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THE UNIT BALL OF BILINEAR FORMS ON $\mathbb{R}^2$ WITH A ROTATED SUPREMUM NORM

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Abstract

Let $0 \leq \theta < \frac{\pi}{2}$ and $l^2_{\infty,\theta}$ be the plane with the rotated supremum norm

$$\|(x,y)\|_{\infty,\theta} = \max\Big\{ \ |(cos\theta)x + (sin\theta)y|, \ |(sin\theta)x - (cos\theta)y| \ \Big\}.$$

We devote to the description of the sets of extreme, exposed and smooth points of the closed unit balls of $\mathcal{L}(^2 l^2_{\infty,\theta})$ and $\mathcal{L}_s(^2 l^2_{\infty,\theta})$, where $\mathcal{L}(^2 l^2_{\infty,\theta})$ is the space of bilinear forms on $l^2_{\infty,\theta}$, and $\mathcal{L}_s(^2 l^2_{\infty,\theta})$ is the subspace of $\mathcal{L}(^2 l^2_{\infty,\theta})$ consisting of symmetric bilinear forms. Let $\mathcal{F} = \mathcal{L}(^2 l^2_{\infty,\theta})$ or $\mathcal{L}_s(^2 l^2_{\infty,\theta})$. First we classify the extreme and exposed points of the closed unit ball of $\mathcal{F}$. We also show that every extreme point of the closed unit ball of $\mathcal{F}$ is exposed. It is shown that $\text{ext}\, B_{\mathcal{L}_s(^2 l^2_{\infty,\theta})} = \text{ext}\, B_{\mathcal{L}(^2 l^2_{\infty,\theta})} \cap \mathcal{L}_s(^2 l^2_{\infty,\theta})$ and $\exp B_{\mathcal{L}_s(^2 l^2_{\infty,\theta})} = \exp B_{\mathcal{L}(^2 l^2_{\infty,\theta})} \cap \mathcal{L}_s(^2 l^2_{\infty,\theta})$. We classify the smooth points of the closed unit ball of $\mathcal{F}$. It is shown that $\text{sm}\, B_{\mathcal{L}(^2 l^2_{\infty,\theta})} \cap \mathcal{L}_s(^2 l^2_{\infty,\theta}) \subsetneq \text{sm}\, B_{\mathcal{L}_s(^2 l^2_{\infty,\theta})}$. As corollary we extend the results of [18, 35].

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1 Introduction

Throughout the paper, we let $n, m \in \mathbb{N}, n, m \geq 2$. We write B_E for the closed unit ball of a real Banach space E and the dual space of E is denoted by E^*. An element $x \in B_E$ is called an *extreme point* of B_E if $y, z \in B_E$ with $x = \frac{1}{2}(y+z)$ implies $x = y = z$. An element $x \in B_E$ is called an *exposed point* of B_E if there is $f \in E^*$ so that $f(x) = 1 = \|f\|$ and $f(y) < 1$ for every $y \in B_E \setminus \{x\}$. It is easy to see that every exposed point of B_E is an extreme point. An element $x \in B_E$ is called a *smooth point* of B_E if there is unique $f \in E^*$ so that $f(x) = 1 = \|f\|$. We denote by $\text{ext}\, B_E$, $\exp B_E$ and $\text{sm}\, B_E$ the set of extreme points, the set of exposed points and the set of smooth points of B_E, respectively. A mapping $P : E \to \mathbb{R}$ is

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a continuous n-homogeneous polynomial if there exists a continuous n-linear form T on the product $E \times \cdots \times E$ such that $P(x) = T(x, \cdots, x)$ for every $x \in E$. We denote by $\mathcal{P}(^nE)$ the Banach space of all continuous n-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$. We denote by $\mathcal{L}(^nE)$ the Banach space of all continuous n-linear forms on E endowed with the norm $\|T\| = \sup_{\|x_k\|=1} |T(x_1, \cdots, x_n)|$. $\mathcal{L}_s(^nE)$ denotes the closed subspace of all continuous symmetric n-linear forms on E. Notice that $\mathcal{L}(^nE)$ is identified with the dual of n-fold projective tensor product $\hat{\bigotimes}_{\pi,n}E$. With this identification, the action of a continuous n-linear form T as a bounded linear functional on $\hat{\bigotimes}_{\pi,n}E$ is given by

$$\Big\langle \sum_{i=1}^{k} x^{(1),i} \otimes \cdots \otimes x^{(n),i},\ T \Big\rangle = \sum_{i=1}^{k} T\Big(x^{(1),i}, \cdots, x^{(n),i}\Big).$$

Notice also that $\mathcal{L}_s(^nE)$ is identified with the dual of n-fold symmetric projective tensor product $\hat{\bigotimes}_{s,\pi,n}E$. With this identification, the action of a continuous symmetric n-linear form T as a bounded linear functional on $\hat{\bigotimes}_{s,\pi,n}E$ is given by

$$\Big\langle \sum_{i=1}^{k} \frac{1}{n!}\Big(\sum_{\sigma} x^{\sigma(1),i} \otimes \cdots \otimes x^{\sigma(n),i}\Big),\ T \Big\rangle = \sum_{i=1}^{k} T\Big(x^{(1),i}, \cdots, x^{(n),i}\Big),$$

where σ goes over all permutations on $\{1, \ldots, n\}$. For more details about the theory of polynomials and multilinear mappings on Banach spaces, we refer to [8].

Let us introduce the history of classification problems of the extreme points, the exposed points and the smooth points of the unit ball of continuous n-homogeneous polynomials on a Banach space.

We let $l_p^n = \mathbb{R}^n$ for every $1 \leq p \leq \infty$ equipped with the l_p-norm. Choi et al. [3, 4, 5] initiated and classified $\operatorname{ext} B_{\mathcal{P}(^2l_p^2)}$ for $p = 1, 2$. Choi and Kim [7] classified $\exp B_{\mathcal{P}(^2l_p^2)}$ for $p = 1, 2, \infty$. Grecu [12] classified $\operatorname{ext} B_{\mathcal{P}(^2l_p^2)}$ for $1 < p < 2$ or $2 < p < \infty$. Kim et al. [46] showed that if E is a separable real Hilbert space with $\dim(E) \geq 2$, then, $\operatorname{ext} B_{\mathcal{P}(^2E)} = \exp B_{\mathcal{P}(^2E)}$. Kim [17] classified $\exp B_{\mathcal{P}(^2l_p^2)}$ for $1 \leq p \leq \infty$. Kim [19, 21] characterized $\operatorname{ext} B_{\mathcal{P}(^2d_*(1,w)^2)}$, where $d_*(1,w)^2 = \mathbb{R}^2$ with the octagonal norm $\|(x,y)\|_w = \max\Big\{|x|, |y|, \frac{|x|+|y|}{1+w}\Big\}$ for $0 < w < 1$. Kim [26] classified $\exp B_{\mathcal{P}(^2d_*(1,w)^2)}$ and showed that $\exp B_{\mathcal{P}(^2d_*(1,w)^2)} \neq \operatorname{ext} B_{\mathcal{P}(^2d_*(1,w)^2)}$. Kim [31, 34, 45] classified $\operatorname{ext} B_{\mathcal{P}(^2\mathbb{R}^2_{h(\frac{1}{2})})}, \exp B_{\mathcal{P}(^2\mathbb{R}^2_{h(\frac{1}{2})})}$ and $\operatorname{sm} B_{\mathcal{P}(^2\mathbb{R}^2_{h(\frac{1}{2})})}$, where $\mathbb{R}^2_{h(\frac{1}{2})} = \mathbb{R}^2$ with the hexagonal norm $\|(x,y)\|_{h(\frac{1}{2})} = \max\Big\{|y|, |x| + \frac{1}{2}|y|\Big\}$.

Parallel to the classification problems of $\operatorname{ext} B_{\mathcal{P}(^nE)}, \exp B_{\mathcal{P}(^nE)}$ and $\operatorname{sm} B_{\mathcal{P}(^nE)}$, it seems to be very natural to study the classification problems of the extreme points, the exposed points and the smooth points of the unit ball of continuous (symmetric) multilinear forms on a Banach space.

Kim [18] initiated and classified $\operatorname{ext} B_{\mathcal{L}_s(^2l_\infty^2)}$, $\exp B_{\mathcal{L}_s(^2l_\infty^2)}$ and $\operatorname{sm} B_{\mathcal{L}_s(^2l_\infty^2)}$. It was shown that $\operatorname{ext} B_{\mathcal{L}_s(^2l_\infty^2)} = \exp B_{\mathcal{L}_s(^2l_\infty^2)}$. Kim [20, 22, 23, 25] classified $\operatorname{ext} B_{\mathcal{L}_s(^2d_*(1,w)^2)}$, $\operatorname{ext} B_{\mathcal{L}(^2d_*(1,w)^2)}$, $\exp B_{\mathcal{L}_s(^2d_*(1,w)^2)}$, and $\exp B_{\mathcal{L}(^2d_*(1,w)^2)}$.

Kim [29, 30] also classified $\operatorname{ext} B_{\mathcal{L}_s(^2l_\infty^3)}$ and $\exp B_{\mathcal{L}_s(^3l_\infty^2)}$. It was shown that $\operatorname{ext} B_{\mathcal{L}_s(^2l_\infty^3)} = \exp B_{\mathcal{L}_s(^2l_\infty^3)}$ and $\operatorname{ext} B_{\mathcal{L}_s(^3l_\infty^2)} = \exp B_{\mathcal{L}_s(^3l_\infty^2)}$. Kim [35] classified $\operatorname{ext} B_{\mathcal{L}(^2l_\infty^2)}, \exp B_{\mathcal{L}(^2l_\infty^2)}$ and $\operatorname{sm} B_{\mathcal{L}(^2l_\infty^2)}$. Kim [33] characterized $\operatorname{ext} B_{\mathcal{L}(^2l_\infty^n)}$ and $\operatorname{ext} B_{\mathcal{L}_s(^2l_\infty^n)}$, and showed that $\exp B_{\mathcal{L}(^2l_\infty^n)} = \operatorname{ext} B_{\mathcal{L}(^2l_\infty^n)}$ and $\exp B_{\mathcal{L}_s(^2l_\infty^n)} = \operatorname{ext} B_{\mathcal{L}_s(^2l_\infty^n)}$. Kim [36] characterized $\operatorname{ext} B_{\mathcal{L}(^2l_\infty^3)}$ and $\exp B_{\mathcal{L}(^2l_\infty^3)}$. Kim [37] characterized $\operatorname{sm} B_{\mathcal{L}_s(^nl_\infty^2)}$. Kim [38] studied $\operatorname{ext} B_{\mathcal{L}(^2l_\infty)}$. Cavalcante et al. [2] characterized $\operatorname{ext} B_{\mathcal{L}(^nl_\infty^m)}$. Kim [41] classified $\operatorname{ext} B_{\mathcal{L}(^nl_\infty^2)}$ and $\operatorname{ext} B_{\mathcal{L}_s(^nl_\infty^2)}$. It was shown that $|\operatorname{ext} B_{\mathcal{L}(^nl_\infty^2)}| = 2^{(2^n)}$ and $|\operatorname{ext} B_{\mathcal{L}_s(^nl_\infty^2)}| = 2^{n+1}$, and that $\exp B_{\mathcal{L}(^nl_\infty^2)} = \operatorname{ext} B_{\mathcal{L}(^nl_\infty^2)}$ and $\exp B_{\mathcal{L}_s(^nl_\infty^2)} = \operatorname{ext} B_{\mathcal{L}_s(^nl_\infty^2)}$. Kim [40, 43] characterize $\operatorname{ext} B_{\mathcal{L}_s(^nl_\infty^m)}$, $\operatorname{ext} B_{\mathcal{L}(^nl_\infty^m)}$, $\exp B_{\mathcal{L}_s(^ml_\infty^m)}$, $\exp B_{\mathcal{L}(^nl_\infty^m)}$, $\operatorname{sm} B_{\mathcal{L}_s(^nl_\infty^m)}$ and $\operatorname{sm} B_{\mathcal{L}(^nl_\infty^m)}$ for every $n, m \geq 2$. Kim [44] characterize $\operatorname{ext} B_{\mathcal{L}(^n\mathbb{R}^m_{\|\cdot\|})}, \operatorname{ext} B_{\mathcal{L}_s(^n\mathbb{R}^m_{\|\cdot\|})}, \exp B_{\mathcal{L}(^n\mathbb{R}^m_{\|\cdot\|})}$, $\exp B_{\mathcal{L}_s(^n\mathbb{R}^m_{\|\cdot\|})}$, where $\mathbb{R}^m_{\|\cdot\|}$ is $\mathbb{R}^m$ with a norm $\|\cdot\|$ such that $|\operatorname{ext} B_{\mathbb{R}^m_{\|\cdot\|}}| = 2m$ for $m \geq 2$. It is shown that every extreme point is exposed.

We refer to [1–7, 9–15, 17–55] and references therein) for some recent work about extremal properties of homogeneous polynomials and multilinear forms on Banach spaces.

Let $0 \leq \theta < \frac{\pi}{2}$ and $l^2_{\infty,\theta}$ be the plane with the rotated supremum norm

$$\|(x,y)\|_{\infty,\theta} = \max\Big\{ \ |(cos\theta)x + (sin\theta)y|, \ |(sin\theta)x - (cos\theta)y| \ \Big\}.$$

Notice that if $\theta = 0$, then $l^2_{\infty,0} = l^2_\infty = \mathbb{R}^2$ with the supremum norm. In this paper, we devote to the description of the sets of extreme, exposed and smooth points of the closed unit balls of $\mathcal{L}(^2l^2_{\infty,\theta})$ and $\mathcal{L}_s(^2l^2_{\infty,\theta})$. Let $\mathcal{F} = \mathcal{L}(^2l^2_{\infty,\theta})$ or $\mathcal{L}_s(^2l^2_{\infty,\theta})$. First we classify the extreme and exposed points of the closed unit ball of $\mathcal{F}$. We also show that every extreme point of the closed unit ball of $\mathcal{F}$ is exposed. It is shown that $\operatorname{ext} B_{\mathcal{L}_s(^2l^2_{\infty,\theta})} = \operatorname{ext} B_{\mathcal{L}(^2l^2_{\infty,\theta})} \cap \mathcal{L}_s(^2l^2_{\infty,\theta})$ and $\exp B_{\mathcal{L}_s(^2l^2_{\infty,\theta})} = \exp B_{\mathcal{L}(^2l^2_{\infty,\theta})} \cap \mathcal{L}_s(^2l^2_{\infty,\theta})$. We classify the smooth points of the closed unit ball of $\mathcal{F}$. It is shown that $\operatorname{sm} B_{\mathcal{L}_s(^2l^2_{\infty,\theta})} = \operatorname{sm} B_{\mathcal{L}(^2l^2_{\infty,\theta})} \cap \mathcal{L}_s(^2l^2_{\infty,\theta})$. As corollary we extend the results of [18, 35] when $\theta = 0$.

2 The extreme points of the unit balls of $\mathcal{L}(^2l^2_{\infty,\theta})$ and $\mathcal{L}_s(^2l^2_{\infty,\theta})$

Throughout the paper we let $0 \leq \theta < \frac{\pi}{2}$ and $l^2_{\infty,\theta}$ be the plane with the rotated supremum norm

$$\|(x,y)\|_{\infty,\theta} = \max\Big\{ \ |(cos\theta)x + (sin\theta)y|, \ |(sin\theta)x - (cos\theta)y| \ \Big\}.$$

If $T \in \mathcal{L}(^2l^2_{\infty,\theta})$, then

$$T((x_1,y_1), \ (x_2,y_2)) = ax_1x_2 + by_1y_2 + cx_1y_2 + dx_2y_1$$

for some $a, b, c, d \in \mathbb{R}$. For simplicity, we denote $T = (a, b, c, d)$.

Let S be a non-empty subset of a real Banach space E. Let

$$\mathrm{conv}(S) := \Big\{ \sum_{j=1}^{k} t_j a_j : 0 \le t_j \le 1, t_1 + \cdots + t_k = 1, a_j \in S \text{ for } k \in \mathbb{N} \text{ and } 1 \le j \le k \Big\}.$$

We call $\mathrm{conv}(S)$ the convex hull of S. Recall that the Krein-Milman Theorem [46] say that every nonempty compact convex subset of a Hausdorff locally convex space is the closed convex hull of its set of extreme points.

Let $A := (cos\theta - sin\theta,\ cos\theta + sin\theta)$ and $B := (cos\theta + sin\theta,\ -cos\theta + sin\theta)$. Notice that

$$\mathrm{ext}\, B_{l^2_{\infty,\theta}} = \{\pm A, \pm B\}.$$

By the Krein-Milman Theorem,

$$B_{l^2_{\infty,\theta}} = \overline{\mathrm{conv}}\Big(\{\pm A, \pm B\}\Big).$$

The following presents an explicit formula for the norm of $T \in \mathcal{L}(^2 l^2_{\infty,\theta})$.

Theorem 1. *Let $T = (a, b, c, d) \in \mathcal{L}(^2 l^2_{\infty,\theta})$ for some $a, b, c, d \in \mathbb{R}$. Then,*

$$\begin{aligned} \|T\| \ &= \ \max\Big\{ \Big|(1 - sin2\theta)a + (1 + sin2\theta)b + (cos2\theta)(c + d)\Big|, \\ & \Big|(1 + sin2\theta)a + (1 - sin2\theta)b - (cos2\theta)(c + d)\Big|, \\ & \Big|(cos2\theta)(a - b) - (1 - sin2\theta)c + (1 + sin2\theta)d\Big|, \\ & \Big|(cos2\theta)(a - b) + (1 + sin2\theta)c - (1 - sin2\theta)d\Big| \Big\}. \end{aligned}$$

Proof. Let $X_1, X_2 \in B_{l^2_{\infty,\theta}}$. By the Krein-Milman Theorem, there exist $t_1^{(j)}, t_2^{(j)} \in \mathbb{R}$ such that

$$|t_1^{(j)}| + |t_2^{(j)}| \le 1 \text{ and } X_j = t_1^{(j)} A + t_2^{(j)} B \quad (j = 1, 2).$$

By the bilinearity of T, it follows that

$$\begin{aligned} & |T(X_1, X_2)| = \Big| T\Big(t_1^{(1)} A + t_2^{(1)} B,\ t_1^{(2)} A + t_2^{(2)} B \Big) \Big| \\ \le & \sum_{1 \le j_k \le 2, 1 \le k \le 2} |t_{j_1}^{(1)}||t_{j_n}^{(2)}| \ \max\{\ |T(A, A)|,\ |T(A, B)|,\ |T(B, A)|,\ |T(B, B)|\ \} \\ = & \ \max\{\ |T(A, A)|,\ |T(A, B)|,\ |T(B, A)|,\ |T(B, B)|\ \} \\ = & \ \max\Big\{ \Big|(1 - sin2\theta)a + (1 + sin2\theta)b + (cos2\theta)(c + d)\Big|, \\ & \Big|(1 + sin2\theta)a + (1 - sin2\theta)b - (cos2\theta)(c + d)\Big|, \\ & \Big|(cos2\theta)(a - b) - (1 - sin2\theta)c + (1 + sin2\theta)d\Big|, \\ & \Big|(cos2\theta)(a - b) + (1 + sin2\theta)c - (1 - sin2\theta)d\Big| \Big\} \le \|T\|, \end{aligned}$$

which completes the proof. □

Notice that if $\theta = \frac{\pi}{4}$ and $T = (a, b, c, d) \in \mathcal{L}(^2 l^2_{\infty, \frac{\pi}{4}})$, then

$$\|T\|_{\frac{\pi}{4}} = 2 \max \Big\{ \ |a|, \ |b| \ , |c|, \ |d| \ \Big\}.$$

Theorem 2. *Let* $0 \leq \theta < \frac{\pi}{2}$ *and* $T = (a, b, c, d) \in \mathcal{L}(^2 l^2_{\infty,\theta})$ *with* $\|T\| = 1$. *Then,* $T \in \operatorname{ext} B_{\mathcal{L}(^2 l^2_{\infty,\theta})}$ *if and only if*

$$1 = |T(A, A)| = |T(B, B)| = |T(A, B)| = |T(B, A)|.$$

Proof. ($\Rightarrow$). Suppose that $T \in \operatorname{ext} B_{\mathcal{L}_s(^2 l^2_{\infty,\theta})}$. Assume the assertion is not true. We have three cases.

Case 1. $|T(A, A)| < 1$.

Let $\theta = \frac{\pi}{4}$.
Since

$$1 > |T(A, A)| = |T((0, \sqrt{2}), \ (0, \sqrt{2}))| = 2|b|,$$

there is $N \in \mathbb{N}$ such that

$$\|T \pm \Big(0, \ \frac{1}{N}, \ 0, \ 0\Big)\| = 1.$$

Let

$$T^{\pm} := T \pm \Big(0, \ \frac{1}{N}, \ 0, \ 0\Big).$$

Hence, T is not extreme. This is a contradiction.

Suppose that $\theta \neq \frac{\pi}{4}$. Let

$$T^{\pm} := T \pm \Big(\frac{(1 - sin2\theta)^2}{n(cos2\theta)^2}, \ \frac{1}{n}, \ \frac{1 - sin2\theta}{ncos2\theta}, \ \frac{1 - sin2\theta}{ncos2\theta}\Big)$$

for a sufficiently large $n \in \mathbb{N}$ so that $\|T^{\pm}\| = 1$ for $j = 1, 2$. Hence, T is not extreme. This is a contradiction.

Case 2. $|T(B, B)| < 1$.

Let $\theta = \frac{\pi}{4}$.
Since

$$1 > |T(B, B)| = |T((\sqrt{2}, 0), \ (\sqrt{2}, 0))| = 2|a|,$$

there is $N \in \mathbb{N}$ such that

$$\|T \pm \Big(\frac{1}{N}, \ 0, \ 0, \ 0\Big)\| = 1.$$

Hence, T is not extreme. This is a contradiction.

Suppose that $\theta \neq \frac{\pi}{4}$. Let

$$T^{\pm} := T \pm \Big(- \frac{(1 + sin2\theta)}{ncos2\theta}, \ \ - \frac{cos2\theta}{n(1 + sin2\theta)}, \ \frac{1}{n}, \ \frac{1}{n}\Big)$$

for a sufficiently large $n \in \mathbb{N}$ so that $\|T^{\pm}\| = 1$ for $j = 1, 2$. Hence, T is not extreme. This is a contradiction.

Case 3. $|T(A, B)| < 1$.

Let $\theta = \frac{\pi}{4}$.
Since

$$1 > |T(A, B)| = |T((0, \sqrt{2}),\ (\sqrt{2}, 0))| = 2|d|,$$

there is $N \in \mathbb{N}$ such that

$$\|T \pm \Big(0,\ 0,\ 0,\ \frac{1}{N}\Big)\| = 1.$$

Hence, T is not extreme. This is a contradiction.

Suppose that $\theta \neq \frac{\pi}{4}$.
Let

$$T^{\pm} := T \pm \Big(\frac{1}{n},\ \ -\frac{1}{n},\ \frac{1 + 3sin2\theta}{ncos2\theta},\ \ -\frac{1 + sin2\theta}{ncos2\theta}\Big)$$

for a sufficiently large $n \in \mathbb{N}$ so that $\|T^{\pm}\| = 1$ for $j = 1, 2$. Hence, T is not extreme. This is a contradiction.

Case 4. $|T(B, A)| < 1$.

Let $\theta = \frac{\pi}{4}$.
Since

$$1 > |T(B, A)| = |T((\sqrt{2}, 0),\ (0, \sqrt{2}))| = 2|c|,$$

there is $N \in \mathbb{N}$ such that

$$\|T \pm \Big(0,\ 0,\ \frac{1}{N},\ 0\Big)\| = 1.$$

Hence, T is not extreme. This is a contradiction.

Suppose that $\theta \neq \frac{\pi}{4}$.
Let

$$T^{\pm} := T \pm \Big(\frac{1}{n},\ \ -\frac{1}{n},\ \frac{1 + 3sin2\theta}{ncos2\theta},\ \frac{-1 + sin2\theta}{ncos2\theta}\Big)$$

for a sufficiently large $n \in \mathbb{N}$ so that $\|T^{\pm}\| = 1$ for $j = 1, 2$. Hence, T is not extreme. This is a contradiction. Therefore, the assertion is true.

($\Leftarrow$). Suppose that $1 = |T(A, A)| = |T(B, B)| = |T(A, B)| = |T(B, A)|$.

Let $R_1, R_2 \in \mathcal{L}(^2 l^2_{\infty,\theta})$ be defined by

$$R_1 = T + (\epsilon,\ \delta,\ \rho,\ t) \text{ and } R_2 = T - (\epsilon,\ \delta,\ \rho,\ t)$$

for some $\epsilon,\ \delta,\ \rho,\ t \in \mathbb{R}$ be such that $\|R_j\| = 1$ for $j = 1, 2$.

Claim. $\epsilon = \delta = \rho = t = 0$.

By Theorem 1, it follows that

$$\begin{aligned} 1 &\geq \max\{|T_1(A,A)|,\ |T_2(A,A)|\} \\ &= |T(A,A)| + |(\epsilon,\ \delta,\ \rho,\ t)(A,A)| \\ &= 1 + |(\epsilon,\ \delta,\ \rho,\ t)(A,A)|, \end{aligned}$$

which shows that

$$0 = (\epsilon,\ \delta,\ \rho,\ t)(A,A) = (1 - sin2\theta)\epsilon + (1 + sin2\theta)\delta + (cos2\theta)\rho + (cos2\theta)t. \quad (*)$$

By Theorem 1, it follows that

$$\begin{aligned} 1 &\geq \max\{|T_1(B,B)|,\ |T_2(B,B)|\} \\ &= |T(B,B)| + |(\epsilon,\ \delta,\ \rho,\ t)(B,B)| \\ &= 1 + |(\epsilon,\ \delta,\ \rho,\ t)(B,B)|, \end{aligned}$$

which shows that

$$0 = (\epsilon,\ \delta,\ \rho, t)(B,B) = (1 + sin2\theta)\epsilon + (1 - sin2\theta)\delta - (cos2\theta)\rho - (cos2\theta)t. \quad (**)$$

By Theorem 1, it follows that

$$\begin{aligned} 1 &\geq \max\{|T_1(A,B)|,\ |T_2(A,B)|\} \\ &= |T(A,B)| + |(\epsilon,\ \delta,\ \rho,\ t)(A,B)| \\ &= 1 + |(\epsilon,\ \delta,\ \rho,\ t)(A,B)|, \end{aligned}$$

which shows that

$$0 = (\epsilon,\ \delta,\ \rho,\ t)(A,B) = (cos2\theta)\epsilon - (cos2\theta)\delta - (1 - sin2\theta)\rho + (1 + sin2\theta)t. \quad (***)$$

By Theorem 1, it follows that

$$\begin{aligned} 1 &\geq \max\{|T_1(B,A)|,\ |T_2(B,A)|\} \\ &= |T(B,A)| + |(\epsilon,\ \delta,\ \rho,\ t)(B,A)| \\ &= 1 + |(\epsilon,\ \delta,\ \rho,\ t)(B,A)|, \end{aligned}$$

which shows that

$$0 = (\epsilon,\ \delta,\ \rho,\ t)(B,A) = (cos2\theta)\epsilon - (cos2\theta)\delta + (1 + sin2\theta)\rho - (1 - sin2\theta)t. \quad (****)$$

Solving the equations of $(*) - (****)$, we get $\epsilon = \delta = \rho = t = 0$. Therefore, T is extreme. We complete the proof. $\square$

Theorem 3. *Let $0 \leq \theta < \frac{\pi}{2}$ and $T = (a,b,c,c) \in \mathcal{L}_s({}^2l^2_{\infty,\theta})$ with $\|T\| = 1$. Then, $T \in \operatorname{ext} B_{\mathcal{L}_s({}^2l^2_{\infty,\theta})}$ if and only if*

$$1 = |T(A,A)| = |T(B,B)| = |T(A,B)|.$$

Proof. ($\Rightarrow$). Suppose that $T \in \operatorname{ext} B_{\mathcal{L}_s(^2 l^2_{\infty,\theta})}$. Assume the assertion is not true. We have three cases.

Case 1. $|T(A,A)| < 1$.

Let $\theta = \frac{\pi}{4}$.
Since

$$1 > |T(A,A)| = |T((0,\sqrt{2}),\ (0,\sqrt{2}))| = 2|b|,$$

there is $N \in \mathbb{N}$ such that

$$\|T \pm \Big(0,\ \frac{1}{N},\ 0,\ 0\Big)\| = 1.$$

Let

$$T^{\pm} := T \pm \Big(0,\ \frac{1}{N},\ 0,\ 0\Big).$$

Hence, $T \notin \operatorname{ext} B_{\mathcal{L}_s(^2 l^2_{\infty,\theta})}$. This is a contradiction.

Suppose that $\theta \neq \frac{\pi}{4}$. Let

$$T^{\pm} := T \pm \Big(\frac{(1-sin2\theta)^2}{n(cos2\theta)^2},\ \frac{1}{n},\ \frac{1-sin2\theta}{ncos2\theta},\ \frac{1-sin2\theta}{ncos2\theta}\Big)$$

for a sufficiently large $n \in \mathbb{N}$ so that $\|T^{\pm}\| = 1$ for $j = 1,2$. Hence, $T \notin \operatorname{ext} B_{\mathcal{L}_s(^2 l^2_{\infty,\theta})}$. This is a contradiction.

Case 2. $|T(B,B)| < 1$.

Let $\theta = \frac{\pi}{4}$.
Since

$$1 > |T(B,B)| = |T((\sqrt{2},0),\ (\sqrt{2},0))| = 2|a|,$$

there is $N \in \mathbb{N}$ such that

$$\|T \pm \Big(\frac{1}{N},\ 0,\ 0,\ 0\Big)\| = 1.$$

Hence, $T \notin \operatorname{ext} B_{\mathcal{L}_s(^2 l^2_{\infty,\theta})}$. This is a contradiction.

Suppose that $\theta \neq \frac{\pi}{4}$. Let

$$T^{\pm} := T \pm \Big(-\frac{(1+sin2\theta)}{ncos2\theta},\ -\frac{cos2\theta}{n(1+sin2\theta)},\ \frac{1}{n},\ \frac{1}{n}\Big)$$

for a sufficiently large $n \in \mathbb{N}$ so that $\|T^{\pm}\| = 1$ for $j = 1,2$. Hence, $T \notin \operatorname{ext} B_{\mathcal{L}_s(^2 l^2_{\infty,\theta})}$. This is a contradiction.

Case 3. $|T(A,B)| < 1$.

Let $\theta = \frac{\pi}{4}$.
Since

$$1 > |T(A,B)| = |T((0,\sqrt{2}),\ (\sqrt{2},0))| = 2|d|,$$

there is $N \in \mathbb{N}$ such that

$$\|T \pm \Big(0,\ 0,\ \frac{1}{N},\ \frac{1}{N}\Big)\| = 1.$$

Hence, $T \notin \operatorname{ext} B_{\mathcal{L}_s(^2 l^2_{\infty,\theta})}$. This is a contradiction.

Suppose that $\theta \neq \frac{\pi}{4}$.
Let

$$T^{\pm} := T \pm \Big(\frac{1}{n},\ \ -\frac{1}{n},\ \frac{tan2\theta}{n},\ \frac{tan2\theta}{n}\Big)$$

for a sufficiently large $n \in \mathbb{N}$ so that $\|T^{\pm}\| = 1$ for $j = 1, 2$. Hence, $T \notin \operatorname{ext} B_{\mathcal{L}_s(^2 l^2_{\infty,\theta})}$. This is a contradiction. Therefore, the assertion is true.

($\Leftarrow$). Suppose that $1 = |T(A, A)| = |T(B, B)| = |T(A, B)|$. Since T is symmetric, $|T(B, A)| = 1$. By Theorem 2, $T \in \operatorname{ext} B_{\mathcal{L}(^2 l^2_{\infty,\theta})}$. Hence, $T \in \operatorname{ext} B_{\mathcal{L}_s(^2 l^2_{\infty,\theta})}$. $\square$

Theorem 4. *Let $0 \leq \theta < \frac{\pi}{2}$. Then,*

$$\begin{aligned}
&\operatorname{ext} B_{\mathcal{L}(^2 l^2_{\infty,\theta})} \\
= \ &\Big\{ \pm\Big(\frac{1}{2}(1+cos2\theta), \frac{1}{2}(1-cos2\theta), \frac{1}{2}sin2\theta, \frac{1}{2}sin2\theta\Big), \\
&\pm\Big(\frac{1}{2}(1-cos2\theta), \frac{1}{2}(1+cos2\theta), -\frac{1}{2}sin2\theta, -\frac{1}{2}sin2\theta\Big), \\
&\pm\Big(\frac{1}{2}(cos2\theta+sin2\theta), -\frac{1}{2}(cos2\theta+sin2\theta), \frac{1}{2}(-cos2\theta+sin2\theta), \\
&\qquad\qquad \frac{1}{2}(-cos2\theta+sin2\theta)\Big), \\
&\pm\Big(\frac{1}{2}(cos2\theta-sin2\theta), -\frac{1}{2}(cos2\theta-sin2\theta), \frac{1}{2}(cos2\theta+sin2\theta), \\
&\qquad\qquad \frac{1}{2}(cos2\theta+sin2\theta)\Big), \\
&\pm\Big(\frac{1}{2}sin2\theta, -\frac{1}{2}sin2\theta, \frac{1}{2}(1-cos2\theta), \frac{1}{2}(1+cos2\theta)\Big), \\
&\pm\Big(\frac{1}{2}sin2\theta, -\frac{1}{2}sin2\theta, -\frac{1}{2}(1+cos2\theta), \frac{1}{2}(1-cos2\theta)\Big), \pm\Big(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\Big), \\
&\pm\Big(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\Big) \Big\}.
\end{aligned}$$

Proof. By Theorem 2, it follows. $\square$

Theorem 5. *Let $0 \leq \theta < \frac{\pi}{2}$. Then,* $\operatorname{ext} B_{\mathcal{L}_s(^2 l^2_{\infty,\theta})} = \operatorname{ext} B_{\mathcal{L}(^2 l^2_{\infty,\theta})} \cap \mathcal{L}_s(^2 l^2_{\infty,\theta})$.

Proof. By Theorems 2 and 3, it follows. $\square$

Theorem 6. *Let $0 \leq \theta < \frac{\pi}{2}$. Then,*

$$
\begin{aligned}
&\text{ext}\, B_{\mathcal{L}_s({}^2 l^2_{\infty,\theta})} \\
= &\Big\{ \pm\Big(\frac{1}{2}(1+cos2\theta), \frac{1}{2}(1-cos2\theta), \frac{1}{2}sin2\theta, \frac{1}{2}sin2\theta\Big), \\
&\pm\Big(\frac{1}{2}(1-cos2\theta), \frac{1}{2}(1+cos2\theta), -\frac{1}{2}sin2\theta, -\frac{1}{2}sin2\theta\Big), \\
&\pm\Big(\frac{1}{2}(cos2\theta+sin2\theta), -\frac{1}{2}(cos2\theta+sin2\theta), \frac{1}{2}(-cos2\theta+sin2\theta), \\
&\qquad\qquad \frac{1}{2}(-cos2\theta+sin2\theta)\Big), \\
&\pm\Big(\frac{1}{2}(cos2\theta-sin2\theta), -\frac{1}{2}(cos2\theta-sin2\theta), \frac{1}{2}(cos2\theta+sin2\theta), \\
&\qquad\qquad \frac{1}{2}(cos2\theta+sin2\theta)\Big) \Big\}.
\end{aligned}
$$

Notice that $\Big|\text{ext}\, B_{\mathcal{L}({}^2 l^2_{\infty,\theta})} \backslash \text{ext}\, B_{\mathcal{L}_s({}^2 l^2_{\infty,\theta})}\Big| = 8 = \Big|\text{ext}\, B_{\mathcal{L}_s({}^2 l^2_{\infty,\theta})}\Big|$.

3 The exposed points of the unit balls of $\mathcal{L}({}^2 l^2_{\infty,\theta})$ and $\mathcal{L}_s({}^2 l^2_{\infty,\theta})$

The following presents an explicit formulae for the norm of $f \in \mathcal{L}({}^2 l^2_{\infty,\theta})^*$.

Theorem 7. *Let $0 \leq \theta < \frac{\pi}{2}$ and $f \in \mathcal{L}({}^2 l^2_{\infty,\theta})^*$ be such that $\alpha := f(x_1x_2),\ \beta := f(y_1y_2),\ \gamma := f(x_1y_2),\ \rho := f(x_2y_1)$. Then,*

$$
\begin{aligned}
\|f\| = \frac{1}{2}\ \max\Big\{ &\Big|(1+cos2\theta)\alpha + (1-cos2\theta)\beta + (sin2\theta)\gamma + (sin2\theta)\rho\Big|, \\
&\Big|(1-cos2\theta)\alpha + (1+cos2\theta)\beta - (sin2\theta)\gamma - (sin2\theta)\rho\Big|, \\
&\Big|(cos2\theta+sin2\theta)\alpha - (cos2\theta+sin2\theta)\beta + (-cos2\theta+sin2\theta)\gamma \\
&\qquad\qquad + (-cos2\theta+sin2\theta)\rho\Big|, \\
&\Big|(cos2\theta-sin2\theta)\alpha - (cos2\theta-sin2\theta)\beta + (cos2\theta+sin2\theta)\gamma \\
&\qquad\qquad + (cos2\theta+sin2\theta)\rho\Big|, \\
&\Big|(sin2\theta)\alpha - (sin2\theta)\beta + (1-cos2\theta)\gamma + (1+cos2\theta)\rho\Big|, \\
&\Big|(sin2\theta)\alpha - (sin2\theta)\beta - (1+cos2\theta)\gamma + (1-cos2\theta)\rho\Big|, |\alpha+\beta| + |\gamma-\rho|\Big\}.
\end{aligned}
$$

Proof. It follows from Theorem 4 and the fact that

$$\|f\| = \sup_{T \in \text{ext}\, B_{\mathcal{L}({}^2 l^2_{\infty,\theta})}} |f(T)|.$$

□

Notice that if $\|f\| = 1$, then

$$|\alpha| \leq 1 + sin2\theta,\ |\beta| \leq 1 + sin2\theta,\ |\gamma| \leq 1 + sin2\theta,\ |\rho| \leq 1 + sin2\theta.$$

Theorem 8. ([23]) *Let E be a real Banach space such that $\mathrm{ext}\, B_E$ is finite. Suppose that $x \in \mathrm{ext}\, B_E$ satisfies that there exists an $f \in E^*$ with $f(x) = 1 = \|f\|$ and $|f(y)| < 1$ for every $y \in \mathrm{ext}\, B_E \backslash \{\pm x\}$. Then $x \in \exp B_E$.*

Theorem 9. $\exp B_{\mathcal{L}(^2 l^2_{\infty,\theta})} = \mathrm{ext}\, B_{\mathcal{L}(^2 l^2_{\infty,\theta})}$ *for $0 \leq \theta < \frac{\pi}{2}$.*

Proof. It suffices to show that if $T \in \mathrm{ext}\, B_{\mathcal{L}(^2 l^2_{\infty,\theta})}$, then T is exposed. Let $T \in \mathrm{ext}\, B_{\mathcal{L}(^2 l^2_{\infty,\theta})}$. We define $f \in \mathcal{L}(^2 l^2_{\infty,\theta})^*$ by

$$\begin{aligned} f &= \frac{1}{4}\Big(\mathrm{sign}(T(A,A))\delta_{A,A} + \mathrm{sign}(T(B,B))\delta_{B,B} + \mathrm{sign}(T(A,B))\delta_{A,B} \\ &\quad + \mathrm{sign}(T(B,A))\delta_{B,A}\Big), \end{aligned}$$

where $\delta_{A,A}(S) := S(A,A)$ for $S \in \mathcal{L}(^2 l^2_{\infty,\theta})$. By Theorem 7, $f(T) = 1 = \|f\|$.

Claim. if $S \in \mathrm{ext}\, B_{\mathcal{L}(^2 l^2_{\infty,\theta})}$ such that $|f(S)| = 1$, then $S = T$ or $S = -T$.

Obviously,

$$\begin{aligned} &\Big(S(A,A) = T(A,A),\ S(B,B) = T(B,B),\ S(A,B) = T(A,B), \\ &S(B,A) = T(B,A)\Big) \text{ or } \Big(S(A,A) = -T(A,A),\ S(B,B) = -T(B,B), \\ &S(A,B) = -T(A,B), S(B,A) = -T(B,A)\Big). \end{aligned}$$

Since $\{A, B\}$ is a basis for $l^2_{\infty,\theta}$, $S = T$ or $S = -T$, respectively. By Theorem 8, T is exposed. We complete the proof. □

Theorem 10. $\exp B_{\mathcal{L}_s(^2 l^2_{\infty,\theta})} = \mathrm{ext}\, B_{\mathcal{L}_s(^2 l^2_{\infty,\theta})}$ *for $0 \leq \theta < \frac{\pi}{2}$.*

Proof. By Theorems 5 and 9,

$$\mathrm{ext}\, B_{\mathcal{L}_s(^2 l^2_{\infty,\theta})} = \mathrm{ext}\, B_{\mathcal{L}(^2 l^2_{\infty,\theta})} \cap \mathcal{L}_s(^2 l^2_{\infty,\theta}) = \exp B_{\mathcal{L}(^2 l^2_{\infty,\theta})} \cap \mathcal{L}_s(^2 l^2_{\infty,\theta}).$$

Let $T \in \mathrm{ext}\, B_{\mathcal{L}_s(^2 l^2_{\infty,\theta})}$. Then, $T \in \exp B_{\mathcal{L}(^2 l^2_{\infty,\theta})}$. Then there is $f \in \mathcal{L}(^2 l^2_{\infty,\theta})^*$ such that $f(T) = 1 = \|f\|$ and $f(S) < 1$ for all $S \in B_{\mathcal{L}(^2 l^2_{\infty,\theta})} \backslash \{T\}$. Let $f_1 := f|_{\mathcal{L}_s(^2 l^2_{\infty,\theta})}$. Obviously, $f_1(T) = 1 = \|f_1\|$ and $f_1(R) < 1$ for all $R \in B_{\mathcal{L}_s(^2 l^2_{\infty,\theta})} \backslash \{T\}$. Hence, $T \in \exp B_{\mathcal{L}_s(^2 l^2_{\infty,\theta})}$. □

Theorem 11. $\exp B_{\mathcal{L}_s(^2 l^2_{\infty,\theta})} = \exp B_{\mathcal{L}(^2 l^2_{\infty,\theta})} \cap \mathcal{L}_s(^2 l^2_{\infty,\theta})$ *for $0 \leq \theta < \frac{\pi}{2}$.*

Proof. It follows from Theorems 5, 9 and 10. □

4 The smooth points of the unit balls of $\mathcal{L}(^2l^2_{\infty,\theta})$ and $\mathcal{L}_s(^2l^2_{\infty,\theta})$

The main result about smooth points is known as "the Mazur density theorem." Recall that the Mazur density theorem [16, p. 171] says that the set of all the smooth points of a solid closed convex subset of a separable Banach space is a residual subset of its boundary.

Theorem 12. *Let $0 \leq \theta < \frac{\pi}{2}$ and $T = (a, b, c, d) \in \mathcal{L}(^2l^2_{\infty,\theta})$ with $\|T\| = 1$. Then, $T \in \mathrm{sm}\, B_{\mathcal{L}(^2l^2_{\infty,\theta})}$ if and only if there is unique $X \in \{(A, A), (B, B), (A, B), (B, A)\}$ such that $|T(X)| = 1$ and $|T(Y)| < 1$ for every $Y \in \{(A, A), (B, B), (A, B), (B, A)\} \setminus\{X\}$.*

Proof. ($\Rightarrow$). Assume the assertion is not true.

Suppose that $|T(A, A)| = 1,\ |T(B, B)| = 1$. Let $f_1 = \mathrm{sign}(T(A, A))\delta_{A,A}$ and $f_2 = \mathrm{sign}(T(B, B))\delta_{B,B}$ be elements of $\mathcal{L}(^2l^2_{\infty,\theta})^*$. Notice that

$$f_1 \neq f_2,\ \|f_j\| = 1 = f_j(T) \text{ for } j = 1, 2.$$

Hence, T is not a smooth point. This is a contradiction. Similarly, we conclude that the other cases reach a contradiction. Therefore, the assertion is true.

($\Leftarrow$). Let $f \in \mathcal{L}(^2l^2_{\infty,\theta})^*$ be such that $1 = \|f\| = f(T)$ with $\alpha := f(x_1x_2),\ \beta := f(y_1y_2),\ \gamma := f(x_1y_2)$ and $\rho := f(x_2y_1)$.

Case 1. $|T(A, A)| = 1,\ |T(B, B)| < 1,\ |T(A, B)| < 1,\ |T(B, A)| < 1.$

Without loss of generality we may assume that $T(A, A) = 1$. Let $\theta = \frac{\pi}{4}$. We will show that $\alpha = \gamma = \rho = 0,\ \beta = 2$. Since

$$T(A, A) = 1,\ |T(B, B)| < 1,\ |T(A, B)| < 1,\ |T(B, A)| < 1$$

we have

$$b = \frac{1}{2},\ |a| < \frac{1}{2},\ |c| < \frac{1}{2},\ |d| < \frac{1}{2}.$$

By Theorem 1, there is $N \in \mathbb{N}$ such that

$$1 = \Big\|T \pm \Big(\frac{1}{N},\ 0,\ 0,\ 0\Big)\Big\| = \Big\|T \pm \Big(0,\ 0,\ \frac{1}{N},\ 0\Big)\Big\| = \Big\|T \pm \Big(0,\ 0,\ 0,\ \frac{1}{N}\Big)\Big\|.$$

It follows that

$$\begin{aligned} 1 &\geq \max\Big\{ \Big|f\Big(T \pm \Big(\frac{1}{N},\ 0,\ 0,\ 0\Big)\Big)\Big|,\ \Big|f\Big(T \pm \Big(0,\ 0,\ \frac{1}{N},\ 0\Big)\Big)\Big|, \\ &\qquad \Big|f\Big(T \pm \Big(0,\ 0,\ 0,\ \frac{1}{N}\Big)\Big)\Big| \Big\} \\ &= \max\{ 1 + \Big|f\Big(\Big(\frac{1}{N},\ 0,\ 0,\ 0\Big)\Big)\Big|,\ 1 + \Big|f\Big(\Big(0,\ 0,\ \frac{1}{N},\ 0\Big)\Big)\Big|, \\ &\qquad 1 + \Big|f\Big(\Big(0,\ 0,\ 0,\ \frac{1}{N}\Big)\Big)\Big| \Big\} \end{aligned}$$

which shows that

$$0 = f\Big(\Big(\frac{1}{N},\ 0,\ 0,\ 0\Big)\Big) = f\Big(\Big(0,\ 0,\ \frac{1}{N},\ 0\Big)\Big) = f\Big(\Big(0,\ 0,\ 0,\ \frac{1}{N}\Big)\Big).$$

Hence, $\alpha = \gamma = \rho = 0$. Since

$$1 = f(T) = a\alpha + b\beta + c\gamma + d\rho = \frac{1}{2}\beta,$$

we have $\beta = 2$. Hence, T is a smooth point.

Suppose that $\theta \neq \frac{\pi}{4}$. Since $T(A, A) = 1$, $|T(B, B)| < 1$, $|T(A, B)| < 1$, $|T(B, A)| < 1$, by Theorem 1, there is $N \in \mathbb{N}$ such that

$$\begin{aligned}
1 &= \Big\|T \pm \Big(\frac{1}{N},\ \frac{-1 + sin2\theta}{N(1 + sin2\theta)},\ 0,\ 0\Big)\Big\|,\\
1 &= \Big\|T \pm \Big(\frac{1}{N},\ 0,\ \frac{-1 + sin2\theta}{Ncos2\theta},\ 0\Big)\Big\|,\\
1 &= \Big\|T \pm \Big(\frac{1}{N},\ 0,\ 0,\ \frac{-1 + sin2\theta}{Ncos2\theta}\Big)\Big\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
1 \ &\geq\ \max\Big\{\Big|f\Big(T \pm \Big(\frac{1}{N}, \frac{-1 + sin2\theta}{N(1 + sin2\theta)},\ 0,\ 0\Big)\Big)\Big|,\\
&\qquad \Big|f\Big(T \pm \Big(\frac{1}{N},\ 0,\ \frac{-1 + sin2\theta}{Ncos2\theta},\ 0\Big)\Big|,\\
&\qquad \Big|f\Big(T \pm \Big(\frac{1}{N},\ 0,\ 0,\ \frac{-1 + sin2\theta}{Ncos2\theta}\Big)\Big)\Big|\ \Big\}\\
&=\ \max\Big\{|f(T)| + \Big|f\Big(\Big(\frac{1}{N},\ \frac{-1 + sin2\theta}{N(1 + sin2\theta)},\ 0,\ 0\Big)\Big)\Big|,\\
&\qquad |f(T)| + \Big|f\Big(\Big(\frac{1}{N},\ 0,\ \frac{-1 + sin2\theta}{Ncos2\theta},\ 0\Big)\Big)\Big|,\\
&\qquad |f(T)| + \Big|f\Big(\Big(\frac{1}{N},\ 0,\ 0,\ \frac{-1 + sin2\theta}{Ncos2\theta}\Big)\Big)\Big|\Big\}\\
&=\ \max\Big\{1 + \Big|f\Big(\Big(\frac{1}{N},\ \frac{-1 + sin2\theta}{N(1 + sin2\theta)},\ 0,\ 0\Big)\Big)\Big|,\\
&\qquad 1 + \Big|f\Big(\Big(\frac{1}{N},\ 0,\ \frac{-1 + sin2\theta}{Ncos2\theta},\ 0\Big)\Big)\Big|,\\
&\qquad 1 + \Big|f\Big(\Big(\frac{1}{N},\ 0,\ 0,\ \frac{-1 + sin2\theta}{Ncos2\theta}\Big)\Big)\Big|\ \Big\},
\end{aligned}$$

which shows that

$$\begin{aligned}
0 &= f\Big(\Big(\frac{1}{N},\ \frac{-1 + sin2\theta}{N(1 + sin2\theta)},\ 0,\ 0\Big)\Big) = f\Big(\Big(\frac{1}{N},\ 0,\ \frac{-1 + sin2\theta}{Ncos2\theta},\ 0\Big)\Big)\\
&= f\Big(\Big(\frac{1}{N},\ 0,\ 0,\ \frac{-1 + sin2\theta}{Ncos2\theta}\Big)\Big).
\end{aligned}$$

Hence,

$$\beta = \Big(\frac{1 + sin2\theta}{1 - sin2\theta}\Big)\alpha,\ \gamma = \Big(\frac{cos2\theta}{1 - sin2\theta}\Big)\alpha,\ \rho = \Big(\frac{cos2\theta}{1 - sin2\theta}\Big)\alpha.$$

It follows that

$$\begin{aligned} 1 &= f(T) = a\alpha + b\beta + c\gamma + d\rho \\ &= \alpha\Big(a + \Big(\frac{1+sin2\theta}{1-sin2\theta}\Big)b + \Big(\frac{cos2\theta}{1-sin2\theta}\Big)c + \Big(\frac{cos2\theta}{1-sin2\theta}\Big)d\Big) \\ &= \frac{\alpha}{1-sin2\theta}\Big((1-sin2\theta)a + (1+sin2\theta)b + (cos2\theta)c + (cos2\theta)d\Big) \\ &= \frac{\alpha}{1-sin2\theta}\ T(A,A) = \frac{\alpha}{1-sin2\theta}, \end{aligned}$$

which shows that

$$\alpha = 1 - sin2\theta,\ \beta = 1 + sin2\theta,\ \gamma = \rho = cos2\theta.$$

Since f is unique, T is a smooth point.

Case 2. $|T(B,B)| = 1,\ |T(A,A)| < 1,\ |T(A,B)| < 1,\ |T(B,A)| < 1.$

Without loss of generality we may assume that $T(B,B) = 1$. Let $\theta = \frac{\pi}{4}$. By analogous arguments in the case 1, $\alpha = 2,\ \beta = \gamma = \rho = 0$. Hence, T is a smooth point.

Suppose that $\theta \neq \frac{\pi}{4}$. By analogous arguments in the case 1,

$$\alpha = 1 + sin2\theta,\ \beta = 1 - sin2\theta,\ \gamma = \rho = -cos2\theta.$$

Hence, T is a smooth point.

Case 3. $|T(A,B)| = 1,\ |T(A,A)| < 1,\ |T(B,B)| < 1,\ |T(B,A)| < 1.$

Notice that if $\theta = \frac{\pi}{4}$, then $\alpha = \beta = \gamma = 0, \rho = 2$.

Suppose that $\theta \neq \frac{\pi}{4}$. Without loss of generality we may assume that $T(A,B) = 1$. Since $T(A,B) = 1,\ |T(A,A)| < 1,\ |T(B,B)| < 1,\ |T(B,A)| < 1$, by Theorem 1, there is $N \in \mathbb{N}$ such that

$$\begin{aligned} 1 &= \Big\|T \pm \Big(\frac{1}{N},\ \frac{1}{N},\ 0,\ 0\Big)\Big\|, \\ 1 &= \Big\|T \pm \Big(\frac{1}{N},\ 0,\ \frac{cos2\theta}{N(1-sin2\theta)},\ 0\Big)\Big\|, \\ 1 &= \Big\|T \pm \Big(\frac{1}{N},\ 0,\ 0,\ -\frac{cos2\theta}{N(1+sin2\theta)}\Big)\Big\|. \end{aligned}$$

It follows that

$$\begin{aligned}
1 &\geq \max\Big\{\Big|f\Big(T \pm \Big(\frac{1}{N},\ \frac{1}{N},\ 0,\ 0\Big)\Big)\Big|,\ \Big|f\Big(T \pm \Big(\frac{1}{N},\ 0,\ \frac{cos2\theta}{N(1-sin2\theta)},\ 0\Big)\Big)\Big|,\\
&\qquad |f\Big(T \pm \Big(\frac{1}{N},\ 0,\ 0,\ \ -\frac{cos2\theta}{N(1+sin2\theta)}\Big)\Big)\Big|\ \Big\}\\
&= \max\Big\{|f(T)| + \Big|f\Big(\Big(\frac{1}{N},\ \frac{1}{N},\ 0,\ 0\Big)\Big)\Big|,\\
&\qquad |f(T)| + \Big|f\Big(\Big(\frac{1}{N},\ 0,\ \frac{cos2\theta}{N(1-sin2\theta)},\ 0\Big)\Big)\Big|,\\
&\qquad |f(T)| + \Big|f\Big(\frac{1}{N},\ 0,\ 0,\ \ -\frac{cos2\theta}{N(1+sin2\theta)}\Big)\Big|\ \Big\}\\
&= \max\Big\{1 + \Big|f\Big(\Big(\frac{1}{N},\ \frac{1}{N},\ 0,\ 0\Big)\Big)\Big|,\ 1 + \Big|f\Big(\Big(\frac{1}{N},\ 0,\ \frac{cos2\theta}{N(1-sin2\theta)},\ 0\Big)\Big)\Big|,\\
&\qquad 1 + \Big|f\Big(\Big(\frac{1}{N},\ 0,\ 0,\ \ -\frac{cos2\theta}{N(1+sin2\theta)}\Big)\Big)\Big|\ \Big\},
\end{aligned}$$

which shows that

$$\begin{aligned}
0 &= f\Big(\Big(\frac{1}{N},\ \frac{1}{N},\ 0,\ 0\Big)\Big) = f\Big(\Big(\frac{1}{N},\ 0,\ \frac{cos2\theta}{N(1-sin2\theta)},\ 0\Big)\Big)\\
&= f\Big(\Big(\frac{1}{N},\ 0,\ 0,\ \ -\frac{cos2\theta}{N(1+sin2\theta)}\Big)\Big).
\end{aligned}$$

Hence,

$$\beta = -\alpha,\ \gamma = \Big(\frac{-1+sin2\theta}{cos2\theta}\Big)\alpha,\ \rho = \Big(\frac{1+sin2\theta}{cos2\theta}\Big)\alpha.$$

It follows that

$$\begin{aligned}
1 &= f(T) = a\alpha + b\beta + c\gamma + d\rho\\
&= \alpha\Big(a - b + \Big(\frac{-1+sin2\theta}{cos2\theta}\Big)c + \Big(\frac{1+sin2\theta}{cos2\theta}\Big)d\Big)\\
&= \frac{\alpha}{cos2\theta}\Big(cos2\theta(a-b) + (-1+sin2\theta)c + (1+sin2\theta)d\Big)\\
&= \frac{\alpha}{cos2\theta}\ T(A,B) = \frac{\alpha}{cos2\theta},
\end{aligned}$$

which shows that

$$\alpha = -\beta = cos2\theta,\ \gamma = -1 + sin2\theta,\ \rho = 1 + sin2\theta.$$

Hence, T is a smooth point.

Case 4. $|T(B,A)| = 1,\ |T(A,A)| < 1,\ |T(B,B)| < 1,\ |T(A,B)| < 1.$

By analogous arguments in the case 1, if $\theta = \frac{\pi}{4}$, then $\alpha = \beta = \rho = 0,\ \gamma = 2$ and if $\theta \neq \frac{\pi}{4}$, then

$$\alpha = -\beta = cos2\theta,\ \gamma = 1 + sin2\theta,\ \rho = -1 + sin2\theta.$$

Hence, T is a smooth point. We complete the proof. $\square$

Theorem 13. *Let $0 \leq \theta < \frac{\pi}{2}$ and $T = (a, b, c, c) \in \mathcal{L}_s({}^2l^2_{\infty,\theta})$ with $\|T\| = 1$. Then, $T \in \operatorname{sm} B_{\mathcal{L}_s({}^2l^2_{\infty,\theta})}$ if and only if there is unique $X \in \{(A, A), (B, B), (A, B)\}$ such that $|T(X)| = 1$ and $|T(Y)| < 1$ for every $Y \in \{(A, A), (B, B), (A, B)\}\backslash\{X\}$.*

Proof. We follow analogous arguments in the proof of Theorem 12.

($\Rightarrow$) follows by the same argument in the proof ($\Rightarrow$) of Theorem 12.

($\Leftarrow$). Let $g \in \mathcal{L}_s({}^2l^2_{\infty,\theta})^*$ be such that $g(T) = 1 = \|g\|$ and $\alpha = g(x_1x_2),\ \beta = g(y_1y_2),\ \gamma = g(x_1y_2 + x_2y_1)$.

Case 1. $|T(A, A)| = 1,\ |T(B, B)| < 1,\ |T(A, B)| < 1.$

Without loss of generality we may assume that $T(A, A) = 1$. Let $\theta = \frac{\pi}{4}$. We will show that $\alpha = \gamma = 0,\ \beta = 2$. Since

$$T(A, A) = 1,\ |T(B, B)| < 1,\ |T(A, B)| < 1$$

we have

$$b = \frac{1}{2},\ |a| < \frac{1}{2},\ |c| < \frac{1}{2}.$$

By Theorem 1, there is $N \in \mathbb{N}$ such that

$$1 \ = \ \Big\|T \pm \Big(\frac{1}{N},\ 0,\ 0,\ 0\Big)\Big\| = \Big\|T \pm \Big(0,\ 0,\ \frac{1}{N},\ \frac{1}{N}\Big)\Big\|.$$

It follows that

$$\begin{aligned} 1 \ &\geq \ \max\Big\{ \ \Big|f\Big(T \pm \Big(\frac{1}{N},\ 0,\ 0,\ 0\Big)\Big)\Big|,\ \Big|f\Big(T \pm \Big(0,\ 0,\ \frac{1}{N},\ \frac{1}{N}\Big)\Big)\Big|, \\ &= \ \max\{\ 1 + \Big|f\Big(\Big(\frac{1}{N},\ 0,\ 0,\ 0\Big)\Big)\Big|,\ 1 + \Big|f\Big(\Big(0,\ 0,\ \frac{1}{N},\ \frac{1}{N}\Big)\Big)\Big|\ \Big\} \end{aligned}$$

which shows that

$$0 = f\Big(\Big(\frac{1}{N},\ 0,\ 0,\ 0\Big)\Big) = f\Big(\Big(0,\ 0,\ \frac{1}{N},\ \frac{1}{N}\Big)\Big).$$

Hence, $\alpha = \gamma = 0$. Since

$$1 = f(T) = a\alpha + b\beta + c\gamma = \frac{1}{2}\beta,$$

we have $\beta = 2$. Hence, T is a smooth point.

Suppose that $\theta \neq \frac{\pi}{4}$. Since $T(A, A) = 1,\ |T(B, B)| < 1,\ |T(A, B)| < 1,\ |T(B, A)| < 1$, by Theorem 1, there is $N \in \mathbb{N}$ such that

$$\begin{aligned} 1 \ &= \ \Big\|T \pm \Big(\frac{1}{N}, \frac{-1 + sin2\theta}{N(1 + sin2\theta)},\ 0,\ 0\Big)\Big\|, \\ 1 \ &= \ \Big\|T \pm \Big(\frac{1}{N},\ 0,\ \frac{-1 + sin2\theta}{2Ncos2\theta},\ \frac{-1 + sin2\theta}{2Ncos2\theta}\Big)\Big\|. \end{aligned}$$

It follows that

$$\begin{aligned}
1 &\geq \max\Big\{\Big|f\Big(T \pm \Big(\frac{1}{N}, \frac{-1+sin2\theta}{N(1+sin2\theta)},\ 0,\ 0\Big)\Big)\Big|, \\
&\Big|f\Big(T \pm \Big(\frac{1}{N},\ 0,\ \frac{-1+sin2\theta}{2Ncos2\theta},\ \frac{-1+sin2\theta}{2Ncos2\theta}\Big)\Big)\Big|, \\
&= \max\Big\{|f(T)| + \Big|f\Big(\Big(\frac{1}{N},\ \frac{-1+sin2\theta}{N(1+sin2\theta)},\ 0,\ 0\Big)\Big)\Big|, \\
&|f(T)| + \Big|f\Big(\Big(\frac{1}{N},\ 0,\ \frac{-1+sin2\theta}{2Ncos2\theta},\ \frac{-1+sin2\theta}{2Ncos2\theta}\Big)\Big)\Big|\Big\} \\
&= \max\Big\{1 + \Big|f\Big(\Big(\frac{1}{N},\ \frac{-1+sin2\theta}{N(1+sin2\theta)},\ 0,\ 0\Big)\Big)\Big|, \\
&1 + \Big|f\Big(\Big(\frac{1}{N},\ 0,\ \frac{-1+sin2\theta}{2Ncos2\theta},\ \frac{-1+sin2\theta}{2Ncos2\theta}\Big)\Big)\Big|\ \Big\},
\end{aligned}$$

which shows that

$$0 = f\Big(\Big(\frac{1}{N},\ \frac{-1+sin2\theta}{N(1+sin2\theta)},\ 0,\ 0\Big)\Big) = f\Big(\Big(\frac{1}{N},\ 0,\ \frac{-1+sin2\theta}{2Ncos2\theta},\ \frac{-1+sin2\theta}{2Ncos2\theta}\Big)\Big).$$

Hence,

$$\beta = \Big(\frac{1+sin2\theta}{1-sin2\theta}\Big)\alpha,\ \gamma = \Big(\frac{2cos2\theta}{1-sin2\theta}\Big)\alpha.$$

It follows that

$$\begin{aligned}
1 &= f(T) = a\alpha + b\beta + c\gamma \\
&= \alpha\Big(a + \Big(\frac{1+sin2\theta}{1-sin2\theta}\Big)b + \Big(\frac{2cos2\theta}{1-sin2\theta}\Big)c\Big) \\
&= \frac{\alpha}{1-sin2\theta}\Big((1-sin2\theta)a + (1+sin2\theta)b + 2(cos2\theta)c\Big) \\
&= \frac{\alpha}{1-sin2\theta}\ T(A,A) = \frac{\alpha}{1-sin2\theta},
\end{aligned}$$

which shows that

$$\alpha = 1 - sin2\theta,\ \beta = 1 + sin2\theta,\ \gamma = 2cos2\theta.$$

Since g is unique, T is a smooth point.

Case 2. $|T(B,B)| = 1,\ |T(A,A)| < 1,\ |T(A,B)| < 1.$

Without loss of generality we may assume that $T(B,B) = 1$. Let $\theta = \frac{\pi}{4}$. By analogous arguments in the case 1, $\alpha = 2,\ \beta = \gamma = 0$. Hence, T is a smooth point.

Suppose that $\theta \neq \frac{\pi}{4}$. By analogous arguments in the case 1,

$$\alpha = 1 + sin2\theta,\ \beta = 1 - sin2\theta,\ \gamma = -2cos2\theta.$$

Hence, T is a smooth point.

Case 3. $|T(A,B)| = 1,\ |T(A,A)| < 1,\ |T(B,B)| < 1.$

Notice that if $\theta=0$, then $\alpha=-\beta=1,\ \gamma=0$ and that if $\theta=\frac{\pi}{4}$, then $\alpha=\beta=0, \gamma=2$.

Suppose that $\theta\neq 0$ and $\theta\neq\frac{\pi}{4}$. Without loss of generality we may assume that $T(A,B)=1$. Since $T(A,B)=1,\ |T(A,A)|<1,\ |T(B,B)|<1$, by Theorem 1, there is $N\in\mathbb{N}$ such that

$$\begin{aligned}1 &= \Big\|T\pm\Big(\frac{1}{N},\ \frac{1}{N},\ 0,\ 0\Big)\Big\|,\\ 1 &= \Big\|T\pm\Big(\frac{1}{N},\ 0,\ \frac{-cos2\theta}{2Nsin2\theta},\ \frac{-cos2\theta}{2Nsin2\theta}\Big)\Big\|.\end{aligned}$$

It follows that

$$\begin{aligned}1 &\geq \max\Big\{\Big|f\Big(T\pm\Big(\frac{1}{N},\ \frac{1}{N},\ 0,\ 0\Big)\Big)\Big|,\Big|f\Big(T\pm\Big(\frac{1}{N},\ 0,\frac{-cos2\theta}{2Nsin2\theta},\frac{-cos2\theta}{2Nsin2\theta}\Big)\Big)\Big|\Big\}\\ &= \max\Big\{|f(T)|+\Big|f\Big(\Big(\frac{1}{N},\ \frac{1}{N},\ 0,\ 0\Big)\Big)\Big|,\\ &\qquad |f(T)|+\Big|f\Big(\Big(\frac{1}{N},\ 0,\ \frac{-cos2\theta}{2Nsin2\theta},\ \frac{-cos2\theta}{2Nsin2\theta}\Big)\Big)\Big|\Big\}\\ &= \max\Big\{1+\Big|f\Big(\Big(\frac{1}{N},\ \frac{1}{N},\ 0,\ 0\Big)\Big)\Big|,1+\Big|f\Big(\Big(\frac{1}{N},\ 0,\ \frac{-cos2\theta}{2Nsin2\theta},\ \frac{-cos2\theta}{2Nsin2\theta}\Big)\Big)\Big|\Big\}\end{aligned}$$

which shows that

$$0 \;=\; f\Big(\Big(\frac{1}{N},\ \frac{1}{N},\ 0,\ 0\Big)\Big)=f\Big(\Big(\frac{1}{N},\ 0,\ \frac{-cos2\theta}{2Nsin2\theta},\ \frac{-cos2\theta}{2Nsin2\theta}\Big)\Big).$$

Hence,

$$\beta=-\alpha,\ \gamma=\Big(\frac{2sin2\theta}{cos2\theta}\Big)\alpha.$$

It follows that

$$\begin{aligned}1 &= f(T)=a\alpha+b\beta+c\gamma\\ &= \alpha\Big(a-b+\Big(\frac{2sin2\theta}{cos2\theta}\Big)c\Big)\\ &= \frac{\alpha}{cos2\theta}\Big(cos2\theta(a-b)+(2sin2\theta)c\Big)\\ &= \frac{\alpha}{cos2\theta}\ T(A,B)=\frac{\alpha}{cos2\theta},\end{aligned}$$

which shows that

$$\alpha=-\beta=cos2\theta,\ \gamma=2sin2\theta.$$

Hence, T is a smooth point. $\square$

Theorem 14. *Let* $0\leq\theta<\frac{\pi}{2}$. *Then,* $\operatorname{sm}B_{\mathcal{L}({}^2l^2_{\infty,\theta})}\cap\mathcal{L}_s({}^2l^2_{\infty,\theta})\subsetneq\operatorname{sm}B_{\mathcal{L}_s({}^2l^2_{\infty,\theta})}$.

Proof. From Theorems 12 and 13, $\operatorname{sm}B_{\mathcal{L}({}^2l^2_{\infty,\theta})}\cap\mathcal{L}_s({}^2l^2_{\infty,\theta})$ is a subset of $\operatorname{sm}B_{\mathcal{L}_s({}^2l^2_{\infty,\theta})}$. Let $T_0\in\operatorname{sm}B_{\mathcal{L}_s({}^2l^2_{\infty,\theta})}$ be such that

$$|T_0(A,B)|=1,\ |T_0(A,A)|<1,\ |T_0(B,B)|<1.$$

Since $|T_0(B,A)|=1$, by Theorem 12, $T_0\notin\operatorname{sm}B_{\mathcal{L}({}^2l^2_{\infty,\theta})}\cap\mathcal{L}_s({}^2l^2_{\infty,\theta})$. We complete the proof. $\square$

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