{u^l u^s \frac{\partial}{\partial x^i}(g_{ls}) - \Gamma^j_{ik} u^k \frac{\partial}{\partial u^i}(u^l u^s g_{ls})\right\} \\ &= h'(r^2) u^l u^s \left\{\frac{\partial}{\partial x^i}(g_{ls}) - \Gamma^k_{il} g_{sk} - \Gamma^k_{is} g_{lk}\right. \\ &= h'(r^2) u^l u^s \left\{\frac{\partial}{\partial x^i}(g_{ls}) - \frac{1}{2}[\frac{\partial g_{ls}}{\partial x^i} + \frac{\partial g_{is}}{\partial x^l} - \frac{\partial g_{il}}{\partial x^s} + \frac{\partial g_{ls}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^s}\right. \\ &\qquad \left. - \frac{\partial g_{is}}{\partial x^l}]\right\} \\ &= 0 \end{aligned}$$

$$\begin{aligned} X^V(h(r^2)) &= X^i h'(r^2) \frac{\partial}{\partial u^i}(u^l u^s g_{ls}) \\ &= 2X^i h'(r^2) u^s g_{is} \\ &= 2h'(r^2) g(X,u) \end{aligned}$$

□

Lemma 2. *Let* (M,g) *be a Riemannian manifold, for* $X, Y \in \Gamma(TM)$

$$\begin{aligned} g(Y,.) : TM &\to \mathbb{R} \\ (x,u) &\to g_x(Y_x, u) \end{aligned}$$

then

$$X^H(g(Y,.)) = g(\nabla_X Y, .)$$
$$X^V(g(X,u)) = g(X,Y)$$

Lemma 3. *Let (M,g) be a Riemannian manifold and (TM,G) its tangent bundle equipped with the Mus-Cheeger-Gromoll metric, then for all vector fields $X,Y,Z\in\Gamma(TM)$ we have:*

1. $X^H G(Y^V,Z^V)=\frac{X(f)}{f}G(Y^V,Z^V)+G((\nabla_X Y)^V,Z^V)+g^f((Y^V,\nabla_X Z)^V)$

2. $$X^V G(Y^V,Z^V)=2\frac{\omega'}{\omega}g(X,u)G(Y^V,Z^V)$$
$$+f\omega\big[\,2\alpha'(r)g(X,u)g(Y,u)g(Z,u)$$
$$+\alpha(r)g(Y,X)g(Z,u)+\alpha(r)g(Z,X)g(Y,u)\,\big].$$

Proof. 1.

$$\begin{aligned}
X^H G(Y^V,Z^V) &= X^H\big(f(x)\omega(r^2)[g(Y,Z)+\alpha(r^2)g(X,u)g(Y,u)]\big)\\
&= X(f)\omega(r^2)[g(Y,Z)+\alpha(r^2)g(X,u)g(Y,u)]\\
&\quad +f(x)\omega\big[g(\nabla_X y,Z)+g(\nabla_X Z,Y)+\\
&\quad +\alpha g(\nabla_X y,u)g(Z,u)+\alpha g(\nabla_X Z,u)g(Y,u)\big]\\
&= \frac{X(f)}{f}G(Y^V,Z^V)+G((\nabla_X Y)^V,Z^V)\\
&\quad +G((\nabla_X Z)^V,Y^V)
\end{aligned}$$

2.

$$\begin{aligned}
X^V G(Y^V,Z^V) &= X^V\big(f(x)\omega(r^2)[g(Y,Z)+\alpha(r^2)g(X,u)g(Y,u)]\big)\\
&= 2\omega' g(X,u)f(x)\big(g(Y,Z)+\alpha g(Y,u)g(Z,u)\\
&\quad +f(x)\omega\big[2\alpha' g(X,u)g(Y,u)g(Z,u)+\alpha g(Y,X)g(Z,u)\\
&\qquad +\alpha g(Z,X)g(Y,u)]\\
&= 2\frac{\omega'}{\omega}g(X,u)G(Y^V,Z^V)+\\
&\quad +f(x)\omega\big[2\alpha' g(X,u)g(Y,u)g(Z,u)\\
&\qquad +\alpha g(Y,X)g(Z,u)+\alpha g(Z,X)g(Y,u)\big]
\end{aligned}$$

□

2.1 Levi-Civita connection of G

Lemma 4. *Let (M,g) be a Riemannian manifold and (TM,G) its tangent bundle equipped with the Mus-Cheeger-Gromoll metric. If ∇ (resp $\overline{\nabla}$) denotes the Levi-Civita connection of (M,g) (resp (TM,G)), then we have:*

1. $G(\overline{\nabla}_{X^H}Y^H,Z^H)=G((\nabla_X Y)^H,Z^H)$

2. $G(\overline{\nabla}_{X^H}Y^H, Z^V) = -\frac{1}{2}G((R(X,Y)u)^V, Z^V)$
3. $G(\overline{\nabla}_{X^H}Y^V, Z^H) = \frac{f\omega}{2}G((R(u,Y)X)^H, Z^H)$
4. $G(\overline{\nabla}_{X^H}Y^V, Z^V) = G\left(\frac{X(f)}{2f}Y^V + (\nabla_X Y)^V, Z^V\right)$
5. $G(\overline{\nabla}_{X^V}Y^H, Z^H) = \frac{f\omega}{2}G((R(u,X)Y)^H, Z^H)$
6. $G(\overline{\nabla}_{X^V}Y^H, Z^V) = \frac{Y(f)}{2f}G(X^V, Z^V)$
7. $G(\overline{\nabla}_{X^V}Y^V, Z^H) = G\big(-\frac{1}{2f}G(X^V, Y^V)(grad_M f)^H, Z^H\big)$
8. $$\begin{aligned} G(\overline{\nabla}_{X^V}Y^V, Z^V) = & \tfrac{\omega'}{\omega}G\left(g(X,u)Y^V + g(Y,u)X^V - \right. \\ & -\tfrac{1}{f\omega(1+\alpha r^2)}G(X^V,Y^V)U, Z^V + \\ & +G(\tfrac{\alpha'}{1+\alpha r^2}g(X,u)g(Y,u)U^V, Z^V) \\ & +G(\tfrac{\alpha}{1+\alpha r^2}g(X,Y)U^V, Z^V), \end{aligned}$$

for all vector fields $X, Y, Z \in \Gamma(TM)$.

Proof. We shall repeatedly make use of the Kozul formula for the Levi-Civita connection $\overline{\nabla}$ stating that

$$\begin{aligned} 2G(\overline{\nabla}_{X^i}Y^j, Z^k) &= X^i(G(Y^j, Z^k)) + Y^j(G(Z^k, X^i)) - Z^k(G(X^i, Y^j)) \\ &- (G(X^i, [Y^j, Z^k])) + (G(Y^j, [Z^k, X^i])) + (G(Z^k, [X^i, Y^j])) \end{aligned}$$

for all vector fields $X, Y, Z \in \Gamma(TM)$ and $i, j, k \in \{H, V\}$.
The result is a direct consequence of the following calculations using Definition 1 and Lemma 3

1.
$$\begin{aligned} 2G(\overline{\nabla}_{X^H}Y^H, Z^H) &= X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ &\quad -g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]) \\ &= 2g\left((\nabla_X Y), Z\right) \\ &= 2G\left((\nabla_X Y)^H, Z^H\right) \end{aligned}$$

2.
$$\begin{aligned} 2G(\overline{\nabla}_{X^H}Y^H, Z^V) &= G(Z^V, [X^H, Y^H]) \\ &= -G(Z^V, (R(X,Y)u)^V) \end{aligned}$$

3.
$$\begin{aligned} 2G(\overline{\nabla}_{X^H}Y^V, Z^H) &= G(Y^V, [Z^H, X^H]) \\ &= -G(Y^V, (R(Z,X)u)^V) \\ &= f\omega[g(R(X,Z)u, Y) + \alpha g(R(X,Z)u, u)g(Y,u)] \\ &= f\omega G((R(u,Y)X)^H, Z^H) \end{aligned}$$

4.

$$\begin{aligned} 2G(\overline{\nabla}_{X^H}Y^V, Z^V) &= X^H(G(Y^V, Z^V)) - G(Y^V, (\nabla_X Z)^V) \\ &\quad + G(Z^V, (\nabla_X Y)^V) \\ &= G\left(\frac{X(f)}{f}Y^V + 2(\nabla_X Y)^V, Z^V\right) \end{aligned}$$

5.

$$\begin{aligned} 2G(\overline{\nabla}_{X^V}Y^H, Z^H) &= -G(X^V, [Y^H, Z^H]) \\ &= G(X^V, (R(Y,Z)u)^V) \\ &= f\omega[g(R(Y,Z)u, X) + \alpha(r^2)g(R(Y,Z)u, u)g(X,u)] \\ &= f\omega G((R(u,X)Y)^H, Z^H) \end{aligned}$$

6.

$$\begin{aligned} 2G(\overline{\nabla}_{X^V}Y^H, Z^V) &= Y^H G(Z^V, X^V) - G(X^V, (\nabla_Y Z)^V) - \\ &\quad - G(Z^V, (\nabla_Y X)^V) \\ &= \frac{Y(f)}{f}G(X^V, Z^V) \end{aligned}$$

7.

$$\begin{aligned} 2G(\overline{\nabla}_{X^V}Y^V, Z^H) &= -Z^H(G(X^V, Y^V)) + G(X^V, (\nabla_Z Y)^V) + \\ &\quad + G(Y^V, (\nabla_Z X)^V) \\ &= -\frac{Z(f)}{f}G(X^V, Y^V) \\ &= G\big(-\frac{1}{f}G(X^V, Y^V)(grad_M f)^H, Z^H\big) \end{aligned}$$

8. we have

$$\begin{aligned} G(X^V, U^V) &= f(x)\omega(r^2)[g(X,u) + \alpha(r^2)g(X,u)r^2] \\ &= f\omega(1 + \alpha r^2)g(X,u) \end{aligned}$$

then

$$\begin{aligned} 2G(\overline{\nabla}_{X^V}Y^V, Z^V) &= 2\frac{\omega'}{\omega}\left[g(X,u)G(Y^V, Z^V) + g(Y,u)G(Z^V, X^V) - \right. \\ &\qquad \left. - g(Z,u)G(X^V, Y^V)\right] \\ &\quad + 2f\omega\big[\alpha' g(X,u)g(Y,u)g(Z,u) + \alpha g(Y,X)g(Z,u)\big] \end{aligned}$$

$$\begin{aligned}2G(\overline{\nabla}_{X^V}Y^V,Z^V) &= 2\frac{\omega'}{\omega}G\Big(g(X,u)Y^V+g(Y,u)X^V-\\ &\qquad -\frac{1}{f\omega(1+\alpha r^2)}G(X^V,Y^V)U,Z^V\Big)\\ &\quad +G(2\frac{\alpha'}{1+\alpha r^2}g(X,u)g(Y,u)U^V,Z^V)+\\ &\quad +G(\frac{2\alpha}{1+\alpha r^2}g(X,Y)U^V,Z^V)\end{aligned}$$

□

Lemma 5. *Let (M,g) be a Riemannian manifold and (TM,G) its tangent bundle equipped with the Mus-Cheeger-Gromoll metric, then for all vector fields $X,U\in\Gamma(TM)$ and $U_x=u=u^i\frac{\partial}{\partial x_i}$, we have:*

1. $\overline{\nabla}_{X^H}U^V=\frac{1}{2f}X(f)U^V$
2. $\overline{\nabla}_{U^V}X^H=\frac{1}{2f}X(f)U^V$
3. $\overline{\nabla}_{X^V}U^V=-\frac{\omega(1+\alpha r^2)}{2}g(X,u)(grad\,f)^H+(\frac{\omega' r^2}{\omega}+1)X^V+(\alpha' r^2+\frac{\alpha}{1+\alpha r^2})g(X,u)U^V$

Proof. 1.

$$\begin{aligned}\overline{\nabla}_{X^H}U^V &= \overline{\nabla}_{X^H}(y^k(\frac{\partial}{\partial x^k})^V)\\ &= X^H(y^k)(\frac{\partial}{\partial x^k})^V+y^k\overline{\nabla}_{X^H}(\frac{\partial}{\partial x^k})^V\\ &= -X^iy^i\Gamma^k_{ij}(\frac{\partial}{\partial x^k})^V+y^k\big[\frac{f\omega}{2}(R(u,\frac{\partial}{\partial x^k})X)^H+(\nabla_X(\frac{\partial}{\partial x^k}))^V\\ &\qquad +\frac{1}{2f}X(f)(\frac{\partial}{\partial x^k})^V\big]\\ &= -(\nabla_XU)^V+\frac{f\omega}{2}(R(u,u)X)^H+(\nabla_XU)^V+\frac{1}{2f}X(f)U^V\\ &= \frac{1}{2f}X(f)U^V\end{aligned}$$

2.

$$\begin{aligned}\overline{\nabla}_{U^V}X^H &= \overline{\nabla}_{y^k(\frac{\partial}{\partial x^k})^V}(X^i(\frac{\partial}{\partial x^i})^H)\\ &= y^k(\frac{\partial}{\partial x^k})^V(X^i(\frac{\partial}{\partial x^i})^H)+y^kX^i\overline{\nabla}_{(\frac{\partial}{\partial x^k})^V}(\frac{\partial}{\partial x^i})^H\\ &= y^kX^i\big[\frac{f\omega}{2}(R(u,\frac{\partial}{\partial x^k})\frac{\partial}{\partial x^i})^H+\frac{1}{2f}\frac{\partial}{\partial x^i}(f)(\frac{\partial}{\partial x^k})^V\big]\\ &= \frac{f\omega}{2}(R(u,u)X)^H+\frac{1}{2f}X(f)U^V\\ &= \frac{1}{2f}X(f)U^V\end{aligned}$$

The last formula is obtained by a similar calculation .

□

Using Lemma 4, we have

Theorem 1. *Let (M, g) be a Riemannian manifold and (TM, G) its tangent bundle equipped with the Mus-Cheeger-Gromoll metric. If $\overline{\nabla}$ is the Levi-Civita connection associated to G, then we have*

$$
\begin{aligned}
(\overline{\nabla}_{X^H} Y^H)_p &= (\nabla_X Y)^H_p - \frac{1}{2}(R(X,Y)u)^V_p, \\
(\overline{\nabla}_{X^H} Y^V)_p &= \frac{f\omega}{2}(R(u,Y)X)^H + \frac{X(f)}{2f} Y^V + (\nabla_X Y)^V, \\
(\overline{\nabla}_{X^V} Y^H)_p &= \frac{f\omega}{2}(R(u,X)Y)^H + \frac{Y(f)}{2f} X^V, \\
(\overline{\nabla}_{X^V} Y^V)_p &= -\frac{1}{2f} G(X^V, Y^V)(grad_M f)^H \\
&\quad + \frac{\omega'}{\omega}\left(g(X,u)Y^V + g(Y,u)X^V - \frac{1}{f\omega(1+\alpha r^2)} G(X^V, Y^V)U^V \right) \\
&\quad + \frac{1}{1+\alpha r^2}\left[\alpha' g(X,u) g(Y,u) U^V + \alpha g(X,Y) U^V \right].
\end{aligned}
$$

for all vector fields $X, Y \in \Gamma(TM)$, $p = (x,u) \in TM$.

Definition 2. *Let (M, g) be a Riemannian manifold and $K : TM \to TTM$ be a smooth bundle endomorphism of the tangent bundle TM. Then we define the vertical and horizontal lifts $K^V : TM \to TTM, K^H : TM \to TTM$ of K by*

$$K^V(\eta) = \sum_{i=1}^{m} \eta_i K(\partial i)^V$$

and

$$K^H(\eta) = \sum_{i=1}^{m} \eta_i K(\partial i)^H$$

where $\sum_{i=1}^{m} \eta_i \partial i \in \pi^{-1}(V)$ is a local representation of $\eta \in \mathcal{C}^\infty(TM)$

Proposition 1. *Let (M, g) be a Riemannian manifold and $\overline{\nabla}$ be a the Levi-Civita connection of the tangent bundle (TM, G). If K is a tensor field of type $(1,1)$ on M, then*

$$
\begin{aligned}
(\overline{\nabla}_{X^H} K^H)_p &= (\nabla_X K)^H_p - \frac{1}{2}(R(X, K(u))u)^V_p \\
(\overline{\nabla}_{X^H} K^V)_p &= \frac{f\omega}{2}(R(u, K(u))X)^H + \frac{X(f)}{2f} K(u)^V + (\nabla_X K)^V \\
(\overline{\nabla}_{X^V} K^H)_p &= (K(X))^H + \frac{f\omega}{2}(R(u,X)K(u))^H + \frac{1}{2f} g(K(u), gradf) X^V
\end{aligned}
$$

$$
\begin{aligned}
(\overline{\nabla}_{X^V}K^V)_p &= -\frac{1}{2f}G(X^V,(K(u))^V)(grad_M f)^H + (K(X))^V \\
&+\frac{\omega'}{\omega}\Big(g(X,u)(K(u))^V + g(K(u),u)X^V - \\
&-\frac{1}{f\omega(1+\alpha r^2)}G(X^V,(K(u))^V)U\Big) \\
&+\frac{1}{1+\alpha r^2}\left[\alpha' g(X,u)g(K(u),u)U^V + \alpha g(X,K(u))U^V\right]
\end{aligned}
$$

for any $X \in \Gamma(TM)$ *and* $p = (x,u) \in TM$.

3 Curvature tensor of Mus-Cheeger-Gromoll metric

Theorem 2. *Let* (M,g) *be a Riemannian manifold and* (TM,G) *its tangent bundle equipped with the Mus-Cheeger-Gromoll metric. If* R *(resp* $\overline{R}$*) denote the Riemannian curvature tensor of* M *(resp*TM*), then we have the following formulas*

$$
\begin{aligned}
\overline{R}(X^H,Y^H)Z^H &= (R(X,Y)Z)^H + \frac{f\omega}{2}(R(u,R(X,Y)u)Z)^H \\
&+\frac{f\omega}{4}(R(u,R(X,Z)u)Y)^H - \frac{f\omega}{4}(R(u,R(Y,Z)u)X)^H \\
&+\frac{1}{2}(\nabla_Z R)(X,Y)u)^V - \frac{X(f)}{4f}(R(u,R(Y,Z)u)^V \\
&+\frac{Y(f)}{4f}(R(u,R(X,Z)u)^V + \frac{Z(f)}{2f}(R(X,Y)u)^V
\end{aligned}
$$

$$
\begin{aligned}
\overline{R}(X^H,Y^V)Z^V &= -\frac{\omega}{2}[g(Y,Z)+\alpha g(Y,u)g(Z,u)](\nabla_X gradf)^H \\
&+\frac{\omega}{4}[g(Y,Z)+\alpha g(Y,u)g(Z,u)](R(X,gradf)u)^V \\
&+\frac{f\omega'}{2}[-g(Y,u)(R(u,Z)X)^H + g(Z,u)(R(u,Y)X)^H] \\
&-\frac{f\omega}{2}(R(Y,Z)X)^H \\
&-\frac{f^2\omega^2}{4}(R(u,Y)R(u,Z)X)^H - \frac{\omega}{4}g(R(u,Z)X,gradf)Y^V \\
&+X(f)\frac{\omega}{4f}[g(Y,Z)+\alpha g(Y,u)g(Z,u)](grad_M f)^H
\end{aligned}
$$

$$\begin{aligned}\overline{R}(X^V,Y^V)Z^H &= \frac{f\omega}{2}(R(X,Y)Z)^H \\ & f\omega'[g(X,u)(R(u,Y)Z)^H - g(Y,u)(R(u,X)Z)^H] \\ & \frac{f^2\omega^2}{4}[(R(u,X)R(u,Y)Z)^H - (R(u,Y)R(u,X)Z)^H] \\ & +\frac{\omega}{4}[g(R(u,Y)Z,gradf)X^V - g(R(u,X)Z,gradf)Y^V]\end{aligned}$$

$$\begin{aligned}\overline{R}(X^H,Y^V)Z^H &= \frac{\omega}{2}X(f)(R(u,Y)Z)^H + \frac{\omega}{4}Z(f)(R(u,Y)X)^H + \\ & +\frac{f\omega}{2}((\nabla_X R)(u,Y)Z)^H - \frac{\omega}{4}g(Y,R(X,Z)u)(gradf)^H + \\ & +\frac{1}{2}(R(X,Z)Y)^V - \frac{f\omega}{4}(R(X,R(u,Y)Z)u)^V \\ & +[\frac{1}{2f}X(Z(f)) - \frac{1}{4f^2}X(f)Z(f) - \frac{1}{2f}(\nabla_X Z)(f)]Y^V \\ & +\frac{\omega'}{2\omega}g(Y,u)(R(X,Z)u)^V \\ & +\frac{1}{2(1+\alpha r^2)}(\alpha - \frac{\omega'}{\omega})g(Y,R(X,Z)u)U^V\end{aligned}$$

$$\begin{aligned}\overline{R}(X^H Y^H)Z^V &= \frac{f\omega}{2}((\nabla_X R)(u,Z)Y)^H - \frac{f\omega}{2}((\nabla_Y R)(u,Z)X)^H \\ & +\frac{\omega}{4}X(f)(R(u,Z)Y)^H - \frac{\omega}{4}Y(f)(R(u,Z)X)^H \\ & -\frac{\omega}{2}g(R(X,Y)u,Z)(gradf)^H + (R(X,Y)Z)^V \\ & -\frac{f\omega}{4}(R(X,R(u,Z)Y)u)^V + \frac{f\omega}{4}(R(Y,R(u,Z)X)u)^V \\ & +\frac{\omega'}{\omega}g(Z,u)(R(X,Y)u)^V + \\ & +(\frac{1}{1+\alpha r^2})(\alpha - \frac{\omega'}{\omega})g(R(X,Y)u,Z)U^V\end{aligned}$$

$$
\begin{aligned}
\overline{R}(X^V,Y^V)Z^V &= A_1 g(Z,u)[g(Y,u)X^V - g(X,u)Y^V] \\
&+A_2[g(Y,Z)X^V - g(X,Z)Y^V] \\
&+A_3[G(Y^V,Z^V)X^V - G(X^V,Z^V)Y^V] \\
&+\frac{\omega}{4}[G(X^V,Z^V)(R(u,Y)gradf)^H - \\
&-G(Y^V,Z^V)(R(u,X)gradf)^H] \\
&+A_4[g(Y,u)G(X^V,Z^V) - g(X,u)G(Y^V,Z^V)]U^V \\
&+\Big(\frac{2\alpha\alpha' r^2+\alpha^2}{1+\alpha r^2} - \alpha'\Big)[g(Y,u)g(X,Z) - g(X,u)g(Y,Z)]U^V
\end{aligned}
$$

with

$$
\begin{aligned}
A_1 &= \left[\frac{-2\omega'\omega + 3(\omega')^2}{\omega^2} + \frac{\alpha'\omega' r^2}{\omega(1+\alpha r^2)} + \frac{\alpha'}{1+\alpha r^2}\right] \\
A_2 &= \left[\frac{\alpha\omega' r^2}{\omega(1+\alpha r^2)} - \frac{\omega'}{\omega} + \frac{\alpha}{1+\alpha r^2}\right] \\
A_3 &= -\left[\frac{1}{4f^2}\|gradf\|^2 + \frac{\omega' r^2}{f\omega^3(1+\alpha r^2)} + \frac{\omega'}{f\omega^2(1+\alpha r^2)}\right] \\
A_4 &= \frac{1}{f\omega^2(1+\alpha r^2)}\left[2\omega'' + \omega'\alpha' r^2 - \frac{(\omega')^2}{\omega} + \frac{\alpha\omega' - 2(\alpha' r^2+\alpha)}{1+\alpha r^2}\right],
\end{aligned}
$$

for all $p=(x,u)\in TM$ *and* $X,Y,Z\in\Gamma(TM)$.

3.1 Sectional curvature of the Mus-Gradient metric

In the following, we consider for all $V,W\in\Gamma(TTM), V\neq W$

$$Q^f(V,W) = g^f(V,V)g^f(W,W) - |g^f(V,W)|^2$$

$$G^f(V,W) = g^f(R^f(V,W)W,V)$$

$$K^f(V,W) = \frac{G^f(V,W)}{Q^f(V,W)}.$$

Lemma 6. *Let* (M,g) *be a Riemannian manifold and* (TM,G) *its tangent bundle equipped with the Mus-Cheeger-Gromoll metric, then for any orthonormal vector fields* $X,Y\in\Gamma(TM)$ *we have*

1. $Q^f(X^H,Y^H)=1$,
2. $Q^f(X^H,Y^V) = f\omega(r^2)[1+\alpha(r^2)|g(Y,u)|^2]$,

3. $Q^f(X^V, Y^V) = f^2\omega^2(1+\alpha(r)|g(X,u)|^2)(1+\alpha(r)|g(Y,u)|^2)$.

Lemma 7. *Let (M, g) be a Riemannian manifold and (TM, G) its tangent bundle equipped with the Mus-Cheeger-Gromoll, then for any orthonormal vector fields $X, Y \in \Gamma(TM)$ we have*

- $G(X^H, Y^H) = g(R(X,Y)Y, X) - \frac{3f\omega}{4}\|R(X,Y)u\|^2$,
- $G(X^H, Y^V) = \frac{f^2\omega^2}{4}\|R(u,Y)X\|^2 + [\frac{|X(f)|^2\omega}{4f} - \frac{\omega}{2}g(\nabla_X gradf, X)][1+\alpha|g(Y,u)|^2]$,
- $G(X^V, Y^V) = C_1|g(X,u)|^2 + C_2|g(Y,u)|^2 + C_3$,

with

$$\begin{aligned}
C_1 &= \frac{f\alpha^2\omega' r^2 + 2f(\alpha r^2+\alpha) - 2f\alpha\alpha'\omega r^2 - f\omega'\alpha + f\omega\alpha^2 + \alpha}{1+\alpha r^2} \\
&\quad - \frac{f\omega'\alpha r^2}{\omega(1+\alpha r^2)} + \frac{f(\omega')^2}{\omega} - f\alpha\omega' - 2f\omega'' \\
&\quad - f\omega'\alpha' r^2 - \frac{\omega^2\alpha}{4}\|gradf\|^2 \\
C_2 &= \frac{f\omega' r^2(\alpha'\omega - \alpha)}{\omega(1+\alpha r^2)} + \frac{3f(\omega')^2}{\omega} + \frac{\alpha'}{1+\alpha r^2} \\
&\quad - 2f\omega' - \frac{\omega'}{f\omega^2(1+\alpha r^2)} - \frac{\omega^2\alpha}{4}\|gradf\|^2 \\
C_3 &= \frac{f\omega' r^2}{1+\alpha r^2}(\omega\alpha - 1) - f\omega' \\
&\quad + \frac{f\omega\alpha - f\omega^2}{1+\alpha r^2} - \frac{\omega^2}{4}\|gradf\|^2
\end{aligned}$$

Proposition 2. *Let (M, g) be a Riemannian manifold and (TM, G) its tangent bundle equipped with the Mus-Cheeger-Gromoll. If K (resp., $\overline{K}$ denote the sectional curvature tensor of (M, g)(resp., (TM, G)) , then for any orthonormal vector fields $X, Y \in \Gamma(TM)$ we have*

1. $\overline{K}(X^H, Y^H) = K(X,Y) - \frac{3f\omega}{4}\|R(X,Y)u\|^2$
2. $\overline{K}(X^H, Y^V) = \frac{f\omega}{4(1+\alpha|g(Y,u)|^2)}\|R(u,Y)X)\|^2 + \frac{|X(f)|^2}{4f^2} - \frac{1}{2f}g(\nabla_X gradf, X)$
3. $\overline{K}(X^V, Y^V) = \frac{1}{f^2\omega^2(1+\alpha|g(X,u)|^2)(1+\alpha|g(Y,u)|^2)}\left[C_1|g(X,u)|^2 + C_2|g(Y,u)|^2 + C_3\right]$.

Proposition 3. *Let (M, g) be a Riemannian manifold of constant sectional curvature λ and (TM, G) its tangent bundle equipped with the mus-Cheeger -Gromoll metric. If $\overline{K}$ denotes the sectional curvature tensor of (TM, G), then for any orthonormal vector fields $X, Y \in \Gamma(TM)$ we have*

1. $\overline{K}(X^H, Y^H) = \lambda - \frac{3\lambda^2 f\omega}{4}[|g(X,u)|^2 + |g(Y,u)|^2]$

2. $\overline{K}(X^H, Y^V) = \frac{f\omega\lambda^2}{4}\frac{|g(X,u)|^2}{(1+\alpha|g(Y,u)|^2)} + \frac{|X(f)|^2}{4f^2} - \frac{1}{2f}g(\nabla_X gradf, X)$

3. $\overline{K}(X^V, Y^V) = \frac{1}{f^2\omega^2(1+\alpha|g(X,u)|^2)(1+\alpha|g(Y,u)|^2)}\left[C_1|g(X,u)|^2 + C_2|g(Y,u)|^2 + C_3\right].$

Proof. The proof of Proposition is deduced from Proposition 2 and the following equations

$$R(X,Y)Z = \lambda[g(Y,Z)X - g(X,Z)Y],$$

$$\begin{aligned}\|R(X,Y)u\|^2 &= \|\lambda[g(Y,u)X - g(X,u)Y]\|^2\\ &= \lambda^2[|g(X,u)|^2 + |g(y,u)|^2],\end{aligned}$$

$$\begin{aligned}\|R(u,Y)X\|^2 &= \|\lambda[g(Y,X)u - g(X,u)Y]\|^2\\ &= \lambda^2|g(X,u)|^2.\end{aligned}$$

□

Lemma 8. *Let (x,u) be a point of TM with $u \neq 0$and $(E_1,...,E_m)$ be a local orthonormal on M such that $E_1 = \frac{u}{\|u\|}$. Then $(F_1,...,F_{2m})$ is a local orthonormal frame on (TM,G). where*
$F_i = E_i^H$, $F_{m+1} = \frac{1}{\sqrt{f\omega(1+\alpha r^2)}}E_1^V$ *and* $F_{m+j} = \sqrt{\frac{1}{f\omega}}E_j^V$, $i = \overline{1,m}$, $j = \overline{2,m}.$

Lemma 9. *Let (M,g) be a Riemannian manifold and (TM,gf) its tangent bundle equipped with the Mus-Cheeger-Gromoll metric. If $(E_1,...,E_m)$(resp.,$(F_1,...,F_{2m})$) is a local orthonormal on M (resp., TM), then for all $i,j = \overline{1,m}$ et $k,l = \overline{2,m}$, we have*

1. $\overline{K}(F_i, F_j) = K(E_i, E_j) - \frac{3f\omega}{4}\|R(E_i,E_j)\|^2$

2. $\overline{K}(F_i, F_{m+1}) = \frac{|E_i(f)|^2}{4f^2} - \frac{1}{2f}g(\nabla_{E_i} gradf, E_i)$

3. $\overline{K}(F_i, F_{m+l}) = \frac{1}{4}\|R(u,E_l)E_i)\|^2 + \frac{|E_i(f)|^2}{4f^2} - \frac{1}{2f}g(\nabla_{E_i} gradf, E_i)$

4. $\overline{K}(F_{m+k}, F_{m+1}) = \frac{1}{f\omega}\frac{1+\alpha r^2}{f\omega(1+\alpha r^2)+\alpha r^2}[\frac{r^2}{f\omega(1+\alpha r^2)}C_2 + C_3]$

5. $\overline{K}(F_{m+k}, F_{m+l}) = \frac{1}{f^2\omega^2}C_3.$

Lemma 10. *Let $(E_1,...,E_m)$ be local orthonormal frame on M, then for all $i,j = \overline{1,m}$ we have*

$$\sum_{i,j=1}^{m}\|R(u,E_i)E_j\|^2 = \sum_{i,j=1}^{m}\|R(E_i,E_j)u\|^2.$$

Proof.

$$\begin{aligned}
\sum_{i,j=1}^{m} \|R(u,E_i)E_j\|^2 &= \sum_{i,j=1}^{m} g(R(u,E_i,E_j,R(u,E_i)E_j) \\
&= \sum_{i,j,k,l,s=1}^{m} u_k u_l g(R(E_k,E_i)E_j,E_s) g(R(E_k,E_i)E_j,E_s) \\
&= \sum_{i,j,k,l,s=1}^{m} u_k u_l g(R(E_j,E_s)E_k,E_i) g(R(E_j,E_s)E_l,E_i) \\
&= \sum_{i,j,s=1}^{m} g(R(E_j,E_s)u, g(E_j,E_s)u,E_i)E_i) \\
&= \sum_{i,j=1}^{m} \|R(E_i,E_j)u\|^2.
\end{aligned}$$

□

Proposition 4. *Let (M,g) be a Riemannian manifold and (TM,gf) its tangent bundle equipped with the Mus-Cheeger-Gromoll metric. If σ (resp.,$\overline{\sigma}$) denote the scalar curvature of (M,g) (resp.,(TM,G)), then for any orthonormal frame $(E_1,...,E_m)$ we have*

$$\begin{aligned}
\overline{\sigma} &= \sigma + \frac{2-3f\omega}{4}\sum_{i,j=1}^{m}\|R(E_i,E_j)\|^2 - \frac{m}{f}\Delta(f) + \frac{2m\|gradf\|^2}{f^2} \\
& 2(m-1)\left[\frac{1}{f\omega}\frac{1+\alpha r^2}{f\omega(1+\alpha r^2)+\alpha r^2}\left(\frac{r^2}{f\omega(1+\alpha r^2)}C_2 + C_3\right) + \frac{m-2}{2f^2\omega^2}C_3\right]
\end{aligned}$$

Proof. Using Lemma 9

$$\begin{aligned}
\overline{\sigma} &= \sum_{s,t=1}^{2m} \overline{K}(F_s,F_t) \\
&= \sum_{i,j=1,i\neq j}^{m} \overline{K}(F_i,F_j) + 2\sum_{i,j=1}^{m}\overline{K}(F_i,F_{m+j}) + \sum_{i,j=1,i\neq j}^{m}\overline{K}(F_{m+i},F_{m+j}) \\
&= \sum_{i,j=1,i\neq j}^{m} \overline{K}(F_i,F_j) + 2\sum_{i=1}^{m}\overline{K}(F_i,F_{m+1}) + 2\sum_{i=1,j=2}^{m}\overline{K}(F_i,F_{m+j}) + \\
&\quad +2\sum_{i=1}^{m}\overline{K}(F_{m+i},F_{m+1}) + \sum_{i,j=2,i\neq j}^{m}\overline{K}(F_{m+i},F_{m+j})
\end{aligned}$$

$$\begin{aligned}
\overline{\sigma} &= \sum_{i,j=1,i\neq j}^{m}[K(E_i,E_j)-\frac{3f\omega}{4}\|R(E_i,E_j)\|^2]\\
&+2\sum_{i=1}^{m}[\frac{|E_i(f)|^2}{4f^2}-\frac{1}{2f}g(\nabla_{E_i}gradf,E_i)]\\
&+2\sum_{i=1,j=2}^{m}[\frac{1}{4}\|R(u,E_j)E_i)\|^2+\frac{|E_i(f)|^2}{4f^2}-\frac{1}{2f}g(\nabla_{E_i}gradf,E_i)]\\
&+2\sum_{i=1}^{m}[\frac{1}{f\omega}\frac{1+\alpha r^2}{f\omega(1+\alpha r^2)+\alpha r^2}[\frac{r^2}{f\omega(1+\alpha r^2)}C_2+C_3]]\\
&+\sum_{i,j=2,i\neq j}^{m}[\frac{1}{f^2\omega^2}C_3]\\
&= \sigma-\sum_{i,j=1,i\neq j}^{m}\frac{3f\omega}{4}\|R(E_i,E_j)\|^2+\frac{2\|gradf\|^2}{f^2}-\frac{1}{f}trace_g(\nabla gradf)\\
&+\frac{2}{4}\sum_{i=1,j=2}^{m}\|R(u,E_j)E_i\|^2+2(m-1)\frac{\|gradf\|^2}{f^2}-\frac{m-1}{f}trace_g(\nabla gradf)\\
&+2(m-1)[\frac{1}{f\omega}\frac{1+\alpha r^2}{f\omega(1+\alpha r^2)+\alpha r^2}[\frac{r^2}{f\omega(1+\alpha r^2)}C_2+C_3]]\\
&+(m-2)(m-1)[\frac{1}{f^2\omega^2}C_3]
\end{aligned}$$

Hence,

$$\begin{aligned}
\overline{\sigma} &= \sigma+\frac{2-3f\omega}{4}\sum_{i,j=1}^{m}\|R(E_i,E_j)\|^2-\frac{m}{f}\Delta(f)+\frac{2m\|gradf\|^2}{f^2}\\
&\quad 2(m-1)\left[\frac{1}{f\omega}\frac{1+\alpha r^2}{f\omega(1+\alpha r^2)+\alpha r^2}(\frac{r^2}{f\omega(1+\alpha r^2)}C_2+C_3)+\frac{m-2}{2f^2\omega^2}C_3\right]
\end{aligned}$$

□

Corollary 1. *Let (M,g) be a Riemannian manifold of constant sectional curvature λ and (TM,G) its tangent bundle equipped with the Mus-Cheeger-Gromoll metric. If $\overline{\sigma}$ denotes the scalar curvature of (TM,G), then for any orthonormal frame $(E_1,...,E_m)$ on M, we have*

$$\begin{aligned}
\overline{\sigma} &= m(m-1)\lambda+2(m-1)\lambda^2r^2\frac{2-3f\omega}{4}-\frac{m}{f}\Delta(f)+\frac{2m\|gradf\|^2}{f^2}\\
&\quad 2(m-1)\left[\frac{1}{f\omega}\frac{1+\alpha r^2}{f\omega(1+\alpha r^2)+\alpha r^2}(\frac{r^2}{f\omega(1+\alpha r^2)}C_2+C_3)+\frac{m-2}{2f^2\omega^2}C_3\right]
\end{aligned}$$

Proof. Taking account that $\sigma = m(m-1)\lambda$ and for any vector fields $X, Y, Z \in TM$

$$R(X,Y)Z = \lambda(g(Z,Y)X - g(X,Z)Y)$$

then we obtain

$$\begin{aligned}
\sum_{i,j=1}^{m} \|R(E_i,E_j)u\|^2 &= \lambda^2 \sum_{i,j=1}^{m} \|g(u,E_j)E_i - g(E_i,u)E_j\|^2 \\
&= \lambda^2 \sum_{i,j=1}^{m} [|g(u,E_j)|^2 - 2g(u,E_j)g(E_i,u)\delta_{ij} + |g(E_i,u)|^2] \\
&= \lambda^2[m\|u\|^2 - 2\|u\|^2 + m\|u\|^2] \\
&= 2\lambda^2(m-1)r^2
\end{aligned}$$

we deduce that

$$\begin{aligned}
\overline{\sigma} &= m(m-1)\lambda + 2(m-1)\lambda^2 r^2 \frac{2-3f\omega}{4} - \frac{m}{f}\Delta(f) + \frac{2m\|gradf\|^2}{f^2} \\
& 2(m-1)\left[\frac{1}{f\omega}\frac{1+\alpha r^2}{f\omega(1+\alpha r^2)+\alpha r^2}\left(\frac{r^2}{f\omega(1+\alpha r^2)}C_2 + C_3\right) + \frac{m-2}{2f^2\omega^2}C_3\right].
\end{aligned}$$

□

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