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# ON DISCRETE q-DERIVATIVES OF q-BERNSTEIN OPERATORS

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**Abstract**

In the present paper we shall investigate the pointwise approximation properties of the q analogue of the Bernstein operators and estimate the rate of pointwise convergence of these operators to the functions $f$ whose q-derivatives are bounded variation on the interval $[0,1]$. We give an estimate for the rate of convergence of the operator $(B_{n,q}f)$ at those points $x$ at which the one sided q-derivatives $D_q^+ f(x), D_q^- f(x)$ exist. We shall also prove that the operators $B_{n,q}f$ converge to the limit $f$. As a continuation of the very recent study of the author on the q-Bernstein Durrmeyer operators [10], the present study will be the first study on the approximation of q analogous of the discrete type operators in the space of $D_qBV$ .

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*Key words:* q-Bernstein operators, pointwise approximation, right and left q-derivatives, convergence rate.

## 1 Introduction

The theory of quantum calculus (or q analysis) has attracted the attention of several researchers (see [5], [12]-[15], [19]-[22] and the references therein), because of its potential for applications, since this theory can be used to investigate black holes, quantum mechanics, nuclear and high energy physics, mathematical physics, functional analysis and especially in the last decades approximation theory and so on. The subject of $q-$analysis concerns mainly the properties of the $q-$special functions, which are the extensions of the classical special functions based on a parameter, or the base, q.

Due to this relation, in the year 1997, Phillips [19] introduced the generalization of Bernstein polynomials based on q-integers to prove the approximation (or superposition) problem of Weierstrass. It is useful to point out that, the first

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person to define the $q-$analog of the Bernstein operators was Lupaş [14], see also [17].

Before getting onto the main subject, we first give definitions of $q-number$, $q-binomial$ coefficient and $q-derivative$, together with their properties of $q-$calculus, which are required in this paper.

Let us recall the definition of $q-integers$. For any fixed real number $q>0$ and non-negative integer $r$ the $q-integer$ of the number $r$ is defined by

$$[r]_q = \begin{cases} (1-q^r)/(1-q) & , \quad q \neq 1 \\ r & , \quad q = 1 \end{cases}.$$

and the set of $q-integers$ is defined as:

$$\mathbb{N}_q = \{[r]_q ; r \in \mathbb{N}\} = \left\{0, 1, 1+q, 1+q+q^2, 1+q+q^2+q^3, ...\right\}.$$

It is clear that the set of $q-integers$ $\mathbb{N}_q$ generalizes the set of non-negative integers $\mathbb{N}$, which we recover by putting $q=1$.

The $q-factorial$ is defined by

$$[r]_q! = \begin{cases} [r]_q [r-1]_q .....[1]_q & , \quad r = 1, 2, 3, ..... \\ 1 & , \quad r = 0. \end{cases},$$

and $q-Binomial$ coefficient is defined as

$$\left[ \begin{array}{c} n \\ r \end{array} \right]_q = \frac{[n]_q!}{[r]_q! [n-r]_q!},$$

for integers $n \geq r \geq 0$.

Phillips [18], [19] introduced the generalization of Bernstein polynomials as follows. For a function defined on the interval $[0,1]$, the $q$-Bernstein operators $B_{n,q}(f)$, $n \geq 1$ are defined by

$$(B_{n,q}f)(x) = \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) p_{n,k,q}(x), 0 \leq x \leq 1 \quad , \quad n \geq 1, \tag{1}$$

where

$$p_{n,k,q}(x) = \left[ \begin{array}{c} n \\ k \end{array} \right]_q x^k \prod_{s=0}^{n-k-1} (1-q^s x)$$

or equivalently

$$p_{n,k,q}(x) = \left[ \begin{array}{c} n \\ k \end{array} \right]_q x^k (1-x)_q^{n-k}.$$

The operators (1) and their different modifications were studied by many researchers (see e.g. [5], [10], [11] and [15] etc.).

It can be easily verified that in case $q=1$, the operators defined by (1) reduce to the well-known Bernstein operators.

Actually several researchers have defined the $q$ analogue of the only discrete type operators. Here our aim is to study the $q$ analogue of summation-integral type operators. Here we estimate the rate of convergence of operators $(B_{n,q}f)$ which have derivative of bounded variation on the interval $[0,1]$. At the point $x$, which is a discontinuity of the first kind of the $q-derivative$, we shall prove that the operator $(B_{n,q}f)$ being defined in (1) converges to the limit $f(x)$. To the best of my knowledge, this study is the first study on the approximation of $q-$Bernstein operators in this space.

Let $q \in (0,1)$ and let $I$ be a real interval containing 0.

**Definition 1.** Let $f : I \to R$ be a function and let $x \in I$. The Jackson's $q-$derivative $D_q$ (see [9]), or Jackson's difference operator, of a function $f$ at $x$ is given by

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \ \ if \ x \neq 0, \qquad D_q f(0) := \lim_{x\to 0} D_q f(x). \tag{2}$$

Here $D_q f(x)$ tends to $f'(x)$ as $q$ tends to 1, provided $f'(x)$ exists.

**Definition 2.** Let $f : I \to R$ be a function with $x \in I$. The left (backward), right (forward) and symmetric $q-derivatives$ of a function $f$ are given by, respectively,

$$D_q^- f(x) := \frac{f(x) - f(qx)}{(1-q)x}, \tag{3}$$

$$D_q^+ f(x) := \frac{f\left(\frac{x}{q}\right) - f(x)}{(1-q)x} \tag{4}$$

and

$$D_q^s f(x) := \frac{f(qx) - f\left(\frac{x}{q}\right)}{\left(q - \frac{1}{q}\right)x}, \tag{5}$$

provided that $x \neq 0$. (See [10], [12] ).

**Remark 1.** Note that the Jackson's $q - derivative$ is sometimes called the left (backward) $q - derivative$. Namely

$$D_q f(x) = D_q^- f(x).$$

It is clear that if a function is $f$ differentiable (in the classical sense) at $x$, then

$$\lim_{q\to 1^-} D_q^- f(x) = \lim_{q\to 1^-} D_q^+ f(x) = f'(x),$$

holds true, where the symbol $f'$denotes the usual derivative.

The generalized form of the backward and forward $q - derivatives$ and their relations can be found in [16].

**Remark 2.** In view of the definition of the Jackson's $q-$derivative, see Definition 1, high $q-$derivatives are defined as

$$D^0 f := f, \quad D^n f := D\left(D^{n-1} f\right) \quad (n = 1, 2, ....).$$

We note that, a continuous function on an interval, which does not include zero, is continuous $q-$differentiable (see [20]).

**Remark 3.** Although the $q-$difference operators convey the same idea, the right choice of q-difference operators provides a lot of convenience during the establishment and application of transformations such as Fourier, Laplace or Mellin, which will be used in the solutions of the encountered $q-$differential equations ( see [[6], p. 1797] and [[7], sections 2-3]).

**Example 1.** Let us consider a $q-$diffusion equation as;

$$D_t^q y(x,t) = \frac{\partial^2}{\partial x^2} y(x,t), \qquad (-\infty < x < \infty, \quad t > 0)$$

with the initial condition

$$y(x,0) = f(x).$$

The main question now is to choose an appropriate integral transform to remove the $q-derivative$, namely how we can solve the aforementioned $q-$diffusion equation. In view of the positivity of the time variable, the two most natural choices are the Laplace and the Mellin transform.

If one uses the Laplace transform of the $q-derivatives$ given in (3), and trying to solve this $q-$diffusion equation, then Ho [[7], sections 2] informed us that the Laplace transform is not useful in solving equations involving $q-derivatives$.

Especially during the last two decades, some authors investigated the convergence problems for linear positive operators for functions in $BV(I)$ and $DBV(I)$, where $I \subset \mathbb{R}$ (See [1]-[4]).

Very recently, the author [10] estimated the rate of convergence of q-Bernstein-Durrmeyer Operators for functions in $DBV[0,\infty)$. As a continuation of this study, in the present paper we shall investigate the pointwise approximation properties of the $q$ analogue of the Bernstein operators and estimate the rate of pointwise convergence of these operators to the functions $f$ whose $q-derivatives$ are bounded variation on the interval $[0,1]$. We give an estimate for the rate of convergence of the operator $(B_{n,q}f)$ at those points $x$ at which the one sided $q-derivatives$ $D_q^+ f(x), D_q^- f(x)$ exist. We shall also prove that the operators $B_{n,q}f$ converge to the limit $f(x)$. To the best of my knowledge, the present study will be the first study on the approximation of q-Bernstein operators in the space of $D_qBV$.

Let $D_qBV[a,b]$ denote the class of real valued $q-differentiable$ functions defined on a set $[a,b]$, whose $q-derivatives$ are bounded variation on $[a,b]$, which can be written as

$$f(x) = C + \int_a^x \Psi(t)dt, \ -\infty < a \leq x \leq b$$

where $C$ is a constant and $\Psi \in BV[a,b]$. It is clear that

$$D_q BV[a,b] := \{f : D_q f = \Psi \in BV[a,b]\}.$$

The first theorem of this paper is stated as:

**Theorem 1.** *Let $f \in D_q BV[0,1]$. Suppose that the right and left q-derivatives exist at a fixed point $x \in (0,1)$. Then at this point $x \in (0,1)$, and $n$ sufficiently large, one has*

$$|(B_{n,q}f)(x) - f(x)| \leq \left|\frac{D_q^+ f(x) - D_q^- f(x)}{2}\right| \sqrt{A_{n,q}(x)} \tag{6}$$

$$+\frac{1}{\sqrt{[n]_q}} \bigvee_{x-\frac{x}{\sqrt{[n]_q}}}^{x+\frac{1-x}{\sqrt{[n]_q}}} (D_q f_x) + \frac{1}{[n]_q} \sum_{k=1}^{\left[\sqrt{[n]_q}\right]} \bigvee_{x-\frac{x}{k}}^{x+\frac{1-x}{k}} (D_q f_x)$$

*where*

$$D_q f_x(t) = \begin{cases} D_q f(t) - D_q^+ f(x) & , \quad x < t \leq 1 \\ 0 & , \quad t = x \\ D_q f(t) - D_q^- f(x) & , \quad 0 \leq t < x \end{cases}, \tag{7}$$

$\bigvee_a^b (f)$ *is the total variation (or Jordan Variation) of $f$ on $[a,b]$ and*

$$A_{n,q}(x) := \frac{x(1-x)}{[n]_q}.$$

In view of Theorem 1 and Remark 2, we get:

**Corollary 1.** *Let $f \in C\,[0,1]$. Then for every $x \in (0,1)$, and $n$ sufficiently large, we have*

$$|(B_{n,q}f)(x) - f(x)| \leq \frac{1}{\sqrt{[n]_q}} \bigvee_{x-\frac{x}{\sqrt{[n]_q}}}^{x+\frac{1-x}{\sqrt{[n]_q}}} (D_q f_x) + \frac{1}{[n]_q} \sum_{k=1}^{\left[\sqrt{[n]_q}\right]} \bigvee_{x-\frac{x}{k}}^{x+\frac{1-x}{k}} (D_q f_x).$$

## 2 Auxiliary results

In this section, we state some basic concepts concerning quantum calculus and some lemmas about the aforementioned operators, which are necessary to prove our theorems.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space and $Z : \Omega \to \mathbb{R}$ be a random variable. The mathematical expectation $E$ of the random variable $Z$ is defined as

$$E(Z) := \int_\Omega Z(\omega)\mathbf{P}(d\omega).$$

If one considers a random variable $Y_n(q;x)$ having the probability distribution

$$\mathbf{P}\left(Y_n(q;x)=\frac{[k]_q}{[n]_q}\right)=p_{n,k,q}\left(x\right),$$

then obviously $\left(B_{n,q}f\right)(x)=E(f(Y_n(q;x)))$ holds true. Indeed;

$$\begin{aligned}\left(B_{n,q}f\right)(x) &= \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right)p_{n,k,q}\left(x\right)\\ &= \sum_{k=0}^{n} f\left(Y_n(q;x)\right)\mathbf{P}\left(Y_n(q;x)\right)\\ &= E(f(Y_n(q;x))).\end{aligned}$$

In view of the relations between the probability theory, the theory of Bernstein polynomials and the theory of singular integrals, $\left(B_{n,q}f\right)(x)$ may be written as a Stieltjes integral in the variable $t$ as follows:

$$\left(B_{n,q}f\right)(x)=\int_0^1 f(t)d\left(K_{n,q}(x,t)\right),$$

where

$$K_{n,q}(x,t)=\begin{cases}\sum\limits_{[k]_q\le[n]_q t} p_{n,k,q}(x) & ,\ 0<t\le 1\\ 0 & ,\ t=0\end{cases},\ k=0,1,...,n-1. \tag{8}$$

For $q=1$, some detailed information about the kernel function can be found in the classical book of Lorentz [13].

We can write also

$$\left(B_{n,q}f\right)(x)=\int_0^1 f(t)H_{n,q}(x,t)dt \tag{9}$$

where

$$H_{n,q}(x,t):=\sum_{k=0}^{n} p_{n,k,q}\left(x\right)\delta(t-[k]_q/[n]_q)(t). \tag{10}$$

and $\delta$ is the Delta function.

**Lemma 1** [15]. *Using the definition of $q-$Bernstein polynomials, we obtain*

$$\left(B_{n,q}1\right)(x)=1,\quad \left(B_{n,q}t\right)(x)=x$$

*and*

$$\left(B_{n,q}t^2\right)(x)=x^2+\frac{X}{[n]_q},$$

*where $X=x(1-x)$.*

**Remark 4.** *By elementary computation, we get*

$$\begin{aligned} \left(B_{n,q}(t-x)\right)(x) &= 0, \\ \left(B_{n,q}(t-x)^2\right)(x) &= \frac{X}{[n]_q} := A_{n,q}(x). \end{aligned} \tag{11}$$

**Lemma 2.** *For all $x \in (0,1)$, and $n$ sufficiently large, then for $0 < t < x$, one obtains*

$$\lambda_{n,q}(x,t) = \int_0^t H_{n,q}(x,u)du \leq \frac{A_{n,q}(x)}{(x-t)^2}. \tag{12}$$

**Proof.** Clearly

$$\lambda_{n,q}(x,t) = \int_0^t H_{n,q}(x,u)du \leq \int_0^t H_{n,q}(x,u)\left(\frac{x-u}{x-t}\right)^2 du$$

$$= \frac{1}{(x-t)^2}\int_0^t H_{n,q}(x,u)(x-u)^2 du \leq \frac{\left(B_{n,q}(u-x)^2\right)(x)}{(x-t)^2}.$$

By (11), one can easily obtain

$$\lambda_{n,q}(x,t) \leq \frac{1}{(x-t)^2} A_{n,q}(x).$$

**Remark 5.** *By Cauchy-Schwarz-Bunyakowsky inequality, one has*

$$\left(B_{n,q}\left|t-x\right|\right)(x) \leq \left(\left(B_{n,q}(t-x)^2\right)(x)\right)^{\frac{1}{2}} \leq \sqrt{A_{n,q}(x)}. \tag{13}$$

# 3 Main results

**Proof of Theorem 1.** Since $(B_{n,q}1)(x) = 1$, in view of the Stieltjes representation of the $q-$Bernstein operators $(B_{n,q}f)$ defined by (9), then clearly

$$\left(B_{n,q}f\right)(x) = \int_0^1 f(x) H_{n,q}(x,t)dt.$$

So, we can write the difference between $(B_{n,q}f)(x)$ and $f(x)$ as follows

$$\begin{aligned} \left(B_{n,q}f\right)(x) - f(x) &= \sum_{k=0}^{n} p_{n,k,q}(x) f\left(\frac{[k]_q}{[n]_q}\right) - f(x) \\ &= \int_0^1 [f(t)-f(x)] H_{n,q}(x,t)dt, \end{aligned}$$

where $K_{n,q}(x,t)$ and $H_{n,q}(x,t)$ being defined in (8) and (10), respectively.
Note that $D_qf=\Psi\in BV[0,1]$,

$$\left(B_{n,q}f\right)(x)-f(x)=\int_0^x\left[f(t)-f(x)\right]H_{n,q}(x,t)dt+\int_x^1\left[f(t)-f(x)\right]H_{n,q}(x,t)dt$$

$$\begin{aligned}&= -\int_0^x\left[\int_t^x D_qf(u)\,du\right]H_{n,q}(x,t)\,dt+\int_x^1\left[\int_x^t D_qf(u)\,du\right]H_{n,q}(x,t)dt\\ &= -E_{1,q}(x)+E_{2,q}(x),\end{aligned}$$

here

$$E_{1,q}(x):=\int_0^x\left[\int_t^x D_qf(u)\,du\right]H_{n,q}(x,t)\,dt \tag{14}$$

and

$$E_{2,q}(x):=\int_x^1\left[\int_x^t D_qf(u)\,du\right]H_{n,q}(x,t)\,dt. \tag{15}$$

In view of the definitions (3)-(4) and (7), for any $D_qf=\Psi\in BV[0,1]$, we decompose $D_qf(t)$ as

$$\begin{aligned}D_qf(t) &= \frac{D_q^+f(x)+D_q^-f(x)}{2}+D_qf_x(t)+\frac{D_q^+f(x)-D_q^-f(x)}{2}sgn(t-x)\\ &\quad+\delta_x(t)\left[D_qf(x)-\frac{D_q^+f(x)+D_q^-f(x)}{2}\right]\end{aligned} \tag{16}$$

where

$$\delta_x(t)=\begin{cases}1\ ,\ x=t\\ 0\ ,\ x\neq t.\end{cases}.$$

If we use (16) in (14) and (15), then the following expressions hold true,

$$\begin{aligned}E_{1,q}(x) &= \int_0^x\left\{\int_t^x \frac{D_q^+f(x)+D_q^-f(x)}{2}+D_qf_x(u)+\frac{D_q^+f(x)-D_q^-f(x)}{2}sgn(u-x)\right.\\ &\quad\left.+\delta_x(u)\left[D_qf(x)-\frac{D_q^+f(x)+D_q^-f(x)}{2}\right]du\right\}H_{n,q}(x,t)\,dt\end{aligned}$$

and

$$\begin{aligned}E_{2,q}(x) &= \int_x^1\left\{\int_x^t \frac{D_q^+f(x)+D_q^-f(x)}{2}+D_qf_x(u)+\frac{D_q^+f(x)-D_q^-f(x)}{2}sgn(u-x)\right.\\ &\quad\left.+\delta_x(u)\left[D_qf(x)-\frac{D_q^+f(x)+D_q^-f(x)}{2}\right]du\right\}H_{n,q}(x,t)\,dt.\end{aligned}$$

At first, we consider $E_{1,q}(x)$ :

$$\begin{aligned}E_{1,q}(x) &= \frac{D_q^+ f(x) + D_q^- f(x)}{2}\int_0^x (x-t)H_{n,q}(x,t)\,dt \\ &+ \int_0^x \left[\int_x^t D_q f_x(u)\,du\right] H_{n,q}(x,t)\,dt \\ &- \frac{D_q^+ f(x) - D_q^- f(x)}{2}\int_0^x (x-t)H_{n,q}(x,t)dt \\ &+ \left[D_q f(x) - \frac{D_q^+ f(x) + D_q^- f(x)}{2}\right]\int_0^x \left[\int_x^t \delta_x(u)\,du\right] H_{n,q}(x,t)\,dt.\end{aligned}$$

Since $\int_x^t \delta_x(u)\,d_q u = 0$, one has

$$\begin{aligned}E_{1,q}(x) &= \frac{D_q^+ f(x) + D_q^- f(x)}{2}\int_0^x (x-t)H_{n,q}(x,t)dt \\ &+ \int_0^x \left[\int_x^t D_q f_x(u)\,d_q u\right] H_{n,q}(x,t)dt \\ &- \frac{D_q^+ f(x) - D_q^- f(x)}{2}\int_0^x (x-t)H_{n,q}(x,t)dt.\end{aligned} \tag{17}$$

Using a similar method, for evaluating $E_{2,q}(x)$, we find that

$$\begin{aligned}E_2(x) &= \frac{D_q^+ f(x) + D_q^- f(x)}{2}\int_x^1 (t-x)H_{n,q}(x,t)\,dt \\ &+ \int_x^1 \left[\int_x^t D_q f_x(u)\,du\right] H_{n,q}(x,t)\,dt \\ &- \frac{D_q^+ f(x) - D_q^- f(x)}{2}\int_x^1 (t-x)H_{n,q}(x,t)\,dt.\end{aligned} \tag{18}$$

Collecting (17) and (18),

$$\begin{aligned} -E_{1,q}(x)+E_{2,q}(x) &= \frac{D_q^+ f(x)+D_q^- f(x)}{2}\int_0^1 (t-x)H_{n,q}(x,t)\,dt \\ &+\frac{D_q^+ f(x)-D_q^- f(x)}{2}\int_0^1 |t-x|\,H_{n,q}(x,t)\,dt \\ &-\int_0^x \left[\int_t^x D_q f_x(u)\,du\right] H_{n,q}(x,t)\,dt \\ &+\int_x^1 \left[\int_x^t D_q f_x(u)\,du\right] H_{n,q}(x,t)dt. \end{aligned} \tag{19}$$

From (19), we can rewrite the difference between $(B_{n,q}f)(x)$ and $f(x)$,

$$\begin{aligned} (B_{n,q}f)(x)-f(x) &= \frac{D_q^+ f(x)+D_q^- f(x)}{2}\int_0^1 (t-x)H_{n,q}(x,t)dt \\ &+\frac{D_q^+ f(x)-D_q^- f(x)}{2}\int_0^1 |t-x|\,H_{n,q}(x,t)dt \\ &-\int_0^x \left[\int_t^x D_q f_x(u)\,du\right] H_{n,q}(x,t)\,dt \\ &+\int_x^1 \left[\int_x^t D_q f_x(u)\,du\right] H_{n,q}(x,t)\,dt. \end{aligned} \tag{20}$$

On the other hand

$$\int_0^1 |t-x|\,H_{n,q}(x,t)\,dt = \left(B_{n,q}\,|t-x|\right)(x) \tag{21}$$

and

$$\int_0^1 (t-x)H_{n,q}(x,t)\,dt = \left(B_{n,q}\,(t-x)\right)(x) = 0, \tag{22}$$

are valid, then using (21) and (22) in (20), we get

$$\begin{aligned} |(B_{n,q}f)(x) - f(x)| \leq\ & \left|\frac{D_q^+ f(x) + D_q^- f(x)}{2}\right| |(B_{n,q}(t-x))(x)| \\ & + \left|\frac{D_q^+ f(x) - D_q^- f(x)}{2}\right| |(B_{n,q}|t-x|)(x)| \qquad (23) \\ & + \left| -\int_0^x \left[\int_t^x D_q f_x(u)\, du\right] H_{n,q}(x,t)\, dt\right| \\ & + \left|\int_x^1 \left[\int_x^t D_q f_x(u)\, du\right] H_{n,q}(x,t) dt\right|. \end{aligned}$$

From the definition $\lambda_{n,q}(x,t)$, we write

$$\int_0^x \left[\int_t^x D_q f_x(u)\, du\right] H_{n,q}(x,t)\, dt = \int_0^x \left[\int_t^x D_q f_x(u)\, du\right] D\left(\lambda_{n,q}(x,t)\right) dt. \qquad (24)$$

Using partial integration, the right hand side of (24), we obtain

$$\int_0^x \left[\int_t^x D_q f_x(u)\, du\right] D\left(\lambda_{n,q}(x,t)\right) dt = \int_0^x D_q f_x(t)\, \lambda_{n,q}(x,t)\, dt.$$

Thus

$$\left| -\int_0^x \left[\int_x^t D_q f_x(u)\, du\right] H_{n,q}(x,t)\, dt\right| \leq \int_0^x |D_q f_x(t)|\, \lambda_{n,q}(x,t)\, dt$$

and

$$\begin{aligned} \left| -\int_0^x \left[\int_x^t D_q f_x(u)\, du\right] H_{n,q}(x,t) dt\right| \leq\ & \int_0^{x-\frac{x}{\sqrt{n}}} |D_q f_x(t)|\, \lambda_{n,q}(x,t)\, dt \\ & + \int_{x-\frac{x}{\sqrt{n}}}^x |D_q f_x(t)|\, \lambda_{n,q}(x,t)\, dt. \end{aligned}$$

Since $D_q f_x(x) = 0$ and $\lambda_{n,q}(x,t) \leq 1$,

$$\begin{aligned} \int_{x-\frac{x}{\sqrt{[n]_q}}}^x |D_q f_x(t)|\, \lambda_{n,q}(x,t)\, dt &= \int_{x-\frac{x}{\sqrt{[n]_q}}}^x |D_q f_x(t) - D_q f_x(x)|\, \lambda_{n,q}(x,t)\, dt \\ &\leq \int_{x-\frac{x}{\sqrt{[n]_q}}}^x \bigvee_t^x (D_q f_x)\, dt. \end{aligned}$$

Owing to (12), we get

$$\begin{aligned}\int\limits_0^{x-\frac{x}{\sqrt{[n]_q}}} |D_q f_x(t)|\; \lambda_{n,q}(x,t)\,dt &\leq A_{n,q}(x) \int\limits_0^{x-\frac{x}{\sqrt{[n]_q}}} |D_q f_x(t)|\, \frac{dt}{(x-t)^2} \\ &= A_{n,q}(x) \int\limits_0^{x-\frac{x}{\sqrt{[n]_q}}} |D_q f_x(t) - D_q f_x(x)|\, \frac{dt}{(x-t)^2} \\ &\leq A_{n,q}(x) \int\limits_0^{x-\frac{x}{\sqrt{[n]_q}}} \bigvee_t^x (D_q f_x) \frac{dt}{(x-t)^2}.\end{aligned}$$

Make the change of variables $t = x - \frac{x}{u}$, then one has

$$\begin{aligned}\int\limits_{x-\frac{x}{\sqrt{[n]_q}}}^{x} \bigvee_t^x (D_q f_x)\,dt &\leq \bigvee_{x-\frac{x}{\sqrt{[n]_q}}}^{x} (D_q f_x) \int\limits_{x-\frac{x}{\sqrt{[n]_q}}}^{x} dt \\ &= \bigvee_{x-\frac{x}{\sqrt{[n]_q}}}^{x} (D_q f_x) \left[x - \left(x - \frac{x}{\sqrt{[n]_q}}\right)\right] \\ &= \frac{x}{\sqrt{[n]_q}} \bigvee_{x-\frac{x}{\sqrt{[n]_q}}}^{x} (D_q f_x)\end{aligned}$$

and

$$\begin{aligned}A_{n,q}(x) \int\limits_0^{x-\frac{x}{\sqrt{[n]_q}}} \bigvee_t^x (D_q f_x) \frac{dt}{(x-t)^2} &= A_{n,q}(x) \int\limits_1^{\sqrt{[n]_q}} \bigvee_{x-\frac{x}{u}}^{x} (D_q f_x) \frac{\left(\frac{x}{u^2}\right)\,du}{(-\frac{x}{u})^2} \\ &= \frac{A_{n,q}(x)}{x} \int\limits_1^{\sqrt{[n]_q}} \bigvee_{x-\frac{x}{u}}^{x} (D_q f_x)\,du \\ &= \frac{A_{n,q}(x)}{x} \sum_{k=1}^{\left[\sqrt{[n]_q}\right]} \int\limits_k^{k+1} \bigvee_{x-\frac{x}{u}}^{x} (D_q f_x)\,du \\ &\leq \frac{A_{n,q}(x)}{x} \sum_{k=1}^{\left[\sqrt{[n]_q}\right]} \bigvee_{x-\frac{x}{k}}^{x} (D_q f_x).\end{aligned}$$

Consequently

$$\left| -\int_0^x \left[ \int_t^x D_q f_x(u)\, du \right] H_{n,q}(x,t)dt \right| \leq \frac{x}{\sqrt{[n]_q}} \bigvee_{x-\frac{x}{\sqrt{[n]_q}}}^{x} (D_q f_x) + A_{n,q}(x) \frac{1}{x} \sum_{k=1}^{\left[\sqrt{[n]_q}\right]} \bigvee_{x-\frac{x}{k}}^{x} (D_q f_x) . \tag{25}$$

By the same way,

$$\int_x^1 \left[ \int_x^t D_q f_x(u)\, du \right] H_{n,q}(x,t)dt \leq \int_x^1 |D_q f_x(t)| \; D\left(\lambda_{n,q}(x,t)\right)\, dt$$

$$= \int_x^{x+\frac{1-x}{\sqrt{[n]_q}}} |D_q f_x(t)| \; \lambda_{n,q}(x,t)\, dt + \int_{x+\frac{1-x}{\sqrt{[n]_q}}}^1 |D_q f_x(t)| \; \lambda_{n,q}(x,t)\, dt$$

$$\leq \int_x^{x+\frac{1-x}{\sqrt{[n]_q}}} |D_q f_x(t)| \; dt + A_{n,q}(x) \int_{x+\frac{1-x}{\sqrt{[n]_q}}}^1 |D_q f_x(t)| \frac{dt}{(x-t)^2}$$

$$= \int_x^{x+\frac{1-x}{\sqrt{[n]_q}}} |D_q f_x(t) - D_q f_x(x)| \; dt + A_{n,q}(x) \int_{x+\frac{1-x}{\sqrt{[n]_q}}}^1 |D_q f_x(t) - D_q f_x(x)| \frac{dt}{(x-t)^2}$$

$$\leq \bigvee_x^{x+\frac{1-x}{\sqrt{[n]_q}}} (D_q f_x) \frac{1-x}{\sqrt{[n]_q}} + A_{n,q}(x) \int_{x+\frac{1-x}{\sqrt{[n]_q}}}^1 \bigvee_x^t (D_q f_x) \frac{dt}{(x-t)^2}$$

Make the change of variables $t = x + \frac{1-x}{u}$, again from (12)

$$\leq \bigvee_x^{x+\frac{1-x}{\sqrt{[n]_q}}} (D_q f_x) \frac{1-x}{\sqrt{[n]_q}} + A_{n,q}(x) \int_{\sqrt{[n]_q}}^1 \bigvee_{x+\frac{1-x}{u}}^{x} (D_q f_x) \frac{-\left(\frac{1-x}{u^2}\right)\, du}{\left(\frac{x-1}{u}\right)^2}$$

$$= \bigvee_x^{x+\frac{1-x}{\sqrt{[n]_q}}} (D_q f_x) \frac{1-x}{\sqrt{[n]_q}} + \frac{A_{n,q}(x)}{(1-x)} \int_1^{\sqrt{[n]_q}} \bigvee_x^{x+\frac{1-x}{u}} (D_q f_x)\, du$$

$$\leq \bigvee_{x}^{x+\frac{1-x}{\sqrt{[n]_q}}} (D_q f_x) \frac{1-x}{\sqrt{[n]_q}} + \frac{A_{n,q}(x)}{(1-x)} \sum_{k=1}^{\left[\sqrt{[n]_q}\right]} \int_{k}^{k+1} \bigvee_{x}^{x+\frac{1-x}{u}} (D_q f_x)\, du$$

$$\leq \bigvee_{x}^{x+\frac{1-x}{\sqrt{[n]_q}}} (D_q f_x) \frac{1-x}{\sqrt{[n]_q}} + \frac{A_{n,q}(x)}{(1-x)} \sum_{k=1}^{\left[\sqrt{[n]_q}\right]} \bigvee_{x}^{x+\frac{1-x}{k}} (D_q f_x)\, .$$

Finally

$$\int_x^1 \left[ \int_x^t D_q f_x(u)\, du \right] H_{n,q}(x,t)\, dt \leq \bigvee_{x}^{x+\frac{1-x}{\sqrt{[n]_q}}} (D_q f_x) \frac{1-x}{\sqrt{[n]_q}} + \frac{A_{n,q}(x)}{(1-x)} \sum_{k=1}^{\left[\sqrt{[n]_q}\right]} \bigvee_{x}^{x+\frac{1-x}{k}} (D_q f_x)\, . \tag{26}$$

Combining (13), (25) and (26) in (23), we get the desired result (6).

Thus the proof is completed.

In order to obtain an approximation theorem, we introduce a new sequence with the following properties. Let $(q_n)$ be a sequence of real numbers such that $0 < q_n < 1$ and

$$\lim_{n\to\infty} q_n = 1.$$

If we replace $q$ by $q_n$, we have immediately

$$[n]_{q_n} \to \infty \qquad (n \to \infty).$$

It is worth mentioning that similar approaches can be found [22] and [11].

On convergence formaIly, Theorem 2 reads:

**Theorem 2.** *Let $(q_n)$ be a sequence of real numbers such that $0 < q_n < 1$ and $\lim_{n\to\infty} q_n = 1$. Let $f$ be a bounded function on $[0,1]$. Suppose that the right and left $q$-derivatives exist at a fixed point $x \in (0,1)$. Then at this point $x \in (0,1)$, and $n$ sufficiently large, we have*

$$\left|(B_{n,q_n} f)(x) - f(x)\right| \leq \left| \frac{D_{q_n}^{+} f(x) - D_{q_n}^{-} f(x)}{2} \right| \sqrt{A_{n,q_n}(x)} + \frac{1}{\sqrt{[n]_{q_n}}} \bigvee_{x-\frac{x}{\sqrt{[n]_{q_n}}}}^{x+\frac{1-x}{\sqrt{[n]_{q_n}}}} (D_{q_n} f_x) + \frac{1}{[n]_{q_n}} \sum_{k=1}^{\left[\sqrt{[n]_{q_n}}\right]} \bigvee_{x-\frac{x}{k}}^{x+\frac{1-x}{k}} (D_{q_n} f_x).$$

*Here we note that*

$$A_{n,q_n}(x) \to 0 \qquad (n \to \infty).$$

In view of Theorem 2 and Remark 2, we get the following estimate:

**Corollary 2.** *Let $f \in C[0,1]$. Then for every $x \in (0,1)$, and $n$ sufficiently large, we have*

$$|(B_{n,q_n} f)(x) - f(x)| \leq \frac{1}{\sqrt{[n]_{q_n}}} \bigvee_{x-\frac{x}{\sqrt{[n]_{q_n}}}}^{x+\frac{1-x}{\sqrt{[n]_{q_n}}}} (D_{q_n} f_x) + \frac{1}{[n]_{q_n}} \sum_{k=1}^{\left[\sqrt{[n]_{q_n}}\right]} \bigvee_{x-\frac{x}{k}}^{x+\frac{1-x}{k}} (D_{q_n} f_x).$$

The proof of Theorem 2 is the same as Theorem 1, so we omit it.

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