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SOME SPECTRAL SETS OF LINEAR OPERATOR PENCILS ON NON-ARCHIMEDEAN BANACH SPACES

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Abstract

In this paper, we define the notions of trace pseudo-spectrum, $\varepsilon-$determinant spectrum and $\varepsilon-trace$ of bounded linear operator pencils on non-Archimedean Banach spaces. Many results are proved about trace pseudo-spectrum, $\varepsilon-$determinant spectrum and $\varepsilon-trace$ of bounded linear operator pencils on non-Archimedean Banach spaces. Examples are given to support our work.

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1 Introduction

The analysis of eigenvalues and eigenvectors had a great effect on mathematics, science, engineering, and many other fields. Then, there are countless applications for this type of analysis. The study of matrix pencils is by now a very thoughtful subject, with the notion of pseudospectrum playing a key role in the theory. However, matrix pencils play an important role in numerical linear algebra, perturbation theory, generalized eigenvalue problems, for more details, we refer to [4] and [11].

Throughout this paper, E_ω is a non-Archimedean (n.a) Hilbert space over a (n.a) non trivially complete valued field $\mathbb{K}$ with valuation $|\cdot|$, ([3], p 54), $\mathcal{L}(E_\omega)$ denote the set of all bounded linear operators on E_ω, $\mathbb{Q}_p$ is the field of p-adic numbers ($p \geq 2$ being a prime) equipped with p-adic valuation $|\cdot|_p$, $\mathbb{Z}_p$ denotes the ring of p-adic integers of $\mathbb{Q}_p$, it is the unit ball of $\mathbb{Q}_p$. For more details and related issues, we refer to [3] and [10]. We denote the completion of algebraic

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closure of $\mathbb{Q}_p$ under the p-adic absolute value $|\cdot|_p$ by $\mathbb{C}_p$ (see [10]). A non-Archimedean Banach space X over $\mathbb{K}$ is said to be a free Banach space if there exists a family $(e_i)_{i\in I}$ of X such that each element $x \in X$ can be written uniquely as $x = \sum_{i\in I} x_i e_i$ and $\|x\| = \sup_{i\in I} |x_i|\|e_i\|$, $(e_i)_{i\in I}$ is called an orthogonal base for X, and if for all $i \in I$, $\|e_i\| = 1$, then $(e_i)_{i\in I}$ is called an orthonormal base. For more details, we refer to [2] and [3]. For $A \in \mathcal{L}(\mathbb{K}^n)$ can be written as a finite matrix. For more details see [3], p. 63.

In this paper, we study the problem of finding the eigenvalues of the generalized eigenvalue problem

$$Ax = \lambda Bx$$

for $\lambda \in \mathbb{K}$ and $x \in \mathbb{K}^n$ or $x \in E_\omega$. $\mathcal{M}_n(\mathbb{K})$ denote the algebra of all $n \times n$ (n.a) matrices, I denote the $n \times n$ identity matrix. Let $A \in \mathcal{M}_n(\mathbb{K})$, the trace, determinant of A are denoted by $Tr(A)$ and $det(A)$ respectively. For more details, we refer to see [2], [3], [7] and [8]. We have the following definitions.

Definition 1. *[6] Let X be a non-Archimedean Banach space. For a pair (A, B) of operators in $\mathcal{L}(X)$, the spectrum $\sigma(A, B)$ of linear operator pencil (A, B) or of the pair (A, B) defined by*

$$\begin{aligned} \sigma(A, B) &= \{\lambda \in \mathbb{K} : \ A - \lambda B \text{ is not invertible in } \mathcal{L}(X)\} \\ &= \{\lambda \in \mathbb{K} : 0 \in \sigma(A - \lambda B)\}. \end{aligned}$$

The resolvent set $\rho(A, B)$ of linear operator pencil (A, B) is the complement of $\sigma(A, B)$ in $\mathbb{K}$ given by

$$\rho(A, B) = \{\lambda \in \mathbb{K} : \ R_\lambda(A, B) = (A - \lambda B)^{-1} \text{ exists in } \mathcal{L}(X)\}.$$

$R_\lambda(A, B)$ is called the resolvent of linear operator pencil (A, B).

Definition 2. *[6] Let X be a non-Archimedean Banach space, let $A, B \in \mathcal{L}(X)$ and $\varepsilon > 0$. The pseudospectrum of bounded linear operator pencil (A, B) on X is defined by*

$$\Sigma_\epsilon(A, B) = \sigma(A, B) \cup \{\lambda \in \mathbb{K} : \ \|(A - \lambda B)^{-1} B\| > \varepsilon^{-1}\}.$$

By convention $\|(A - \lambda B)^{-1} B\| = \infty$ if, and only if, $\lambda \in \sigma(A, B)$.

2 p-Adic spectral sets of matrix pencils

We introduce the following definition.

Definition 3. *Let $A \in \mathcal{M}_n(\mathbb{K})$, The trace of A is*

$$Tr(A) = \sum_{k=1}^{n} a_{k,k},$$

where for all $k \in \{1, \cdots, n\}$, $a_{k,k} \in \mathbb{K}$ are diagonal coefficients of A.

We have the following proposition.

Proposition 1. *Let $A, B \in \mathcal{M}_n(\mathbb{K})$, and $\lambda \in \mathbb{K}$. Then*

(i) $Tr(A + \lambda B) = Tr(A) + \lambda Tr(B)$,

(ii) $Tr(AB) = Tr(BA)$.

Furthermore, the map $Tr : \mathcal{M}_n(\mathbb{K}) \to \mathbb{K}$ is a continuous linear functional with $|Tr(A)| \leq \|A\|$.

Proof. Let $A, B \in \mathcal{M}_n(\mathbb{K})$ and $\lambda \in \mathbb{K}$ with $A = (a_{i,j})_{1\leq i\leq n,\, 1\leq j\leq n}$ and $B = (b_{i,j})_{1\leq i\leq n,\, 1\leq j\leq n}$. Then

(i)

$$\begin{aligned} Tr(A + \lambda B) &= \sum_{k=1}^{n} \Big(a_{k,k} + \lambda b_{k,k}\Big) \\ &= \sum_{k=1}^{n} a_{k,k} + \lambda \sum_{k=1}^{n} b_{k,k} = Tr(A) + \lambda Tr(B). \end{aligned}$$

(ii)

$$Tr(AB) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j} b_{j,i},$$

and

$$Tr(BA) = \sum_{j=1}^{n}\sum_{i=1}^{n} a_{i,j} b_{j,i}.$$

Thus,

$$Tr(AB) = Tr(BA).$$

Furthermore,

$$\begin{aligned} |Tr(A)| &= |\sum_{k=1}^{n} a_{k,k}| \\ &\leq \sup_{1\leq k\leq n} |a_{k,k}| \\ &\leq \sup_{1\leq i,j\leq n} |a_{i,j}| = \|A\|. \end{aligned}$$

Hence, the map $Tr : \mathcal{M}_n(\mathbb{K}) \to \mathbb{K}$ is a continuous linear functional with $|Tr(A)| \leq \|A\|$. □

We have the following definition.

Definition 4. *Let $A, B \in \mathcal{M}_n(\mathbb{K})$ and $\varepsilon > 0$. The trace pseudo-spectrum of the matrix pencil (A, B) of the form $A - \lambda B$ is denoted by $Tr_\varepsilon(A, B)$ and is defined as*

$$Tr_\varepsilon(A, B) = \sigma(A, B) \cup \{\lambda \in \mathbb{K} : |Tr(A - \lambda B)| \leq \varepsilon\}.$$

The trace pseudo-resolvent of the matrix pencil of the form $A - \lambda B$ is denoted by $Tr\rho_\varepsilon(A, B)$ and is defined by

$$Tr\rho_\varepsilon(A, B) = \rho(A, B) \cap \{\lambda \in \mathbb{K} : |Tr(A - \lambda B)| > \varepsilon\}.$$

We begin with the following theorems.

Theorem 1. *Let $A, B \in \mathcal{M}_n(\mathbb{K})$ and $\varepsilon > 0$. Then,*

(i) If $0 < \varepsilon_1 \leq \varepsilon_2$, $Tr_{\varepsilon_1}(A, B) \subset Tr_{\varepsilon_2}(A, B)$,

(ii) If $\alpha \in \mathbb{K}$ and $\beta \in \mathbb{K}\backslash\{0\}$, then $Tr_\varepsilon(\beta A + \alpha B, B) = \beta Tr_{\frac{\varepsilon}{|\beta|}}(A, B) + \alpha$,

(iii) For all $\lambda, \alpha \in \mathbb{K}$, and $Tr(A) \neq 0$,

$$Tr_\varepsilon(\alpha A, A) = \Big\{\lambda \in \mathbb{K} : |\lambda - \alpha| \leq \frac{\varepsilon}{|Tr(A)|}\Big\}.$$

Proof. (i) It is clear from the definition of trace pseudo-spectrum of matrix pencil.

(ii) Let $\alpha \in \mathbb{K}$ and $\beta \in \mathbb{K}\backslash\{0\}$, then it is easy to see that

$$\sigma(\beta A + \alpha B, B) = \alpha + \beta\sigma(A, B),$$

and

$$\begin{aligned} Tr_\varepsilon(\beta A + \alpha B, B) &= \Big\{\lambda \in \mathbb{K} : |Tr(\beta A + \alpha B - \lambda B)| \leq \varepsilon\Big\} \\ &= \Big\{\lambda \in \mathbb{K} : |\beta||Tr(A - \frac{(\lambda - \alpha)}{\beta}B)| \leq \varepsilon\Big\} \\ &= \Big\{\lambda \in \mathbb{K} : |Tr(A - \frac{(\lambda - \alpha)}{\beta}B)| \leq \frac{\varepsilon}{|\beta|}\Big\}. \end{aligned}$$

Hence $\lambda \in Tr_\varepsilon(\beta A + \alpha B, B)$ if, and only if, $\frac{\lambda - \alpha}{\beta} \in Tr_{\frac{\varepsilon}{|\beta|}}(A, B)$ i.e $\lambda \in \beta Tr_{\frac{\varepsilon}{|\beta|}}(A, B) + \alpha$.

(iii) Let $\alpha, \lambda \in \mathbb{K}$, then

$$|Tr(\alpha A - \lambda A)| = |\lambda - \alpha||tr(A)| \leq \varepsilon.$$

Thus

$$Tr_\varepsilon(\alpha A, A) = \Big\{\lambda \in \mathbb{K} : |\lambda - \alpha| \leq \frac{\varepsilon}{|Tr(A)|}\Big\}.$$

□

Theorem 2. *Let $A, B, U, V \in \mathcal{M}_n(\mathbb{K})$ and $\varepsilon > 0$.*
(i) If $U = VAV^{-1}$ and $BV = VB$, then $Tr_\varepsilon(A, B) = Tr_\varepsilon(U, B)$.
(ii) If $U = VAV^{-1}$ and $A = VBV^{-1}$, then $Tr_\varepsilon(U, A) = Tr_\varepsilon(A, B)$.

Proof. Let $A, B, U, V \in \mathcal{M}_n(\mathbb{K})$ and $\varepsilon > 0$.

(i) If $U = VAV^{-1}$ and $BV = VB$, then

$$\begin{array}{rcl} \lambda \notin \sigma(A, B) & \iff & (A - \lambda B) \textit{ invertible} \\ & \iff & V^{-1}UV - \lambda V^{-1}BV \textit{ invertible} \\ & \iff & V^{-1}(U - \lambda B)V \textit{ invertible} \\ & \iff & (U - \lambda B) \textit{ invertible} \\ & \iff & \lambda \notin \sigma(U, B). \end{array}$$

Thus $\sigma(A, B) = \sigma(U, B)$. Furthermore

$$\begin{array}{rcl} Tr_\varepsilon(A, B) & = & \sigma(A, B) \cup \Big\{\lambda \in \mathbb{K} : |Tr(A - \lambda B)| \leq \varepsilon\Big\} \\ & = & \sigma(U, B) \cup \Big\{\lambda \in \mathbb{K} : |Tr(V^{-1}UV - \lambda V^{-1}BV)| \leq \varepsilon\Big\} \\ & = & \sigma(U, B) \cup \Big\{\lambda \in \mathbb{K} : |Tr(V^{-1}(U - \lambda B)V)| \leq \varepsilon\Big\} \\ & = & \sigma(U, B) \cup \Big\{\lambda \in \mathbb{K} : |Tr(U - \lambda B)| \leq \varepsilon\Big\} \\ & = & Tr_\varepsilon(U, B). \end{array}$$

(ii) It is easy to see that $\sigma(A, B) = \sigma(U, A)$. Then

$$\begin{array}{rcl} Tr_\varepsilon(U, A) & = & \sigma(U, A) \cup \Big\{\lambda \in \mathbb{K} : |Tr(U - \lambda A)| \leq \varepsilon\Big\} \\ & = & \sigma(A, B) \cup \Big\{\lambda \in \mathbb{K} : |Tr(VAV^{-1} - \lambda VBV^{-1})| \leq \varepsilon\} \\ & = & \sigma(A, B) \cup \Big\{\lambda \in \mathbb{K} : |Tr\Big(V(A - \lambda B)V^{-1}\Big)| \leq \varepsilon\Big\} \\ & = & \sigma(A, B) \cup \Big\{\lambda \in \mathbb{K} : |Tr(A - \lambda B)| \leq \varepsilon\Big\} \\ & = & Tr_\varepsilon(A, B). \end{array}$$

□

We have the following example.

Example 1. *Let $\mathbb{K} = \mathbb{Q}_p$ and $\varepsilon > 0$. If*

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \textit{ and } U = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$Tr_\varepsilon(A, U) = Tr_\varepsilon(B, U) = \{1\} \cup \{\lambda \in \mathbb{Q}_p : |\lambda - 2|_p \leq \varepsilon\}.$$

Let $r > 0$, $B_f(0, r) = \{\lambda \in \mathbb{K} : |\lambda| \leq r\}$ is the closed ball centered at zero with radius r. We have the following theorem.

Theorem 3. *Let $A, B \in \mathcal{M}_n(\mathbb{K})$ and $\varepsilon > 0$. Then*

$$Tr_\delta(A,B) + B_f\left(0, \frac{\varepsilon}{|Tr(B)|}\right) \subseteq Tr_\gamma(A,B)$$

where $\gamma = \max\{\delta, \varepsilon\}$. If $\delta < \varepsilon$, we have

$$Tr_\delta(A,B) + B_f\left(0, \frac{\varepsilon}{|Tr(B)|}\right) \subseteq Tr_\varepsilon(A,B).$$

Proof. Let $A, B \in \mathcal{M}_n(\mathbb{K})$ and $\varepsilon > 0$. Let $\lambda \in Tr_\delta(A,B) + B_f(0, \frac{\varepsilon}{|Tr(B)|})$, then there exists $\lambda_0 \in Tr_\delta(A,B)$ and $\lambda_1 \in B_f(0, \frac{\varepsilon}{|Tr(B)|})$ such that $\lambda = \lambda_0 + \lambda_1$, hence

$$|Tr(A - \lambda_0 B)| \leq \delta$$

and

$$|\lambda_1||Tr(B)| \leq \varepsilon.$$

Thus,

$$\begin{aligned} |Tr(A - \lambda B)| &= |Tr(A - \lambda_0 B - \lambda_1 B)| \\ &\leq \max\Big\{|Tr(A - \lambda_0 B)|,\ |\lambda_1||Tr(B)|\Big\} \\ &\leq \max\Big\{\delta,\ \varepsilon\Big\}. \end{aligned}$$

We set $\gamma = \max\Big\{\delta,\ \varepsilon\Big\}$. Then,

$$Tr_\delta(A,B) + B_f\left(0, \frac{\varepsilon}{|Tr(B)|}\right) \subseteq Tr_\gamma(A,B).$$

If $\delta < \varepsilon$, then $\gamma = \varepsilon$. Consequently,

$$Tr_\delta(A,B) + B_f\left(0, \frac{\varepsilon}{|Tr(B)|}\right) \subseteq Tr_\varepsilon(A,B).$$

□

We introduce the following $\varepsilon-$*trace* set of matrix pencils in non-Archimedean finite dimensional Banach spaces.

Definition 5. *Let $A, B \in \mathcal{M}_n(\mathbb{K})$ and $\varepsilon > 0$. The $\varepsilon-$trace set of matrix pencil (A,B) is denoted by $tr_\varepsilon(A,B)$ and is defined as*

$$tr_\varepsilon(A,B) = \{\lambda \in \mathbb{K} : |Tr(A - \lambda B)| \leq \varepsilon\}.$$

Remark 1. *For all $\varepsilon > 0$, $tr_\varepsilon(A,B) \subseteq Tr_\varepsilon(A,B)$.*

Consequently we have the following results.

Theorem 4. *Let* $A, B, C \in \mathcal{M}_n(\mathbb{K})$ *and* $\varepsilon > 0$. *Then,*

(i) $tr_\varepsilon(AB, C) = tr_\varepsilon(BA, C)$

(ii) $tr_\varepsilon(A, C) + tr_\varepsilon(B, C) \subseteq tr_\varepsilon(A + B, C)$.

Proof.

(i) Let $\lambda \in tr_\varepsilon(AB, C)$, hence

$$\begin{aligned} \varepsilon &\geq |Tr(AB - \lambda C)| \\ &= |Tr(AB) + Tr(-\lambda C)| \\ &= |Tr(BA) + Tr(-\lambda C)| \\ &= |Tr(BA - \lambda C)|. \end{aligned}$$

Thus, $\lambda \in tr_\varepsilon(BA, C)$. Then, $tr_\varepsilon(AB, C) \subseteq tr_\varepsilon(BA, C)$. Similarly, we obtain $tr_\varepsilon(BA, C) \subseteq tr_\varepsilon(AB, C)$. Hence, $tr_\varepsilon(AB, C) = tr_\varepsilon(BA, C)$.

(ii) Let $\lambda \in tr_\varepsilon(A, C) + tr_\varepsilon(B, C)$, then there exists $\lambda_0 \in tr_\varepsilon(A, C)$ and $\lambda_1 \in tr_\varepsilon(B, C)$ such that $\lambda = \lambda_0 + \lambda_1$. Then

$$|Tr(A - \lambda_0 C)| \leq \varepsilon \textit{ and } |Tr(B - \lambda_1 C)| \leq \varepsilon.$$

Thus,

$$\begin{aligned} |Tr(A + B - \lambda C)| &= |Tr(A + B - \lambda_0 C - \lambda_1 C)| \\ &= |Tr(A - \lambda_0 C) + Tr(B - \lambda_1 C)| \\ &\leq \max\Big\{|Tr(A - \lambda_0 C)|,\ |Tr(B - \lambda_1 C)|\Big\} \\ &\leq \varepsilon. \end{aligned}$$

Consequently, $\lambda \in tr_\varepsilon(A + B, C)$. Hence,

$$tr_\varepsilon(A, C) + tr_\varepsilon(B, C) \subseteq tr_\varepsilon(A + B, C).$$

□

The following proposition shows that the $\varepsilon - trace$ of a matrix pencil is a convex set in the non-Archimedean valued field $\mathbb{K}$.

Proposition 2. *Let* $A, B \in \mathcal{M}_n(\mathbb{K})$ *and* $\varepsilon > 0$. *Let* $\lambda, \mu \in tr_\varepsilon(A, B)$ *and* $\alpha \in \mathbb{K}$ *such that* $|\alpha| \leq 1$. *Then* $\alpha\lambda + (1 - \alpha)\mu \in tr_\varepsilon(A, B)$.

Proof. Let $A, B \in \mathcal{M}_n(\mathbb{K})$ and $\varepsilon > 0$. Let $\lambda, \mu \in tr_\varepsilon(A, B)$ and $\alpha \in \mathbb{K}$ such that $|\alpha| \leq 1$. Then

$$|Tr(A - \lambda B)| \leq \varepsilon,$$

and

$$|Tr(A - \mu B)| \leq \varepsilon.$$

Hence

$$\begin{aligned} |Tr(A-\Big(\alpha\lambda+(1-\alpha)\mu\Big)B)| &= |Tr(A-\mu B+\alpha\Big(A-\lambda B-(A-\mu B)\Big)| \\ &= \max\Big\{|Tr(A-\mu B)|, |\alpha||Tr(A-\lambda B)|, \\ &\qquad |\alpha||Tr(A-\mu B)|\Big\} \\ &\leq \varepsilon. \end{aligned}$$

Thus,

$$\alpha\lambda+(1-\alpha)\mu \in tr_{\varepsilon}(A,B).$$

□

Proposition 3. *Let $A, B \in \mathcal{M}_n(\mathbb{K})$ and $\varepsilon > 0$ such that $\|A\| < \varepsilon$. Let $\lambda, \mu \in tr_{\varepsilon}(A,B)$. Then*

$$\lambda-\mu \in tr_{\varepsilon}(A,B).$$

Proof. Let $A, B \in \mathcal{M}_n(\mathbb{K})$ and $\varepsilon > 0$ such that $\|A\| < \varepsilon$. Let $\lambda, \mu \in tr_{\varepsilon}(A,B)$. By Proposition 1, we have $|Tr(A)| \leq \|A\|$. Then

$$|Tr(A-\lambda B)| \leq \varepsilon,$$

and

$$|Tr(A-\mu B)| \leq \varepsilon.$$

Hence

$$\begin{aligned} |Tr(A-\Big(\lambda-\mu\Big)B)| &= |Tr(A-\lambda B)-\Big(A-\mu B\Big)+Tr(A)|, \\ &= \max\Big\{|Tr(A-\mu B)|, |Tr(A-\lambda B)|, |Tr(A)|\Big\}, \\ &\leq \varepsilon. \end{aligned}$$

Thus,

$$\lambda-\mu \in tr_{\varepsilon}(A,B).$$

□

3 Determinant spectrum of non-Archimedean matrix pencils

We introduce the following definition.

Definition 6. *Let $A, B \in \mathcal{M}_n(\mathbb{K})$ and $\varepsilon > 0$. The $\varepsilon-$determinant spectrum of matrix pencil (A,B) is denoted by $d_{\varepsilon}(A,B)$ and is defined as*

$$d_{\varepsilon}(A,B) = \{\lambda \in \mathbb{K} : |det(A-\lambda B)| \leq \varepsilon\}.$$

Remark 2. *The definition of determinant spectrum gives for each $A, B \in \mathcal{M}_n(\mathbb{K})$ and $\varepsilon > 0$, $\sigma(A,B) \subseteq d_{\varepsilon}(A,B)$ and $d_0 = \sigma(A,B)$.*

Proposition 4. *Let $A, B \in \mathcal{M}_n(\mathbb{K})$ and $\varepsilon > 0$. Then*

(i) For all $0 < \varepsilon_1 \leq \varepsilon_2$, we have $d_{\varepsilon_1} \subseteq d_{\varepsilon_2}$,

(ii) $d_\varepsilon(\alpha I, I) = \{\lambda \in \mathbb{K} : |\lambda - \alpha| \leq \varepsilon^{\frac{1}{n}}\}$,

(iii) For all $\alpha, \beta \in \mathbb{K}$ such that $\beta \neq 0$, $d_\varepsilon(\alpha I + \beta A, I) = \alpha + \beta d_{\frac{\varepsilon}{|\beta|^n}}(A, I)$.

Proof. Let $A, B \in \mathcal{M}_n(\mathbb{K})$ and $\varepsilon > 0$.

(i) For $0 < \varepsilon_1 \leq \varepsilon_2$. Let $\lambda \in d_{\varepsilon_1}(A, B)$, then $|det(A - \lambda B)| \leq \varepsilon_1 \leq \varepsilon_2$. Hence $\lambda \in d_{\varepsilon_2}(A,\ B)$.

(ii) Let $\alpha \in \mathbb{K}$, then $|\det(\alpha I - \lambda I)| = |\lambda - \alpha|^n$. Hence

$$\begin{aligned} d_\varepsilon(\alpha I, I) &= \{\lambda \in \mathbb{K} : |\det(\alpha I - \lambda I)| \leq \varepsilon\} \\ &= \{\lambda \in \mathbb{K} : |\lambda - \alpha|^n \leq \varepsilon\} \\ &= \{\lambda \in \mathbb{K} : |\lambda - \alpha| \leq \varepsilon^{\frac{1}{n}}\}. \end{aligned}$$

(iii) Let $\alpha, \beta \in \mathbb{K}$ such that $\beta \neq 0$, we have

$$\begin{aligned} d_\varepsilon(\alpha I + \beta A, I) &= \{\lambda \in \mathbb{K} : |\det(\alpha I + \beta A - \lambda I)| \leq \varepsilon\} \\ &= \{\lambda \in \mathbb{K} : |\det(\beta A - (\lambda - \alpha) I)| \leq \varepsilon\} \\ &= \{\lambda \in \mathbb{K} : |\beta|^n |\det(A - \frac{(\lambda - \alpha)}{\beta} I)| \leq \varepsilon\} \\ &= \{\lambda \in \mathbb{K} : \det(A - \frac{(\lambda - \alpha)}{\beta} I)| \leq \frac{\varepsilon}{|\beta|^n}\}. \end{aligned}$$

Hence,

$$\begin{aligned} \lambda \in d_\varepsilon(\alpha I + \beta A, I) &\iff \frac{\lambda - \alpha}{\beta} \in d_{\frac{\varepsilon}{|\beta|^n}}(A, I) \\ &\iff \lambda \in \alpha + \beta d_{\frac{\varepsilon}{|\beta|^n}}(A, I). \end{aligned}$$

□

We give some examples of $\varepsilon-$determinant spectrum.

Example 2. *Let $\mathbb{K} = \mathbb{Q}_p$ and $\varepsilon > 0$. We consider*

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}.$$

It is easy to see that

$$\begin{aligned} d_\varepsilon(A, B) &= \{\lambda \in \mathbb{Q}_p : |\det(A - \lambda B)|_p \leq \varepsilon\} \\ &= \{\lambda \in \mathbb{Q}_p : |1 - \lambda|_p |1 - 2\lambda|_p \leq \varepsilon\}. \end{aligned}$$

Example 3. *Let $\mathbb{K} = \mathbb{Q}_p$ with $p \geq 2$ and $\varepsilon > 0$. We consider*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It is easy to see that

$$\begin{aligned} d_\varepsilon(A,B) &= \{\lambda \in \mathbb{Q}_p : |\det(A - \lambda B)|_p \leq \varepsilon\} \\ &= \{\lambda \in \mathbb{Q}_p : |1-\lambda|_p |2+\lambda|_p \leq \varepsilon\}. \end{aligned}$$

Example 4. *Let $\mathbb{K} = \mathbb{Q}_p$ with $p \neq 2$ and $\varepsilon > 0$. Let $a, b \in \mathbb{Q}_p$ such that $a \neq 0$ and $a^2 + b^2 \neq 0$, we consider*

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

It is easy to see that

$$\begin{aligned} d_\varepsilon(A,B) &= \{\lambda \in \mathbb{Q}_p : |\det(A - \lambda B)|_p \leq \varepsilon\} \\ &= \{\lambda \in \mathbb{Q}_p : |a^2 - 2\lambda a + b^2|_p \leq \varepsilon\} \\ &= \{\lambda \in \mathbb{Q}_p : |det(A) - \lambda Tr(A)|_p \leq \varepsilon\}. \end{aligned}$$

Example 5. *Let $\mathbb{K} = \mathbb{Q}_p$ with $p \geq 2$ and $\varepsilon > 0$. We consider*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix} \text{ with } a \in \mathbb{Q}_p.$$

It is easy to see that

$$\begin{aligned} d_\varepsilon(A,B) &= \{\lambda \in \mathbb{Q}_p : |\det(A - \lambda B)|_p \leq \varepsilon\} \\ &= \{\lambda \in \mathbb{Q}_p : |\lambda - 1|_p \leq \varepsilon\}. \end{aligned}$$

We have the following proposition.

Proposition 5. *Let $D \in \mathcal{L}(\mathbb{Q}_p^n)$ be a diagonal operator such that for all $i \in \{1, \cdots, n\}$, $De_i = \lambda_i e_i$ with $\lambda_i \in \mathbb{Q}_p$, $\lambda_i \neq \lambda_{i+1}$. Then*

$$d_\varepsilon(D, I) = \{\lambda \in \mathbb{Q}_p : |\lambda - \lambda_1|_p \cdots |\lambda - \lambda_n|_p \leq \varepsilon\}.$$

Proof. Let $\varepsilon > 0$. Then $D - \lambda I$ has the form

$$for\ all\ i \in \{1, \cdots, n\},\ (D - \lambda)e_i = (\lambda_i - \lambda)e_i$$

where $(e_i)_{1 \leq i \leq n}$ is the canonical base of $\mathbb{Q}_p^n$.

Then, $|\det(D - \lambda I)|_p = |a_1 - \lambda_1|_p \cdots |a_n - \lambda_n|_p$.

$$d_\varepsilon(D, I) = \{\lambda \in \mathbb{Q}_p : |\det(D - \lambda I)|_p \leq \varepsilon\} = \{\lambda \in \mathbb{Q}_p : |\lambda - \lambda_1|_p \cdots |\lambda - \lambda_n|_p \leq \varepsilon\}.$$

□

4 Trace pseudo-spectrum of linear operator pencils on p-adic Hilbert space E_ω

We have the following definition.

Definition 7. *([2]) For $A \in \mathcal{L}(E_\omega)$, we define the trace of A to be*

$$Tr(A) = \sum_{k=0}^{\infty} \frac{\langle Ae_s, e_s \rangle}{\omega_k}$$

if this series converges in $\mathbb{K}$. We denote by $\mathcal{TC}(E_\omega)$ the subspace of all Trace class operators, namely, those bounded operators for which the trace exists.

Remark 3. *([2]) Let $A \in \mathcal{L}(E_\omega)$. Then $A \in \mathcal{TC}(E_\omega)$ if and only if $\lim_{k\to\infty} a_{k,k} = 0$. Moreover,*

$$Tr(A) = \sum_{k=0}^{\infty} a_{k,k}.$$

We introduce the following definition.

Definition 8. *Let $A, B \in \mathcal{TC}(E_\omega)$ and $\varepsilon > 0$. The $\varepsilon-trace$ set of bounded linear operator pencil $A - \lambda B$ is denoted by $tr_\varepsilon(A, B)$ and is defined as*

$$tr_\varepsilon(A, B) = \{\lambda \in \mathbb{K} : |Tr(A - \lambda B)| \leq \varepsilon\}.$$

Remark 4. *Let X be a free Banach space over $\mathbb{K}$, let $A, B \in \mathcal{TC}(X)$, the Definition 8, is valid.*

We start with the following statements.

Theorem 5. *Let $A, B \in \mathcal{TC}(E_\omega)$ and $\varepsilon > 0$. Then,*

(i) If $0 < \varepsilon_1 \leq \varepsilon_2$, $tr_{\varepsilon_1}(A, B) \subset tr_{\varepsilon_2}(A, B)$

(ii) If $\alpha \in \mathbb{K}$ and $\beta \in \mathbb{K}\backslash\{0\}$, then $tr_\varepsilon(\beta A + \alpha B, B) = \beta tr_{\frac{\varepsilon}{|\beta|}}(A, B) + \alpha$,

(iii) For all $\lambda, \alpha \in \mathbb{K}$, and $Tr(A) \neq 0$, we have

$$tr_\varepsilon(\alpha B, B) = \Big\{\lambda \in \mathbb{K} : |\lambda - \alpha| \leq \frac{\varepsilon}{|Tr(A)|}\Big\}.$$

Proof. (i) It is clear from the definition of $\varepsilon-trace$ set of linear operator pencils having trace.

(ii) Let $\alpha \in \mathbb{K}$ and $\beta \in \mathbb{K}\backslash\{0\}$, then

$$\begin{aligned} tr_\varepsilon(\beta A + \alpha B, B) &= \Big\{\lambda \in \mathbb{K} : |Tr(\beta A + \alpha B - \lambda B)| \leq \varepsilon\Big\} \\ &= \Big\{\lambda \in \mathbb{K} : |\beta||Tr(A - \frac{(\lambda - \alpha)}{\beta}B)| \leq \varepsilon\Big\} \\ &= \Big\{\lambda \in \mathbb{K} : |Tr(A - \frac{(\lambda - \alpha)}{\beta}B)| \leq \frac{\varepsilon}{|\beta|}\Big\}. \end{aligned}$$

Hence $\lambda \in tr_\varepsilon(\beta A + \alpha B, B)$ if, and only if, $\frac{\lambda-\alpha}{\beta} \in tr_{\frac{\varepsilon}{|\beta|}}(A,B)$ i.e $\lambda \in \beta Tr_{\frac{\varepsilon}{|\beta|}}(A,B) + \alpha$.

(iii) Let $\alpha, \lambda \in \mathbb{K}$, then

$$|Tr(\alpha A - \lambda A)| = |\lambda - \alpha||tr(A)| \leq \varepsilon.$$

Thus

$$tr_\varepsilon(\alpha A, A) = \Big\{\lambda \in \mathbb{K} : |\lambda - \alpha| \leq \frac{\varepsilon}{|Tr(A)|}\Big\}.$$

□

Example 6. *Let $\mathbb{K} = \mathbb{Q}_p$, and $a, b \in \mathbb{Q}_p$ such that $|a|_p < 1$ and $|b|_p < 1$, and suppose for all $s \in \mathbb{N}$, $|\omega_s| = 1$. Let $A = (a_{i,j})$ and $B = (b_{i,j})$ two linear operators are defined on E_ω respectively by*

$$\begin{aligned} a_{i,j} &= 1 \, if \; i < j \\ &= a^i \, if \; i = j \\ &= 0 \, if \; i > j. \end{aligned}$$

And

$$\begin{aligned} b_{i,j} &= 1 \, if \; i < j \\ &= b^i \, if \; i = j \\ &= 0 \, if \; i > j. \end{aligned}$$

It is easy to see that

$$\begin{aligned} Tr(A - \lambda B) &= \sum_{k=0}^{\infty} a_{k,k} - \lambda \sum_{k=0}^{\infty} b_{k,k} \\ &= \frac{1}{1-a} - \lambda \frac{1}{1-b} \\ &= \frac{1 - b - \lambda(1-a)}{(1-a)(1-b)}. \end{aligned}$$

Then,

$$\begin{aligned} tr_\varepsilon(A,B) &= \{\lambda \in \mathbb{Q}_p : |Tr(A - \lambda B)|_p \leq \varepsilon\} \\ &= \{\lambda \in \mathbb{Q}_p : |1 - b - \lambda(1-a)|_p \leq \varepsilon\} \\ &= \{\lambda \in \mathbb{Q}_p : |\lambda - \frac{1-b}{1-a}|_p \leq \varepsilon\}. \end{aligned}$$

We finish with the following theorems.

Theorem 6. *Let $A, B \in \mathcal{TC}(E_\omega)$ and $\varepsilon > 0$. Then*

$$tr_\delta(A,B) + B_f(0, \frac{\varepsilon}{|Tr(B)|}) \subseteq tr_\gamma(A,B),$$

where $\gamma = \max\{\delta, \varepsilon\}$. If $\delta < \varepsilon$, we have

$$tr_\delta(A,B) + B_f(0, \frac{\varepsilon}{|Tr(B)|}) \subseteq tr_\varepsilon(A,B).$$

Proof. Let $A, B \in \mathcal{TC}(E_\omega)$ and $\varepsilon > 0$. Let $\lambda \in tr_\delta(A,B) + B_f(0, \frac{\varepsilon}{|Tr(B)|})$, then there exists $\lambda_0 \in tr_\delta(A,B)$ and $\lambda_1 \in B_f(0, \frac{\varepsilon}{|Tr(B)|})$ such that $\lambda = \lambda_0 + \lambda_1$. Then

$$|Tr(A - \lambda_0 B)| \leq \delta,$$

and

$$|\lambda_1 Tr(B)| \leq \varepsilon.$$

Thus

$$\begin{aligned} |Tr(A-\lambda B)| &= |Tr(A - \lambda_0 B - \lambda_1 B)| \\ &\leq \max\Big\{|Tr(A-\lambda_0 B)|,\ |\lambda_1||Tr(B)|\Big\} \\ &\leq \max\Big\{\delta,\ \varepsilon\Big\}. \end{aligned}$$

Setting, $\gamma = \max\Big\{\delta,\ \varepsilon\Big\}$. Hence,

$$tr_\delta(A,B) + B_f(0, \frac{\varepsilon}{|Tr(B)|}) \subseteq tr_\gamma(A,B).$$

If $\delta < \varepsilon$, then $\gamma = \varepsilon$. Consequently,

$$Tr_\delta(A,B) + B_f(0, \frac{\varepsilon}{|Tr(B)|}) \subseteq Tr_\varepsilon(A,B).$$

□

$C(E_\omega)$ denote the set of all completely continuous operators on E_ω. We have the following theorem.

Theorem 7. *[2] Suppose $A, B \in C(E_\omega)$, then $Tr(AB) = Tr(BA)$.*

Theorem 8. *Let $A, B, C \in \mathcal{TC}(E_\omega)$ and $\varepsilon > 0$. We have the following statements:*

(i) If $A, B \in C(E_\omega)$, then $tr_\varepsilon(AB, C) = tr_\varepsilon(BA, C)$,

(ii) $tr_\varepsilon(A,C) + tr_\varepsilon(B,C) \subseteq tr_\varepsilon(A+B, C)$.

Proof.

(i) Let $\lambda \in tr_\varepsilon(AB, C)$. Since $A, B \in C(E_\omega)$, by Theorem 7, we have $Tr(AB) = Tr(BA)$, hence:

$$\begin{aligned} \varepsilon &\geq |Tr(AB - \lambda C)| \\ &= |Tr(AB) + Tr(-\lambda C)| \\ &= |Tr(BA) + Tr(-\lambda C)| \\ &= |Tr(BA - \lambda C)|. \end{aligned}$$

Thus, $\lambda \in Tr_\varepsilon(BA, C)$. Then, $Tr_\varepsilon(AB, C) \subseteq Tr_\varepsilon(BA, C)$. Similarly, we obtain $Tr_\varepsilon(BA, C) \subseteq Tr_\varepsilon(AB, C)$. Hence, $Tr_\varepsilon(AB, C) = Tr_\varepsilon(BA, C)$.

(ii) Let $\lambda \in Tr_\varepsilon(A,C) + Tr_\varepsilon(B,C)$, then there exists $\lambda_0 \in Tr_\varepsilon(A,C)$ and $\lambda_1 \in Tr_\varepsilon(B,C)$ such that $\lambda = \lambda_0 + \lambda_1$. Then

$$|Tr(A - \lambda_0 C)| \leq \varepsilon \text{ and } |Tr(B - \lambda_1 C)| \leq \varepsilon.$$

Thus,

$$\begin{aligned} |Tr(A + B - \lambda C)| &= |Tr(A + B - \lambda_0 C - \lambda_1 C)| \\ &= |Tr(A - \lambda_0 C) + Tr(B - \lambda_1 C)| \\ &\leq \max\Big\{|Tr(A - \lambda_0 C)|,\ |Tr(B - \lambda_1 C)|\Big\} \\ &\leq \varepsilon. \end{aligned}$$

Consequently, $\lambda \in Tr_\varepsilon(A + B, C)$. Hence,

$$Tr_\varepsilon(A,C) + Tr_\varepsilon(B,C) \subseteq Tr_\varepsilon(A + B, C).$$

□

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