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ON PARA-COMPLEX STRUCTURES OF HORIZONTAL SASAKI GRADIENT METRIC

Hichem EL HENDI[*1] **and Abderrahim ZAGANE**[2]

Abstract

In the present paper, we consider the horizontal Sasaki gradient metric on the tangent bundle TM over an m-dimensional Riemannian manifold (M, g). We construct almost para-complex Norden structures on TM and investigate conditions for these structures to be para-Kähler. We finally study some properties of horizontal Sasaki gradient Metric which is pure with respect to some para-complex structures on TM.

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1 Introduction

The geometry of tangent bundle of a Riemannian manifold (M, g) has undergone an intense study since the 1970s, especially due to the fact that it is the basic manifold of a Finsler space (M, F). The main ingredient in such studies is is that of nonlinear connection intrisecly related by the fundamental function F. Such a nonlinear connection is that of Cartan $N_i^j = \dfrac{\partial G^i}{\partial y^j}$, where $G^i = \dfrac{1}{2}\gamma^i_{00}$ are the spray coefficients. Any nonlinear connection spans a horizontal distribution by the adapted frames $\left\{\dfrac{\delta}{\delta x^j} = \dfrac{\partial}{\partial x^j} - N_j^i \dfrac{\partial}{\partial y^i}\right\}$, orthogonal to the vertical distribution spanned by $\left\{\dfrac{\partial}{\partial y^i}\right\}$. The fundamental function determines a metric tensor $g_{ij} = \dfrac{\partial^2 F^2}{\partial y^i \partial y^j}$ and then $g = g_{ij} dx^i \otimes dx^j + g_{ij}\delta y^i \otimes \delta y^j$ is a metric structure on TM, where $\delta y^i = dy^i + N_j^i dx^j$.

[1*] *Corresponding author*, Department of Mathematics, University of Bechar, PO Box 417, 08000, Bechar, Algeria,e-mail: elhendihichem@yahoo.fr
[2]Department of Mathematics, Relizane University, 48000, Relizane-Algeria, e-mail: abderrahim.zagane@univ-relizane.dz

In recent years, a lot of studies about their local or global geometric properties have been published in the literature. When the authors studied this topic, they used different metrics which are called natural metrics on the tangent bundle. Firstly, the geometry of a tangent bundle has been studied by using a new metric g^s, which is called Sasaki metric, with the aid of a Riemannian metric g on a differential manifold M in 1958 by Sasaki [19]. It is uniquely determined by.

$$\begin{aligned} g^s(X^H, Y^H) &= g(X,Y)\circ\pi \\ g^s(X^H, Y^V) &= 0 \\ g^s(X^V, Y^V) &= g(X,Y)\circ\pi \end{aligned} \tag{1}$$

for all vector fields X and Y on M. More intuitively, the metric g^s is constructed in such a way that the vertical and horizontal sub bundles are orthogonal and the bundle map $\pi : (TM, g^s) \longrightarrow (M, g)$ is a Riemannian submersion.

After that, the tangent bundle could be split into its horizontal and vertical subbundles with the aid of Levi Civita connection ∇ on (M, g). Later, the Lie bracket of the tangent bundle TM, the Levi Civita connection ∇^s on TM and its Riemannian curvature tensor R^s have been obtained in [7] and [14]. Furthermore, the explicit formulae of another natural metric g_{CG}, which is called Cheeger-Gromoll metric, on the tangent bundle TM of a Riemannian manifold (M, g). It is uniquely determined by

$$\begin{aligned} &g_{CG}(X^H, Y^H) = g(X,Y)\circ\pi \\ &g_{CG}(X^H, Y^V) = 0 \\ &g_{CG}(X^V, Y^V) = \frac{1}{\alpha}\Big(g(X,Y) + g(X,u)g(Y,u)\Big)\circ\pi, \end{aligned} \tag{2}$$

where $X, Y \in \Gamma(TM)$, $(x, u) \in TM$, $\alpha = 1 + g_x(u, u)$. This metric has been given by Musso and Tricerri in [18], using Cheeger and Gromoll's study [2]. The Levi Civita connection ∇^{CG} and the Riemannian curvature tensor R^{CG} of (TM, g_{CG}) have been obtained in [20] and [11], respectively.

Let g be a Riemannian metric, that is a particular case of Finsler space, if $g_{ij}(x)$ is a metric structure on M, then, $F^2 = g_{ij}(x)y^iy^j$ is a Finsler function, and consequently we can apply the above technique. On TM endowed with a nonlinear connection (derived from a Finsler structure) there exists a natural almost complex structure given by $F\Big(\frac{\delta}{\delta x^j}\Big) = -\frac{\partial}{\partial y^j}$ and $F\Big(\frac{\partial}{\partial y^j}\Big) = \frac{\delta}{\delta x^j}$, which is not always integrable. Similarly, $J\Big(\frac{\delta}{\delta x^j}\Big) = -\frac{\partial}{\partial y^j}$ and $J\Big(\frac{\partial}{\partial y^j}\Big) = \frac{\delta}{\delta x^j}$ define a natural almost product structure on TM. Obviously, such structures can be considered more general on TM. Naturally, if $g(FX, FY) = g(X, Y)$ then (TM, F, g) is almost complex structure on TM, respectively if $g(JX, JY) = g(X, Y)$ then (TM, F, g) is an almost para-complex (Norden) structure. If the structure is integrable $(\nabla J) = 0$ then it is called an para-Kahler-Norden structure.

A Tachibana operator Φ_J applied to the pure metric g is given by [?]

$$\begin{aligned}\Phi_J g(X,Y,Z) &= (JX)(g(Y,Z)) - X(g(JY,Z)) + g((L_Y J)X,Z)\\ &\quad + g((L_Z J)X,Y),\end{aligned} \tag{3}$$

for all $X,Y,Z \in \Gamma(TM)$. It is well known that the theorem ($\nabla J = 0$ is equivalent to $\Phi_J g = 0$), which was proven in [22].
The notion of almost para-complex structure (or almost product structure) on a smooth manifold was introduced in [17] and a survey of further results on para-complex geometry (including para-Hermitian and para-Kähler geometry) can be found for instance in [6],[5]. Also, other further signifiant developments are due in some recent surveys [1], [3], [4], where some aspects concerning the geometry of para-complex manifolds are presented systematically by analogy with the geometry of complex manifolds using some para-holomorphic coordinate systems. See also [9], [13], [15], [16].

Motivated by the above studies, we define a new class of naturally metric on TM given by

$$\begin{aligned} g_f^H(X^H,Y^H)_{(x,u)} &= g_x(X,Y) + X_x(f)Y_x(f),\\ g_f^H(X^H,Y^V)_{(x,u)} &= 0,\\ g_f^H(X^V,Y^V)_{(x,u)} &= g_x(X,Y),\end{aligned} \tag{4}$$

where f is a strictly positive smooth function on M and any vector fields X and Y on M. Let f be a constant then g_f^H is exactly the Sasaki metric. We call g_f^H the horizontal Sasaki gradient metric.

In section 2, we study the Levi-Civita connection ∇^f of a horizontal Sasaki gradient metric. In Section3, we consider the almost product structure $J(X^H) = -X^V$ and $J(X^V) = X^H$ and prove that g_f^H is an almost para-complex (Norden) structure (Lemma 2), and (TM, g_f^H) is an almost para-complex (Norden) manifold (Theorem 2). The problem of integrability, (TM, J, g_f^H) be a paraKahler-Norden manifold is studied in Theorem 3, namely it is proved this is happening if and only if (M,g) is flat. In Section 4 is devoted to the study of Almost product connection symmetric. Let ∇^f be the Levi-Civita connection from Theorem 1, and S a tensor field of type (1,2) defined by formula (14). Then $\overline{\nabla} = \nabla^f - S$ is a symmetric almost product connection on TM if and only if (M,g) is flat (Theorem 6).

2 Horizontal Sasaki gradient metric.

Let (M^m, g) be an m-dimensional Riemannian manifold and (TM,π,M) be its tangent bundle. A local chart $(U,x^i)_{i=\overline{1,m}}$ on M induces a local chart $(\pi^{-1}(U), x^i, y^i)_{i=\overline{1,m}}$ on TM. Denote by Γ_{ij}^k the Christoffel symbols of g and by ∇ the Levi-Civita connection of g.

Let $X = X^i \frac{\partial}{\partial x^i}$ be a local vector field on M. The vertical and the horizontal lifts of X are defined by

$$X^V = X^i \frac{\partial}{\partial y^i}, \tag{5}$$
$$X^H = X^i \frac{\delta}{\delta x^i} = X^i \Big\{ \frac{\partial}{\partial x^i} - y^j \Gamma_{ij}^k \frac{\partial}{\partial y^k} \Big\}. \tag{6}$$

For consequences, we have $\Big(\frac{\partial}{\partial x^i}\Big)^H = \frac{\delta}{\delta x^i}$ and $\Big(\frac{\partial}{\partial x^i}\Big)^V = \frac{\partial}{\partial y^i}$.

Then $\Big(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\Big)_{i=\overline{1,m}}$ is a local adapted frame on TTM.

Lemma 1. *([21]). Let (M, g) be a Riemannian manifold and R its tensor curvature, then for all vector fields $X, Y \in \Gamma(TM)$ we have:*

1. $[X^H, Y^H]_p = [X, Y]_p^H - (R_x(X, Y)u)^V$,
2. $[X^H, Y^V]_p = (\nabla_X Y)_p^V$,
3. $[X^V, Y^V]_p = 0$,

where $p = (x, u) \in TM$.

Definition 1. *Let (M, g) be a Riemannian manifold and $f : M \to]0, +\infty[$. On the tangent bundle TM, we define a horizontal Sasaki gradient metric noted g_f^H by*

$$\begin{aligned} g_f^H(X^H, Y^H)_{(x,u)} &= g_x(X, Y) + X_x(f)Y_x(f), \\ g_f^H(X^H, Y^V)_{(x,u)} &= 0, \\ g_f^H(X^V, Y^V)_{(x,u)} &= g_x(X, Y), \end{aligned} \tag{7}$$

where $X, Y \in \Gamma(TM)$, $(x, u) \in TM$.
From Definition 1, g_f^H is a Riemannian metric, because for each $(x, u) \in TM$, g_f^H satisfies

1 $g_f^H(U, V) = g_f^H(V, U)$ *for all* $U, V \in T_{(x,u)}(TM)$,

2 $g_f^H(U, U) = g(U, U)^2 + (U(f))^2 \geq 0$ *for all* $U \in T_{(x,u)}(TM)$,

3 $g_f^H(U, U) = 0$ *if and only if* $U = 0$.

Remark 1.

1. *If f is a constant, then g_f^H is the Sasaki metric [21],*

2. *In the following, we consider $\alpha = 1 + \|grad\, f\|^2$,where $\|.\|$ denote the norm with respect to (M, g).*

2.1 The Levi-Civita connection.

We shall calculate the Levi-Civita connection ∇^f of TM with horizontal Sasaki gradient metric g_f^H. This connection is characterized by the Koszul formula:

$$\begin{aligned} 2g_f^H(\nabla^f_{\widetilde{X}}\widetilde{Y},\widetilde{Z}) &= \widetilde{X}g_f^H(\widetilde{Y},\widetilde{Z})+\widetilde{Y}g_f^H(\widetilde{Z},\widetilde{X})-\widetilde{Z}g_f^H(\widetilde{X},\widetilde{Y}) \\ &\quad +g_f^H(\widetilde{Z},[\widetilde{X},\widetilde{Y}])+g_f^H(\widetilde{Y},[\widetilde{Z},\widetilde{X}])-g_f^H(\widetilde{X},[\widetilde{Y},\widetilde{Z}]). \end{aligned} \tag{8}$$

for all $\widetilde{X},\widetilde{Y},\widetilde{Z}\in\Gamma(TM)$.

Theorem 1. *Let (M,g) be a Riemannian manifold and (TM,g_f^H) its tangent bundle equipped with the horizontal Sasaki gradient metric. If ∇ (resp ∇^f) denote the Levi-Civita connection of (M,g) (resp (TM,g_f^H)), then we have:*

$$\begin{aligned} 1.\,(\nabla^f_{X^H}Y^H)_p &= (\nabla_X Y)^H_p+\frac{1}{\alpha}g_x(Y,\nabla_X grad\, f)(grad\, f)^H_P \\ &\quad -\frac{1}{2}(R_x(X,Y)u)^V, \\ 2.\,(\nabla^f_{X^H}Y^V)_p &= (\nabla_X Y)^V_p-\frac{1}{2\alpha}g_x(R(u,Y)X,grad\, f)(grad\, f)^H_p \\ &\quad +\frac{1}{2}(R_x(u,Y)X)^H, \\ 3.\,(\nabla^f_{X^V}Y^H)_p &= \frac{1}{2}(R_x(u,X)Y)^H-\frac{1}{2\alpha}g_x(R(u,X)Y,grad\, f)(grad\, f)^H_p, \\ 4.\,(\nabla^f_{X^V}Y^V)_p &= 0, \end{aligned}$$

for all vector fields $X,Y\in\Gamma(TM)$ and $p=(x,u)\in TM$, where R denotes the curvature tensor of (M,g).

Proof.
The proof of Theorem 1 follows directly from Kozul formula (8), Lemma 1 and Definition 1. □

3 Para-Kähler-Norden structures.

Let (M,g) be a Riemannian manifold and (TM,g_f^H) be its tangent bundle equipped with the horizontal Sasaki gradient metric. Consider an almost para-complex structure J on TM defined by

$$\begin{cases} JX^H &= -X^H \\ JX^V &= X^V \end{cases} \tag{9}$$

for all $X\in\Gamma(TM)$.

Lemma 2. *The metric g_f^H is pure with respect to almost para-complex structure J defined by* (9)*.*
i.e for all $X,Y\in\Gamma(TM)$ and $k,h\in\{H,V\}$,

$$g_f^H(JX^k,Y^h)=g_f^H(X^k,JY^h).$$

Proof.

1. $g_f^H(JX^H, Y^H) = g_f^H(-X^H, Y^H) = g_f^H(X^H, -Y^H) = g_f^H(X^H, JY^H)$.

2. $g_f^H(JX^H, Y^V) = g_f^H(-X^H, Y^V) = 0 = g_f^H(X^H, Y^V) = g_f^H(X^H, JY^V)$.

3. $g_f^H(JX^V, Y^V) = g_f^H(X^V, Y^V) = g_f^H(X^H, JY^V)$.

□

Hence we have the following theorem.

Theorem 2. *Let (M, g) be a Riemannian manifold, (TM, g_f^H) be its tangent bundle equipped with the horizontal Sasaki gradient metric and the almost para-complex structure J defined by (9). The triple (TM, J, g_f^H) is an almost para-complex Norden manifold.*

Proposition 1. *Let (M, g) be a Riemannian manifold, (TM, g_f^H) be its tangent bundle equipped with the horizontal Sasaki gradient metric and the almost para-complex structure J defined by (9). for all $X, Y, Z \in \Gamma(TM)$, then we get*

1. $(\Phi_J g_f^H)(X^H, Y^H, Z^H) = 0$,

2. $(\Phi_J g_f^H)(X^V, Y^H, Z^H) = 0$,

3. $(\Phi_J g_f^H)(X^H, Y^V, Z^H) = 2g(R(Z, X)u, Y)$,

4. $(\Phi_J g_f^H)(X^H, Y^H, Z^V) = 2g(R(Y, X)u, Z)$,

5. $(\Phi_J g_f^H)(X^V, Y^V, Z^H) = 0$,

6. $(\Phi_J g_f^H)(X^V, Y^H, Z^V) = 0$,

7. $(\Phi_J g_f^H)(X^H, Y^V, Z^V) = 0$,

8. $(\Phi_J g_f^H)(X^V, Y^V, Z^V) = 0$.

where R denotes the curvature tensor of (M, g).

Proof.
We have for all $\widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \Gamma(TTM)$

$$\begin{aligned}(\Phi_J g_f^H)(\widetilde{X}, \widetilde{Y}, \widetilde{Z}) &= (J\widetilde{X})g_f^H(\widetilde{Y}, \widetilde{Z}) - \widetilde{X}g_f^H(J\widetilde{Y}, \widetilde{Z}) + g_f^H\big((L_{\widetilde{Y}}J)\widetilde{X}, \widetilde{Z}\big) \\ &\quad + g_f^H\big(\widetilde{Y}, (L_{\widetilde{Z}}J)\widetilde{X}\big).\end{aligned}$$

$$
\begin{aligned}
1.\ (\Phi_J g_f^H)(X^H, Y^H, Z^H) &= (JX^H) g_f^H(Y^H, Z^H) - X^H g_f^H(JY^H, Z^H) \\
&\quad + g_f^H\big((L_{Y^H} J)X^H, Z^H\big) + g_f^H\big(Y^H, (L_{Z^H} J)X^H\big) \\
&= -X^H g_f^H(Y^H, Z^H) + X^H g_f^H(Y^H, Z^H) \\
&\quad + g_f^H\big(L_{Y^H} JX^H - J(L_{Y^H} X^H), Z^H\big) \\
&\quad + g_f^H\big(Y^H, L_{Z^H} JX^H - J(L_{Z^H} X^H)\big) \\
&= -g_f^H\big([Y^H, X^H], Z^H\big) - g_f^H\big(J[Y^H, X^H], Z^H\big) \\
&\quad - g_f^H\big(Y^H, [Z^H, X^H]\big) - g_f^H\big(Y^H, J[Z^H, X^H]\big) \\
&= 0.
\end{aligned}
$$

$$
\begin{aligned}
2.\ (\Phi_J g_f^H)(X^V, Y^H, Z^H) &= (JX^V) g_f^H(Y^H, Z^H) - X^V g_f^H(JY^H, Z^H)) \\
&\quad + g_f^H\big((L_{Y^H} J)X^V, Z^H\big) + g_f^H\big(Y^H, (L_{Z^H} J)X^V\big) \\
&= + g_f^H\big([Y^H, X^V], Z^H\big) - g_f^H\big(J[Y^H, X^V], Z^H\big) \\
&\quad + g_f^H\big(Y^H, [Z^H, X^V]\big) - g_f^H\big(Y^H, J[Z^H, X^V]\big) \\
&= 2g_f^H\big([Y^H, X^V], Z^H\big) + 2g_f^H\big(Y^H, [Z^H, X^V]\big) \\
&= 2g_f^H\big((\nabla_Y X)^V, Z^H\big) + 2g_f^H\big(Y^H, (\nabla_Z X)^V\big) \\
&= 0.
\end{aligned}
$$

$$
\begin{aligned}
3.\ (\Phi_J g_f^H)(X^H, Y^V, Z^H) &= (JX^H) g_f^H(Y^V, Z^H) - X^H g_f^H(JY^V, Z^H) \\
&\quad + g_f^H\big((L_{Y^V} J)X^H, Z^H\big) + g_f^H\big(Y^V, (L_{Z^H} J)X^H\big) \\
&= -g_f^H\big([Y^V, X^H], Z^H\big) - g_f^H\big(J[Y^V, X^H], Z^H\big) \\
&\quad - g_f^H\big(Y^V, [Z^H, X^H]\big) - g_f^H\big(Y^V, J[Z^H, X^H]\big) \\
&= -2g_f^H\big(Y^V, [Z^H, X^H]\big) \\
&= 2g_f^H\big(Y^V, (R(Z, X)u)^V\big) \\
&= 2g(R(Z, X)u, Y).
\end{aligned}
$$

$$
\begin{aligned}
4.\ (\Phi_J g_f^H)(X^H, Y^H, Z^V) &= (JX^H) g_f^H(Y^H, Z^V) - X^H g_f^H(JY^H, Z^V) \\
&\quad + g_f^H\big((L_{Y^H} J)X^H, Z^V\big) + g_f^H\big(Y^H, (L_{Z^V} J)X^H\big) \\
&= -g_f^H\big([Y^H, X^H], Z^V\big) - g_f^H\big(J[Y^H, X^H], Z^V\big) \\
&\quad - g_f^H\big(Y^H, [Z^V, X^H]\big) - g_f^H\big(Y^H, J[Z^V, X^H]\big) \\
&= -2g_f^H\big([Y^H, X^H], Z^V\big) \\
&= 2g(R(Y, X)u, Z).
\end{aligned}
$$

The other formulas are obtained by a similar calculation. □

Therefore, we have the following result.

Theorem 3. *Let (M,g) be a Riemannian manifold, (TM, g_f^H) be its tangent bundle equipped with the horizontal Sasaki gradient metric and the almost para-complex structure J defined by* (9). *The triple (TM, J, g_f^H) is a para-Kähler-Norden (or para-holomorphic Norden) manifold if and only if M is flat.*

Proof.
For all $X, Y, Z \in \Gamma(TM)$, $k, h, l \in \{H, V\}$ and $(x,u) \in TM$

$$(\phi_J g_f^S)(X^h, Y^k, Z^l) = 0 \quad \Leftrightarrow \quad \begin{cases} g(R(Z,X)Y, u) & = 0 \\ g(R(Y,X)Z, u) & = 0 \end{cases}$$
$$\Leftrightarrow \quad \begin{cases} R(Z,X)Y & = 0 \\ R(Y,X)Z & = 0 \end{cases}$$
$$\Leftrightarrow \quad R = 0$$

□

Remark 2. *Let (M,g) be a Riemannian manifold, (TM, g_f^H) be its tangent bundle equipped with the horizontal Sasaki gradient metric. Another almost para-complex structure $\bar{J}$ on TM is defined by*

$$\begin{cases} \bar{J}X^H & = X^V \\ \bar{J}X^V & = X^H \end{cases} \tag{10}$$

The horizontal Sasaki gradient metric g_f^H is pure with respect to $\bar{J}$ if and only if f is constant function (g_f^H is the Sasaki metric).

Now we study a generalization of the almost para-complex structure defined by 9. Let (M,g) be a Riemannian manifold and (TM, g_f^H) be its tangent bundle equipped with the horizontal Sasaki gradient metric. We define an endomorphism $J : TTM \to TTM$ by, for all $X \in \Gamma(TM)$

$$\begin{cases} JX^H & = -X^H + \eta X(f)(grad\, f)^H \\ \\ JX^V & = X^V + \mu X(f)(grad\, f)^V \end{cases} \tag{11}$$

where $\eta, \mu : M \to \mathbb{R}$ are smooth functions.

Remark 3. **1.** *If $f = constant$ or $\eta = \mu = 0$, we have the almost para-complex structure defined by* (9),

2. $J(grad\, f)^H = (-1 + \eta(\alpha - 1))(grad\, f)^H$,

3. $J(grad\, f)^V = (1 + \mu(\alpha - 1))(grad\, f)^V$,

where $\alpha = 1 + \|grad\, f\|^2$.

In the following, we consider $f \neq constant$ and $\eta \neq 0 \neq \mu$.

Lemma 3. *Let (M,g) be a Riemannian manifold and (TM,g_f^H) be its tangent bundle equipped with the horizontal Sasaki gradient metric. The endomorphism J defined by* (15) *is an almost para-complex structure on TM if and only if*

$$\eta=-\mu=\frac{2}{\alpha-1}.$$

Proof.
1) Let $Z\in T_pTM$, then $Z=X^H+Y^V$, $X,Y\in T_xM$ and $p=(x,u)\in TM$.

$$\begin{aligned} J^2Z &= J(J(X^H+Y^V))\\ &= J(JX^H+JY^V)\\ &= J\big(-X^H+\eta X(f)(grad\, f)^H+Y^V+\mu Y(f)(grad\, f)^V\big)\\ &= -(-X^H+\eta X(f)(grad\, f)^H)+\eta X(f)(-1+\eta(\alpha-1))(grad\, f)^H\\ &\quad +Y^V+\mu Y(f)(grad\, f)^V+\mu Y(f)(1+\mu(\alpha-1))(grad\, f)^V \end{aligned}$$

$$\begin{aligned} J^2Z &= X^H+\eta X(f)(-2+\eta(\alpha-1))(grad\, f)^H\\ &\quad +Y^V+\mu Y(f)(2+\mu(\alpha-1))(grad\, f)^V\\ &= Z+\eta X(f)(-2+\eta(\alpha-1))(grad\, f)^H\\ &\quad +\mu Y(f)(2+\mu(\alpha-1))(grad\, f)^V \end{aligned}$$

J is an almost product structure on TM if and only if

$$-2+\eta(\alpha-1)=2+\mu(\alpha-1)=0,$$

i.e

$$\eta=-\mu=\frac{2}{\alpha-1}.$$

in this case J is written in the following form:

$$\begin{cases} JX^H &= -X^H+\dfrac{2}{\alpha-1}X(f)(grad\, f)^H\\ JX^V &= X^V-\dfrac{2}{\alpha-1}X(f)(grad\, f)^V \end{cases}\tag{12}$$

and

$$\begin{cases} J(grad\, f)^H &= (grad\, f)^H\\ J(grad\, f)^V &= -(grad\, f)^V. \end{cases}$$

2) Let $\{e_1,\cdots,e_m\}$ be a local basis on M, $A_i=e_i^H-\dfrac{1}{\alpha-1}e_i(f)(grad\, f)^H$ and

$B_i = -e_i^V + \frac{1}{\alpha-1} e_i(f)(grad\, f)^V$, for all $i = \overline{1,m}$.

$$\begin{aligned} J(A_i) &= J\big(e_i^H - \frac{1}{\alpha-1} e_i(f)(grad\, f)^H\big) \\ &= J(e_i^H) - \frac{1}{\alpha-1} e_i(f) J(grad\, f)^H \\ &= -e_i^H + \frac{2}{\alpha-1} e_i(f)(grad\, f)^H - \frac{1}{\alpha-1} e_i(f)(grad\, f)^H \\ &= -e_i^H + \frac{1}{\alpha-1} e_i(f)(grad\, f)^H \\ &= -A_i. \end{aligned}$$

Then

$$TTM^- = \{Z \in T_pTM, JZ = -Z\} =< (A_i)_{i=\overline{1,m}} > .$$

Similarly we have:

$$J(B_i) = B_i.$$

Then

$$TTM^+ = \{Z \in T_pTM, JZ = Z\} =< (B_i)_{i=\overline{1,m}} > .$$

□

Proposition 2. *Let (M,g) be a Riemannian manifold, (TM, g_f^H) be its tangent bundle equipped with the horizontal Sasaki gradient metric and the almost para-complex structure J defined by (12), then g_f^H is pure with respect to J i.e (TM, J, g_f^H) is an almost para-complex Norden manifold.*

Proof.

$$\begin{aligned} 1.\ g_f^H(JX^H, Y^H) &= g_f^H(-X^H + \frac{2}{\alpha-1} X(f)(grad\, f)^H, Y^H) \\ &= -g_f^H(X^H, Y^H) + \frac{2}{\alpha-1} X(f) g_f^H((grad\, f)^H, Y^H) \\ &= -g_f^H(X^H, Y^H) + \frac{2}{\alpha-1} g_f^H(X^H, (grad\, f)^H) Y(f) \\ &= g_f^H(X^H, -Y^H + \frac{2}{\alpha-1} Y(f)(grad\, f)^H) \\ &= g_f^H(X^H, JY^H). \end{aligned}$$

$$\begin{aligned} 2)\ g_f^H(JX^H, Y^V) &= g_f^H(-X^H + \frac{2}{\alpha-1} X(f)(grad\, f)^H, Y^V) \\ &= 0 \\ &= g_f^H(X^H, Y^V - \frac{2}{\alpha-1} Y(f)(grad\, f)^V) \\ &= g_f^H(X^H, JY^V). \end{aligned}$$

$$\begin{aligned}
3)\ g_f^H(JX^V,Y^V) &= g_f^H(X^V - \frac{2}{\alpha-1}X(f)(grad\,f)^V, Y^V)\\
&= g_f^H(X^V,Y^V) - \frac{2}{\alpha-1}X(f)g_f^H((grad\,f)^V,Y^V)\\
&= g_f^H(X^V,Y^V) - \frac{2}{\alpha-1}g_f^H(X^V,(grad\,f)^V)Y(f)\\
&= g_f^H(X^V,Y^V - \frac{2}{\alpha-1}Y(f)(grad\,f)^V)\\
&= g_f^H(X^V,JY^V).
\end{aligned}$$

□

Now we study a generalization of the almost para-complex structure defined by 10. We define an endomorphism $J : TTM \to TTM$ by, for all $X \in \Gamma(TM)$

$$\begin{cases} JX^H &= X^V + \eta X(f)(grad\,f)^V \\ JX^V &= X^H + \mu X(f)(grad\,f)^H \end{cases} \tag{13}$$

where $\eta, \mu : M \to \mathbb{R}$ are smooth functions.

Remark 4. **1.** *If $f = constant$ or $\eta = \mu = 0$, we have the almost para-complex structure defined by* (10)*,*

2. $J(grad\,f)^H = (1+\eta(\alpha-1))(grad\,f)^V$*,*

3. $J(grad\,f)^V = (1+\mu(\alpha-1))(grad\,f)^H$*,*

where $\alpha = 1 + \|grad\,f\|^2$*.*

In the following, we consider $f \neq constant$ and $\eta \neq 0 \neq \mu$.

Lemma 4. *Let (M,g) be a Riemannian manifold and (TM, g_f^H) be its tangent bundle equipped with the horizontal Sasaki gradient metric. The endomorphism J defined by* (13) *is an almost para-complex structure on TM if and only if*

$$\mu = \frac{-\eta}{1+\eta(\alpha-1)}.$$

Proof.
1) Let $Z \in T_pTM$, then $Z = X^H + Y^V$, $X, Y \in T_xM$ and $p = (x,u) \in TM$.

$$\begin{aligned}
J^2Z &= J(J(X^H+Y^V))\\
&= J(JX^H + JY^V)\\
&= J\big(X^V + \eta X(f)(grad\,f)^V + Y^H + \mu Y(f)(grad\,f)^H\big)\\
&= X^H + \mu X(f)(grad\,f)^H) + \eta X(f)(1+\mu(\alpha-1))(grad\,f)^H\\
&\quad + Y^V + \eta Y(f)(grad\,f)^V + \mu Y(f)(1+\eta(\alpha-1))(grad\,f)^V\\
&= Z + X(f)[\mu + \eta(1+\mu(\alpha-1))](grad\,f)^H\\
&\quad + Y(f)[\eta + \mu(1+\eta(\alpha-1))](grad\,f)^V
\end{aligned}$$

J is an almost product structure on TM if and only if

$$\mu + \eta(1 + \mu(\alpha - 1)) = \eta + \mu(1 + \eta(\alpha - 1)) = 0,$$

i.e

$$\mu = \frac{-\eta}{1 + \eta(\alpha - 1)}.$$

in this case we have

$$\begin{cases} J(grad\, f)^H &= \dfrac{-\eta}{\mu}(grad\, f)^V \\ \\ J(grad\, f)^V &= \dfrac{-\mu}{\eta}(grad\, f)^H. \end{cases}$$

2) Let $\{e_1, \cdots, e_m\}$ be a local basis on M, and for all $i = \overline{1,m}$ we put,
$A_i = e_i^H + \frac{\mu}{2} e_i(f)(grad\, f)^H - e_i^V - \frac{\eta}{2} e_i(f)(grad\, f)^V$,
$B_i = e_i^H + \frac{\mu}{2} e_i(f)(grad\, f)^H + e_i^V + \frac{\eta}{2} e_i(f)(grad\, f)^V$.

$$\begin{aligned} J(A_i) &= J(e_i^H + \frac{\mu}{2} e_i(f)(grad\, f)^H - e_i^V - \frac{\eta}{2} e_i(f)(grad\, f)^V) \\ &= Je_i^H + \frac{\mu}{2} e_i(f) J(grad\, f)^H - Je_i^V - \frac{\eta}{2} e_i(f) J(grad\, f)^V \\ &= e_i^V + \eta e_i(f)(grad\, f)^V + \frac{\mu}{2} e_i(f) \frac{-\eta}{\mu}(grad\, f)^V \\ &\quad - e_i^H - \mu e_i(f)(grad\, f)^H - \frac{\eta}{2} e_i(f) \frac{-\mu}{\eta}(grad\, f)^H \\ &= e_i^V + \frac{\eta}{2} e_i(f)(grad\, f)^V - e_i^H - \frac{\mu}{2} e_i(f)(grad\, f)^H \\ &= -A_i. \end{aligned}$$

Then

$$TTM^- = \{Z \in T_pTM, JZ = -Z\} =< (A_i)_{i=\overline{1,m}} > .$$

Similarly we have:

$$J(B_i) \;=\; B_i.$$

Then

$$TTM^+ = \{Z \in T_pTM, JZ = Z\} =< (B_i)_{i=\overline{1,m}} > .$$

□

Proposition 3. *Let (M, g) be a Riemannian manifold, (TM, g_f^H) be its tangent bundle equipped with the horizontal Sasaki gradient metric and the almost para-complex structure J defined by* (13)*, then*

1. *If $\eta \neq 1 + \mu\alpha$, g_f^H is never pure with respect to J.*

2. *If $\eta = 1 + \mu\alpha$, g_f^H is pure with respect to J.*

Proof.

$$
\begin{aligned}
i)\ g_f^H(JX^H, Y^H) &= g_f^H(X^V + \eta X(f)(grad\, f)^V, Y^H)\\
&= 0\\
&= g_f^H(X^H, Y^V + \eta Y(f)(grad\, f)^V)\\
&= g_f^H(X^H, JY^H).
\end{aligned}
$$

$$
\begin{aligned}
ii)\ g_f^H(JX^V, Y^V) &= g_f^H(X^H + \mu X(f)(grad\, f)^H, Y^V)\\
&= 0\\
&= g_f^H(X^V, Y^H + \mu Y(f)(grad\, f)^H)\\
&= g_f^H(X^V, JY^V).
\end{aligned}
$$

$$
\begin{aligned}
iii)\ g_f^H(JX^H, Y^V) &= g_f^H(X^V + \eta X(f)(grad\, f)^V, Y^V)\\
&= g(X, Y) + \eta X(f)Y(f).
\end{aligned}
$$

$$
\begin{aligned}
iv)\ g_f^H(X^H, JY^V) &= g_f^H(X^H, Y^H + \mu Y(f)(grad\, f)^H)\\
&= g(X, Y) + X(f)Y(f) + \alpha X(f)\mu Y(f)\\
&= g(X, Y) + (1 + \mu\alpha)X(f)Y(f).
\end{aligned}
$$

From *iii*) and *iv*) then:

1. If $\eta \neq 1 + \mu\alpha$, g_f^H is never pure with respect to J.

2. If $\eta = 1 + \mu\alpha$, g_f^H is pure with respect to J. In this case we have,

$$
\begin{cases}
\eta &= \dfrac{-1 + \varepsilon\sqrt{\alpha}}{\alpha - 1}\\[2ex]
\mu &= \dfrac{-\alpha + \varepsilon\sqrt{\alpha}}{\alpha(\alpha - 1)}
\end{cases}
\quad , \quad \varepsilon = \pm 1.
$$

□

Therefore, we have the following result.

Theorem 4. *Let (M, g) be a Riemannian manifold, (TM, g_f^H) be its tangent bundle equipped with the horizontal Sasaki gradient metric and the almost para-complex structure J defined by* (13)*, then*

1. *If $\eta \neq 1 + \mu\alpha$, the triple (TM, J, g_f^H) is never an almost para-complex Norden manifold.*

2. *If $\eta = 1 + \mu\alpha$, the triple $(TM, J, g_f^H)$is an almost para-complex Norden manifold.*

4 Almost product connection symmetric.

Let (M,g) be a Riemannian manifold, (TM,g_f^H) be its tangent bundle equipped with the horizontal Sasaki gradient metric and the almost product structure J defined by (9). ∇^f denotes the Levi-Civita connection of (TM,g_f^H) given by Theorem 1. We define a tensor field S of type $(1,2)$ and linear connection $\overline{\nabla}$ on TM by, for all $\widetilde{X},\widetilde{Y}\in\Gamma(TTM)$

$$S(\widetilde{X},\widetilde{Y})=\frac{1}{2}\big[(\nabla^f_{J\widetilde{Y}}J)\widetilde{X}+J\big((\nabla^f_{\widetilde{Y}}J)\widetilde{X}\big)-J\big((\nabla^f_{\widetilde{X}}J)\widetilde{Y}\big)\big], \tag{14}$$

$$\overline{\nabla}_{\widetilde{X}}\widetilde{Y}=\nabla^f_{\widetilde{X}}\widetilde{Y}-S(\widetilde{X},\widetilde{Y}). \tag{15}$$

$\overline{\nabla}$ is an almost product connection on TM, (see [8]).

Lemma 5. *Let (M,g) be a Riemannian manifold, (TM,g_f^H) be its tangent bundle equipped with the horizontal Sasaki gradient metric and the almost product structure J defined by* (9)*. Then tensor field S is as follows, for all $X,Y\in\Gamma(TM)$.*

1. $S(X^H,Y^H)=-\frac{1}{2}(R(X,Y)u)^V$,

2. $S(X^H,Y^V)=\frac{1}{2}(R(u,Y)X)^H-\frac{1}{2\alpha}g(R(u,Y)X,grad\,f)(grad\,f)^H$,

3. $S(X^V,Y^H)=-(R(u,X)Y)^H+\frac{1}{\alpha}g(R(u,X)Y,grad\,f)(grad\,f)^H$,

4. $S(X^V,Y^V)=0$.

Proof.
Using Theorem 1, we have for all $X,Y\in\Gamma(TM)$

$$\begin{aligned}
1.\ S(X^H,Y^H) &= \frac{1}{2}\big[(\nabla^f_{JY^H}J)X^H+J\big((\nabla^f_{Y^H}J)X^H\big)-J\big((\nabla^f_{X^H}J)Y^H\big)\big]\\
&= \frac{1}{2}\big[J(\nabla^f_{X^H}Y^H)+\nabla^f_{X^H}Y^H\big]\\
&= \frac{1}{2}\big[-(\nabla_XY)^H-\frac{1}{\alpha}g(Y,\nabla_Xgrad\,f)(grad\,f)^H\\
&\quad -\frac{1}{2}(R(X,Y)u)^V+(\nabla_XY)^H\\
&\quad +\frac{1}{\alpha}g(Y,\nabla_Xgrad\,f)(grad\,f)^H-\frac{1}{2}(R(X,Y)u)^V\big]\\
&= -\frac{1}{2}(R(X,Y)u)^V.
\end{aligned}$$

$$\begin{aligned}
2.\ S(X^H,Y^V) &= \frac{1}{2}\big[(\nabla^f_{JY^V}J)X^H + J\big((\nabla^f_{Y^V}J)X^H\big) - J\big((\nabla^f_{X^H}J)Y^V\big)\big] \\
&= -J\big(\nabla^f_{Y^V}X^H\big) - \nabla^f_{Y^V}X^H + \frac{1}{2}\big[-J\big(\nabla^f_{X^H}Y^V\big) + \nabla^f_{X^H}Y^V\big] \\
&= \frac{1}{2}(R(u,Y)X)^H - \frac{1}{2\alpha}g(R(u,Y)X, grad\, f)(grad\, f)^H \\
&\quad -\frac{1}{2}(R(u,Y)X)^H + \frac{1}{2\alpha}g(R(u,Y)X, grad\, f)(grad\, f)^H \\
&\quad +\frac{1}{2}\big[-(\nabla_X Y)^V - \frac{1}{2\alpha}g(R(u,Y)X, grad\, f)(grad\, f)^H \\
&\quad +\frac{1}{2}(R(u,Y)X)^H + (\nabla_X Y)^V \\
&\quad -\frac{1}{2\alpha}g(R(u,Y)X, grad\, f)(grad\, f)^H + \frac{1}{2}(R(u,Y)X)^H\big] \\
&= \frac{1}{2}(R(u,Y)X)^H - \frac{1}{2\alpha}g(R(u,Y)X, grad\, f)(grad\, f)^H.
\end{aligned}$$

The other formulas are obtained by a similar calculation. □

Theorem 5. *Let (M,g) be a Riemannian manifold, (TM, g^H_f) be its tangent bundle equipped with the horizontal Sasaki gradient metric and the almost product structure J defined by* (9)*. Then the almost product connection $\overline{\nabla}$ defined by* (15) *is as follows, for all $X,Y \in \Gamma(TM)$*

$$\begin{aligned}
1.\ \overline{\nabla}_{X^H}Y^H &= (\nabla_X Y)^H + \frac{1}{\alpha}g(Y, \nabla_X grad\, f)(grad\, f)^H, \\
2.\ \overline{\nabla}_{X^H}Y^V &= (\nabla_X Y)^V, \\
3.\ \overline{\nabla}_{X^V}Y^H &= \frac{3}{2}(R(u,X)Y)^H - \frac{3}{2\alpha}g(R(u,X)Y, grad\, f)(grad\, f)^H, \\
4.\ \overline{\nabla}_{X^V}Y^V &= 0.
\end{aligned}$$

Proof.
Using Theorem 1 and Lemma 5, we have for all $X,Y \in \Gamma(TM)$.

$$\begin{aligned}
1.\ \overline{\nabla}_{X^H}Y^H &= \nabla^f_{X^H}Y^H - S(X^H,Y^H) \\
&= (\nabla_X Y)^H + \frac{1}{\alpha}g(Y, \nabla_X grad\, f)(grad\, f)^H - \frac{1}{2}(R(X,Y)u)^V \\
&\quad +\frac{1}{2}(R(X,Y)u)^V \\
&= (\nabla_X Y)^H + \frac{1}{\alpha}g(Y, \nabla_X grad\, f)(grad\, f)^H.
\end{aligned}$$

$$\begin{aligned}
2.\ \overline{\nabla}_{X^H}Y^V &= \nabla^f_{X^H}Y^V - S(X^H,Y^V)\\
&= (\nabla_X Y)^V - \frac{1}{2\alpha}g(R(u,Y)X, grad\, f)(grad\, f)^H\\
&\quad +\frac{1}{2}(R(u,Y)X)^H\\
&\quad -\frac{1}{2}(R(u,Y)X)^H + \frac{1}{2\alpha}g(R(u,Y)X, grad\, f)(grad\, f)^H\\
&= (\nabla_X Y)^V.
\end{aligned}$$

The other formulas are obtained by a similar calculation. □

Lemma 6. *Let (M,g) be a Riemannian manifold, (TM, g^H_f) be its tangent bundle equipped with the horizontal Sasaki gradient metric and the almost product structure J defined by* (9). *Then the torsion tensor $\overline{T}$ of $\overline{\nabla}$, is as follows, for all $X, Y \in \Gamma(TM)$*

1. $\overline{T}(X^H, Y^H) = (R(X,Y)u)^V,$

2. $\overline{T}(X^H, Y^V) = -\frac{3}{2}(R(u,Y)X)^H + \frac{3}{2\alpha}g(R(u,Y)X, grad\, f)(grad\, f)^H,$

3. $\overline{T}(X^V, Y^H) = \frac{3}{2}(R(u,X)Y)^H - \frac{3}{2\alpha}g(R(u,X)Y, grad\, f)(grad\, f)^H,$

4. $\overline{T}(X^V, Y^V) = 0.$

Proof.
We have for all $\widetilde{X}, \widetilde{Y} \in \Gamma(TTM)$

$$\begin{aligned}
\overline{T}(\widetilde{X},\widetilde{Y}) &= \overline{\nabla}_{\widetilde{X}}\widetilde{Y} - \overline{\nabla}_{\widetilde{Y}}\widetilde{X} - [\widetilde{X},\widetilde{Y}]\\
&= \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} - S(\widetilde{X},\widetilde{Y}) - \widetilde{\nabla}_{\widetilde{Y}}\widetilde{X} + S(\widetilde{Y},\widetilde{X}) - [\widetilde{X},\widetilde{Y}]\\
&= S(\widetilde{Y},\widetilde{X}) - S(\widetilde{X},\widetilde{Y})
\end{aligned}$$

Using Lemma 5, we have

$$\begin{aligned}
1.\ \overline{T}(X^H,Y^H) &= S(Y^H,X^H) - S(X^H,Y^H)\\
&= -\frac{1}{2}(R(Y,X)u)^V + \frac{1}{2}(R(X,Y)u)^V\\
&= (R(X,Y)u)^V.
\end{aligned}$$

$$\begin{aligned}
2.\ \overline{T}(X^H,Y^V) &= S(Y^V,X^H) - S(X^H,Y^V)\\
&= -(R(u,Y)X)^H + \frac{1}{\alpha}g(R(u,X)Y, grad\, f)(grad\, f)^H\\
&\quad -\frac{1}{2}(R(u,Y)X)^H + \frac{1}{2\alpha}g(R(u,Y)X, grad\, f)(grad\, f)^H\\
&= -\frac{3}{2}(R(u,Y)X)^H + \frac{3}{2\alpha}g(R(u,Y)X, grad\, f)(grad\, f)^H.
\end{aligned}$$

The other formulas are obtained by a similar calculation. □

Theorem 6. *Let (M, g) be a Riemannian manifold, (TM, g_f^H) be its tangent bundle equipped with the horizontal Sasaki gradient metric, the almost product structure J defined by* (9) *and the almost product connection $\overline{\nabla}$ defined by* (15), *then*
$\overline{\nabla}$ is symmetric if and only if M is flat.

Proof.
For all $X, Y \in \Gamma(TM)$.

$$\overline{\nabla} \text{ } is \text{ } symmetric \quad \Leftrightarrow \quad \overline{T}(X^k, Y^h) = 0 \;,\; k, h \in \{H, V\}$$

Then

$$\begin{cases} (R(X,Y)u)^V & = 0 \\ -\dfrac{3}{2}(R(u,Y)X)^H + \dfrac{3}{2\alpha} g(R(u,Y)X, grad\, f)(grad\, f)^H & = 0 \\ \dfrac{3}{2}(R(u,X)Y)^H - \dfrac{3}{2\alpha} g(R(u,X)Y, grad\, f)(grad\, f)^H & = 0 \end{cases}$$

Hence we have $R = 0$. □

References

[1] Alekseevsky D. V., Medori C. and Tomassini A., *Homogeneous para-Kähler Einstein manifolds*, Russ. Math. Surv **64** (2009), no. 1.

[2] Cheeger J. and Gromoll D., *On the structure of complete manifolds of non-negative curvature*, Ann. of Math. **96** (1972), no. 2, 413-443.

[3] Cortes V., Mayer C., Mohaupt T. and Saueressig F., *Special geometry of Euclidean supersymmetry. I. Vector multiplets*, J. High Energy Phys **2004**, JHEP03(2004).

[4] Cortes V., Mayer C., Mohaupt T. and Saueressig F., *Special geometry of Euclidean supersymmetry. II. Hypermultiplets and the c-map*, J. High Energy Phys. Springer, **6**, 025(2005), 37 p.

[5] Cruceanu V., Fortuny P. and Gadea P. M., *A survey on paracomplex geometry*, Rocky Mountain J. Math. **26** (1996), 83-115.

[6] Cruceanu V., Gadea P. M. and Munoz Masque J., *Para-Hermitian and para-Kähler manifolds*, Quaderni Inst. Mat. Univ. Messina **1** (1995), 1-72.

[7] Dombrowski P., *On the Geometry of the Tangent Bundle*, J. Reine Angew. Math. **210** (1962), 73-88.

[8] De León M. and Rodrigues P. R., *Methods of differential geometry in analytical mechanics*, North-Holland Mathematics Studies, (1989).

[9] Erdem S., *Paraholomorphic structures and the connections of vector bundles over paracomplex manifolds*, New Zealand J. of Math. **30** (2001), no.1, 41-50.

[10] Gezer A., *On the Tangent Bundle With Deformed Sasaki Metric*, International Electronic Journal of Geometry **6** , no. 2 (2013), 19-31.

[11] Gudmundsson S. and Kappos E., *On the Geometry of the Tangent Bundle with the Cheeger-Gromoll Metric*, Tokyo J. Math. **25** (2002), no. 1.

[12] Jian W.and Yong W., *On the geometry of tangent bundles with the rescaled metric*, arXiv:1104.5584v1 [math.DG] 29 Apr 2011.

[13] Kaneyuki S. and Kozai M., *Paracomplex structures and affine symmetric spaces*, Tokyo J. Math. **8** (1985), 81-98.

[14] Kowalski O. and Sekizawa M., *On Riemannian geometry of tangent sphere bundles with arbitrary constant radius*, Arch. Math. (Brno) **44** (2008), 391-401.

[15] Krahe M., *Para-pluriharmonic maps and twistor spaces*, Ch. 15, pag. 497-558, Handbook of pseudo-Riemannian Geometry and Supersymmetry, Ed. V. Cortes, 2010.

[16] Lawn M.A. and SchKäfer L., *Decompositions of para-complex vector bundles and paracomplex affine immersions*, Results Math. **48** (2005), 246-274.

[17] Libermann P., *Sur les structures proofsque paracomplexes*, C. R. Acad. Sci. Paris **234** (1952), 2517-2519.

[18] Musso E. and Tricerri F., *Riemannian metrics on tangent bundles*, Ann. Mat. Pura. Appl. **(4)** 150 (1988), 1-19.

[19] Sasaki S., *On the differential geometry of tangent bundles of Riemannian manifolds II*, Tohoku Math. J. **14** (1962), 146-155.

[20] Sekizawa M., *Curvatures of tangent bundles with Cheeger-Gromoll metric*, Tokyo J. Math. **14** (1991), no. 2, 407-417.

[21] Yano K. and Ishihara S., *Tangent and cotangent bundles*, Marcel Dekker INC. New York 1973, 1-173.

[22] Salimov, A. A., Iscan, M., Etayo, F., *Paraholomorphic B-manifold and its properties*, Topology Appl., **154** (2007), no. 4, 925-933.

[23] Tachibana, S., *Analytic tensor and its generalization*, Tohoku Math. J., **12** (1960), no.2, 208-221.