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ON THE PELL-EISENSTEIN SERIES OF POWER m

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Abstract

In this paper, we introduce the Pell-Eisenstein Series which are obtained by Pell numbers and they are a new class of Eisenstein-type series. First we see that they are well-defined and then we prove that the Pell-Eisenstein series satisfies some functional equations. Proofs are based on properties of Pell numbers and calculations.

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1 Introduction

Since number sequences and especially Fibonacci sequences are observed in nature, they find application in many fields of science [6, 10, 14, 15]. In fact, many generalizations of Fibonacci sequences have been made and the relationships between them have been studied for many years [5, 7, 13, 16]. One of these generalizations is Pell numbers. There are also many studies with Pell numbers [3, 4, 11, 17]. This study is about Pell numbers and Eisenstein series and a similar procedure is used in [12]. Having relation with several different branches of mathematics, even in string theory of physics, modular forms have been receiving much attention for decades. In this work, we stick to the number theory side. In this case, Eisenstein series are, indeed, one of the most important examples for modular forms. Furthermore, their Fourier coefficients can be calculated easily by

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divisor sigma function, hence they are computation-friendly! For a comprehensive consultation on modular forms one can have a look at recent sources by Cohen and Strömberg at [2] as a classical approach and see [18] for computational approach by Stein. From a different perspective, we can think that classical Eisenstein series are not far away from "integer theory" since their Fourier coefficients are exactly integers. So it is not surprising that, in a brand new paper in [8], some families called "semi-modular forms" are introduced by Eisenstein-like series that are defined on integer partitions. As a subsequent paper, in [1], a similar problem with Fibonacci number setting is considered. One of the latest works is [19] where Pell-Lucas-Eisenstein Series was defined and gave some properties about them. In this paper, we introduce the Pell-Eisenstein Series of weight m. We give the functional equalities for these series and convergent on its domain. First of all, we will start with definitions of Eisenstein series and Pell numbers.

Definition 1. *[2]*
For each $z \in \mathbb{H} = \{z \in \mathbb{C} : \mathfrak{Im} z > 0\}$ for even $k \geq 4$ the weight k Eisenstein series is indicated as $G_k(z)$ and given by

$$G_k(z) := \sum_{(m,n)\in\mathbb{Z}^2/(0,0)} 1/(mz+n)^k.$$

Definition 2. *[9] We will indicate the nth-Pell number as P_n and defined as $P_0 = 0$, $P_1 = 1$ and $P_2 = 2$ and for $n \geq 3$*

$$P_n = 2P_{n-1} + P_{n-2}.$$

For a framework on Pell numbers, see [9]. It is easy to conclude the following result which is essential for the rest of the work.

Theorem 1. *[9]*
Let P_n be the nth-Pell number. We have

$$P_{-n} = (-1)^n P_n.$$

2 Main results

Definition 3. *The Pell-Eisenstein Series of weight m is indicated by $P_m(z)$ and defined by*

$$P_m(z) := \sum_{j=-\infty}^{\infty} (P_j z + P_{j-1})^{-m}.$$

First of all, let us illustrate the definition by an example.

Example 1. *For $m = 5$, the Pell-Eisenstein series of weight 5 is obtained as*

$$P_5(z) = \sum_{j=-\infty}^{\infty} (P_j z + P_{j-1})^{-5}$$

$= ... + (P_{-2}z + P_{-3})^{-5} + (P_{-1}z + P_{-2})^{-5} + (P_0 z + P_{-1})^{-5} + (P_1 z + P_0)^{-5} + (P_2 z + P_1)^{-5} + (P_3 z + P_2)^{-5} + ... = ... + (2z - 5)^{-5} + (-z + 2)^{-5} + (-1)^{-5} + (z)^{-5} + (2z + 1)^{-5} + (5z + 2)^{-5} + ...$

It is a natural question to ask whether $P_m(z)$ is well-defined and the answer is positive by the following theorem.

Theorem 2. *For all integers $m > 1$, $P_m(z)$ are convergent on its domain.*

Proof. It is well-known that for $\mathfrak{Re}(s) > 1$ the Riemann-zeta function is denoted $\zeta(s)$ and defined as

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

On the other hand, it is clear that

$$P_m(z) = \sum_{j=-\infty}^{\infty} \frac{1}{(P_j z + P_{j-1})^m} \leq \sum_{n=1}^{\infty} \frac{1}{n^m} = \zeta(m).$$

By Weierstrass criterion the theorem follows. □

Theorem 3. *The following functional equalities hold for the Pell-Eisenstein Series:*

1. $P_{2k}(\frac{-1}{z}) = z^{2k} P_{2k}(z)$
2. $P_{2k}(z+2) = z^{-2k} P_{2k}(\frac{1}{z})$
3. $P_{2k}(-z) = z^{-2k} P_{2k}(\frac{1}{z})$
4. $P_{2k}(2-z) = P_{2k}(z)$.

This section is reserved for the proof of Theorem 3 Note that the main technique is direct calculation.

Proof. 1. Let P_{2k} be the Pell-Eisenstein series of weight $2k$. we can clearly write

$$P_{2k}\left(\frac{-1}{z}\right) = \sum_{j=-\infty}^{\infty} \left(P_j \left(\frac{-1}{z}\right) + P_{j-1}\right)^{-2k}.$$

Then, if we expand the series we get

$$P_{2k}\left(\frac{-1}{z}\right) = \cdots + \left(-\frac{5}{z} - 12\right)^{-2k} + \left(\frac{2}{z} + 5\right)^{-2k} + \left(-\frac{1}{z} - 2\right)^{-2k} + (1)^{-2k} + \left(-\frac{1}{z}\right)^{-2k} + \left(-\frac{2}{z} + 1\right)^{-2k} + \left(-\frac{5}{z} + 2\right)^{-2k} + \left(-\frac{12}{z} + 5\right)^{-2k} + \dots$$

When we shall equate the denominator and after taking common brackets, we have

$$
\begin{aligned}
P_{2k}\left(\frac{-1}{z}\right) &= z^{2k}[\cdots + (12z+5)^{-2k} + (5z+2)^{-2k} + (2z+1)^{-2k} + (z)^{-2k} \\
&+(1)^{-2k} + (z-2)^{-2k} + (-2z+5)^{-2k} + (-12+5z)^{-2k} + \dots] = z^{2k}P_{2k}(z)
\end{aligned}
$$

as desired.

2. Let us define

$$P_m(z) = P_m^-(z) + P_m^+(z)$$

where $P_m^-(z) := \sum_{-\infty<n\leq 0}(P_n z + P_{n-1})^{-m}$ and $P_m^+(z) := \sum_{1\leq n\leq\infty}(P_n z + P_{n-1})^{-m}$.

One can see clearly,

$$
\begin{aligned}
P_{2k}^+(z+2) &= \sum_{n\geq 1}(P_n(z+2) + P_{n-1})^{-2k} = P_{2k}^+(z+2) \\
&= \sum_{n\geq 1}(P_n z + 2P_n + P_{n-1})^{-2k} = \sum_{n\geq 1}(P_n z + P_{n+1})^{-2k}.
\end{aligned}
$$

When we are expanding the series and multiplying by z^{2k} both sides, we obtain

$$\sum_{n\geq 1}(P_n z + P_{n+1})^{-2k} = \sum_{n\geq 1} P_n \left(\frac{1}{z} + P_{n-1}\right)^{-2k},$$

hence we get $P_{2k}^+(z+2) = z^{-2k}P_{2k}^+(\frac{1}{z}) - 1$. In a similar way, we have $P_{2k}^-(z+2) = z^{-2k}P_{2k}^- + (\frac{1}{z}) + 1$.

In the end, combining results above, we find

$$P_{-2k}(z+2) = z^{2k}P_{2k}\left(\frac{1}{z}\right),$$

as desired.

3. Let's suppose the above and let us think

$$P_{2k}^-(-z) = \sum_{n\leq 0}(P_n(-z) + P_{n-1})^{-2k}$$

Expanding the series and later multiplying by z^{2k} then it is easy that we can deduce that

$$P_{2k}^{-}(-z) = z^{-2k} P_{2k}^{-}\left(\frac{1}{z}\right).$$

If we repeat the process above for part $P_{2k}^{+}(-z)$, we have the desired result.

4. It is clear from 2 and 3.

□

3 Result and discussion

In this paper, we deduced the definition of the Pell-Eisenstein series and gave basic properties for them. Although, new series are missing to be semi-modular forms, they provide some certain interesting functional equations. Of course, since we only focus on the basic facts in this short note, there are some open questions about the Pell-Eisenstein series. More precisely, studying other arithmetic properties of, for instance, questions about their coefficients are open.

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