Bulletin of the *Transilvania* University of Braşov
Series III: Mathematics and Computer Science, Vol. 2(64), No. 1 - 2022, 23-40
https://doi.org/10.31926/but.mif.2022.2.64.1.3

GEOMETRY OF KENMOTSU MANIFOLDS ADMITTING Z-TENSOR

Ajit BARMAN [1] and İnan ÜNAL [*2]

Abstract

The object of this paper is to study Kenmotsu manifolds admitting $\mathcal{Z}-$tensor, which is a generalization of Einstein tensor that comes from general relativity. We define a special type of quarter-symmetric non-metric ϕ and η-connection on a Kenmotsu manifold and we examine some geometric properties of such manifolds with $\mathcal{Z}-$tensor. Some semi-symmetry conditions related to $\mathcal{Z}-$tensor are studied on Kenmotsu manifolds and finally, we observe our results on a 5-dimensional Kenmotsu manifold.

2020 *Mathematics Subject Classification:* 53C05, 53C10, 53C25.
Key words: Kenmotsu manifold, semi-symmetry, connections, Einstein tensor, $\mathcal{Z}-$tensor.

1 Introduction

The study of differential equations on a (pseudo)-Riemannian manifold is an important topic of differential geometry. An important one is the Einstein field equation. On a (pseudo)-Riemannian manifold M, the Einstein tensor G which comes from the Einstein fields equation is defined by

$$G(U_1, U_2) = Ric(U_1, U_2) - \frac{1}{2}g(U_1, U_2)scal$$

for all $U_1, U_2 \in \Gamma(TM)$, where Ric is the Ricci curvature tensor and $scal$ is the scalar curvature of M. This tensor describes the curvature of the spacetime and it is a contracted piece of the Riemann curvature tensor that has vanishing divergence.

In 2012 Mantica and Molinari [23] introduce a new $(0,2)-$tensor as a generalization of Einstein tensor. This tensor is called the $\mathcal{Z}$-tensor, and it is also

[1]Department of Mathematics, *Ramthakur* College, Arundhuti Nagar-799003, West Tripura, Tripura, India, e-mail: ajitbarmanaw@yahoo.in

[2*] *Corresponding author*, Department of Computer Engineering, *Munzur* University, Aktuluk, Tunceli, Turkey, e-mail: inanunal@munzur.edu.tr

the trace of $Q-$tensor, which is a generalization of concircular curvature tensor. Mantica and Molinari [23] defined weakly $\mathcal{Z}$-symmetric manifolds and this was a generalization of the notion of weakly Ricci symmetric manifolds, pseudo Ricci symmetric manifolds, pseudo projective Ricci symmetric manifolds. Other authors have studied on $\mathcal{Z}$-symmetric manifolds the notion of weakly cyclic $\mathcal{Z}$-symmetric manifolds [1], pseudo $\mathcal{Z}$-tensor-symmetric Riemannian manifolds with harmonic curvature tensors [23], or almost pseudo $\mathcal{Z}$-tensor-symmetric manifolds [2]. In [3] the authors studied $\mathcal{Z}$-tensor-symmetric manifold admitting concircular Ricci symmetric tensor. In [31] Ünal studied the $N(k)-$quasi Einstein manifolds with $\mathcal{Z}$-tensor.

Contact manifolds have both physical applications and important geometric properties. The definition of a contact manifold occurred while searching special solutions of differential equation systems. With the study of contact manifolds, taking into account the properties of complex manifolds, important developments have been recorded with tensorial perspective. Over time, different classes of contact manifolds have been defined. One of them is Kenmotsu manifolds. Kenmotsu manifolds are not compact and they have negative scalar curvature. Also, these manifolds are normal, but not Sasakian. An almost contact metric manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ is called an almost Kenmotsu manifold if $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$ [19]. Obviously a normal almost Kenmotsu manifold is a Kenmotsu manifold. Kenmotsu manifolds have been studied in [7, 8, 14, 20, 24].

In Riemannian geometry we use advanced calculus tools to classify Riemannian manifolds. We use linear connection work with calculus tools. The correct one, the Levi-Civita connection, is a major object to understand the differential geometric properties of Riemannian manifolds. Levi-Civita connection is a metric connection, that means it is invariant under the change of the metric, and also it is a symmetric connection that means it has no torsion. Except for the Levi-Civita connection, there are many different kinds of connections which are metric or non-metric and symmetric or non-symmetric. In general, on a Riemannian manifold M, such connection $\mathcal{D}$ is defined as

$$\mathcal{D}_{U_1}U_2 = \nabla_{U_1}U_2 + H(U_1, U_2)$$

for all vector fields $U_1, U_2 \in \Gamma(TM)$, where ∇ is the Levi-Civita connection and H is a $(1,2)-$type tensor field. Based on different properties of $\mathcal{D}$, this connection is named as semi-symmetric metric, semi-symmetric non-metric, semi-symmetric quarter-metric etc. In [29], a generalization of such connections was given. Here, the notation $\mathcal{D}$ is used to state the general connection and $\widetilde{\nabla}, \widehat{\nabla}$ etc. will be used state special connections.

A semi-symmetric connection $\widetilde{\widetilde{\nabla}}$ on a differentiable manifold M was defined by Friedmann and Schouten [15]. The torsion of $\widetilde{\widetilde{\nabla}}$ is given by

$$Tor(U_1, U_2) = u(U_2)U_1 - u(U_1)U_2$$

where u is a 1-form and ρ is a vector field defined by $u(U_1) = g(U_1, \rho)$, for all vector fields $U_1, U_2 \in \Gamma(TM)$. A semi-symmetric metric connection on a Riemannian manifold was defined by Hayden [18]. Then, Yano [32] gave a different type of semi-symmetric metric connection as the form $\widetilde{\nabla}_{U_1}U_2 = \nabla_{U_1}U_2 + u(U_2)U_1 - g(U_1, U_2)\rho$, where $u(U_1) = g(U_1, \rho)$. The another type of semi-symmetric connection which is not metric was defined by Prvanović [27] with the name pseudo-metric semi-symmetric connection and was just followed by Andonie [6]. These types of connections are called semi-symmetric non-metric connections.

Agashe and Chafle [4] defined another kind of semi-symmetric non-metric connections which has the non-invariant metric

$$(\widetilde{\nabla}_{U_1}g)(U_2, U_3) = -u(U_2)g(U_1, U_3) - u(U_3)g(U_1, U_2).$$

Later, some different kind of semi-symmetric non-metric connections have been defined and studied in [10] and [21]. Golab [17] defined and studied quarter-symmetric connection in differentiable manifolds with affine connections. A quarter-symmetric connection $\widehat{\nabla}$ has the torsion

$$Tor(U_1, U_2) = \eta(U_2)\phi U_1 - \eta(U_1)\phi U_2$$

where η is a 1-form and ϕ is a (1,1) tensor field. $\widehat{\nabla}$ reduces $\widetilde{\nabla}$ if $\phi U_1 = U_1$. Barman [9] studied another type of a quarter-symmetric non-metric connection $\widehat{\nabla}$ for which we get $(\widehat{\nabla}_{U_1}g)(U_2, U_3) = 2\eta(U_1)g(U_2, U_3)$, where η is a non-zero 1-form and the author called this a quarter-symmetric non-metric ϕ-connection. Some geometric properties of Riemannian manifolds with special structures such as complex, contact, golden, statistical, etc. have been changed when consider different types of connections. Kenmotsu manifolds with different types of connections have been studied in [5, 13, 28, 25, 33].

In this paper, we define a new type of quarter-symmetric non-metric connection $\widehat{\nabla}$ which satisfies the conditions $(\widehat{\nabla}_{U_1}\phi)U_2 = 0$ and $(\widehat{\nabla}_{U_1}\eta)U_2 = 0$. We call this connection as *quarter-symmetric non-metric ϕ and η- connection*. This connection will be a good reference for future works on manifolds with special contact structures. We investigate the geometry of Kenmotsu manifolds under special conditions of the $\mathcal{Z}$-tensor using this connection. In Section 3, we obtain basic curvature properties of Kenmotsu manifolds admitting quarter-symmetric non-metric ϕ and η- connection. In Section 4, we examine geometric properties of Kenmotsu manifolds satisfying some special conditions on $\mathcal{Z}-$tensor. We proved the following results:

- *A Kenmotsu manifold is $\mathcal{Z}$-semi-symmetric if and only if it is Ricci semi-symmetric with respect to ∇.*
- *On an Einstein Kenmotsu manifold endowed with $\widehat{\nabla}$, we have $\widehat{R} \cdot \widehat{\mathcal{Z}} = 0$.*
- *On a Kenmotsu manifold $\mathcal{Z}(U_1, \xi) \cdot R = 0$ cannot be satisfied.*
- *If a Kenmotsu manifold satisfies $\widehat{Z}(U_1, \xi) \cdot \widehat{R} = 0$ condition, then it is $\eta-$Einstein.*

On the other hand, we consider $\mathcal{T}-$tensor which is a general curvature tensor and we obtain a classification for special curvature tensor such as conformal, quasi-conformal, conharmonic, concircular and projective curvature tensor. Finally, we consider an example of 5-dimensional Kenmotsu manifolds with respect to $\widehat{\nabla}$ to observe our results.

2 Kenmotsu manifolds

Let M be a $(2n+1)$-dimensional differentiable manifold with $(1,1)-$tensor field ϕ, a vector field ξ, a 1-form η and the Riemannian metric g. If the following conditions are satisfied for all vector fields U_1, U_2 on $\Gamma(TM)$, then M is called an almost contact metric manifold [11]:

$$\phi^2 U_1 = -U_1 + \eta(U_1)\xi, \tag{1}$$

$$\eta(\xi) = 1,\ \phi(\xi) = 0,\ \eta \circ \phi = 0,\ g(U_1, \xi) = \eta(U_1), \tag{2}$$

$$g(\phi U_1, \phi U_2) = g(U_1, U_2) - \eta(U_1)\eta(U_2). \tag{3}$$

We state 4-tuple as $\mathcal{S} = (\phi, \xi, \eta, g)$ for an almost contact metric structure. $\mathcal{S}$ is reduced to some special structures via different properties of the structure. $(M, \mathcal{S})$ is called an almost Kenmotsu manifold if the following conditions are satisfied

$$d\eta = 0; \quad d\Omega = 2\eta \wedge \Omega$$

where Ω is the 2-form defined by $\Omega(U_1, U_2) = g(U_1, \phi U_2)$.

Theorem 1. *$(M, \mathcal{S})$ is a Kenmotsu manifold if and only if*

$$(\nabla_{U_1}\phi)U_2 = g(\phi U_1, U_2)\xi - \eta(U_2)\phi U_1 \tag{4}$$

for all $U_1, U_2 \in \Gamma(TM)$[19].

From now on, we use only the notation M instead of Kenmotsu manifold $(M, \mathcal{S})$. By using the above relations, it follows that

$$\nabla_{U_1}\xi = U_1 - \eta(U_1)\xi, \tag{5}$$

$$(\nabla_{U_1}\eta)(U_2) = g(U_1, U_2) - \eta(U_1)\eta(U_2), \tag{6}$$

$$R(U_1, U_2)\xi = \eta(U_1)U_2 - \eta(U_2)U_1, \tag{7}$$

$$R(\xi, U_1)U_2 = \eta(U_2)U_1 - g(U_1, U_2)\xi, \tag{8}$$

$$\eta(R(U_1, U_2)U_3) = g(U_1, U_3)\eta(U_2) - g(U_2, U_3)\eta(U_1), \tag{9}$$

$$Ric(\phi U_1, \phi U_2) = Ric(U_1, U_2) + 2n\eta(U_1)\eta(U_2), \tag{10}$$

$$Ric(U_1, \xi) = -2n\eta(U_1), \tag{11}$$

where R and Ric denote the curvature tensor and the Ricci tensor of M, respectively. M is said to be a η-Einstein manifold if there exists the real valued functions λ_1, λ_2 such that $Ric(U_1, U_2) = \lambda_1 g(U_1, U_2) + \lambda_2 \eta(U_1)\eta(U_2)$. From (11), it is obvious that $\lambda_1 + \lambda_2 = -2n$.

3 Quarter-symmetric non-metric ϕ and η-connection on Kenmotsu manifolds

This section deals with a special type of quarter-symmetric non-metric ϕ and η-connection on a Kenmotsu manifold. Let us define a map $\widehat{\nabla} : \Gamma(TM) \to \Gamma(TM)$ on a Kenmotsu manifold by

$$\widehat{\nabla}_{U_1}U_2 = \nabla_{U_1}U_2 - \eta(U_1)\phi U_2 + g(U_1,U_2)\xi - \eta(U_2)U_1 - \eta(U_1)U_2 + \eta(U_1)\eta(U_2)\xi. \quad (12)$$

for all $U_1, U_2 \in \Gamma(TM)$, where ∇ is the Levi-Civita connection of M. It is easy to verify that $\widehat{\nabla}$ is a linear connection.

Using (12), the torsion tensor Tor of $\widehat{\nabla}$ is given by

$$\mathrm{Tor}(U_1,U_2) = \widehat{\nabla}_{U_1}U_2 - \widehat{\nabla}_{U_2}U_1 - [U_1,U_2] = \eta(U_2)\phi U_1 - \eta(U_1)\phi U_2. \quad (13)$$

On the other hand, by using basics on tensor calculus we get

$$\begin{aligned}(\widehat{\nabla}_{U_1}g)(U_2,U_3) &= \widehat{\nabla}_{U_1}g(U_2,U_3) - g(\widehat{\nabla}_{U_1}U_2,U_3) - g(U_2,\widehat{\nabla}_{U_1}U_3)\\ &= 2\eta(U_1)g(U_2,U_3) - 2\eta(U_1)\eta(U_2)\eta(U_3) \neq 0. \quad (14)\end{aligned}$$

This shows that $\widehat{\nabla}$ is a quarter-symmetric non-metric connection. By making use of (4) and (12), it is obvious that

$$(\widehat{\nabla}_{U_1}\phi)(U_2) = \widehat{\nabla}_{U_1}\phi U_2 - \phi(\widehat{\nabla}_{U_1}U_2) = 0, \quad (15)$$

and by using (6) and (12) , we see that

$$(\widehat{\nabla}_{U_1}\eta)(U_2) = \widehat{\nabla}_{U_1}\eta U_2 - \eta(\widehat{\nabla}_{U_1}U_2) = 0. \quad (16)$$

Thus, $\widehat{\nabla}$ defined by (12) to (16) is a special type of quarter-symmetric non-metric ϕ and η-connection on Kenmotsu manifolds.

Conversely, let us show that $\widehat{\nabla}$ is defined on M satisfying (13) to (16) is given by (12). Let H be a tensor field of type $(1,2)$ and write

$$\widehat{\nabla}_{U_1}U_2 = \nabla_{U_1}U_2 + H(U_1,U_2). \quad (17)$$

Then, we conclude that

$$\mathrm{Tor}(U_1,U_2) = H(U_1,U_2) - H(U_2,U_1). \quad (18)$$

Further, using (17), it follows that

$$(\widehat{\nabla}_{U_1}g)(U_2,U_3) = -g(H(U_1,U_2),U_3) - g(U_2,H(U_1,U_3)). \quad (19)$$

In view of (14) and (19) one obtains,

$$g(H(U_1,U_2),U_3) + g(U_2,H(U_1,U_3)) = -2\eta(U_1)g(U_2,U_3) + 2\eta(U_1)\eta(U_2)\eta(U_3). (20)$$

Also using (18) and (20), we derive that

$$g(\mathrm{Tor}(U_1,U_2),U_3)+g(\mathrm{Tor}(U_3,U_1),U_2)+g(\mathrm{Tor}(U_3,U_2),U_1)=2g(H(U_1,U_2),U_3)$$
$$+2\eta(U_1)g(U_2,U_3)+2\eta(U_2)g(U_1,U_3)-2\eta(U_3)g(U_1,U_2)-2\eta(U_1)\eta(U_2)\eta(U_3).$$

From the above equation yields,

$$\begin{aligned} g(H(U_1,U_2),U_3) &= \frac{1}{2}[g(\mathrm{Tor}(U_1,U_2),U_3)+g(\mathrm{Tor}(U_3,U_1),U_2) \\ &\quad +g(\mathrm{Tor}(U_3,U_2),U_1)]-\eta(U_1)g(U_2,U_3)-\eta(U_2)g(U_1,U_3) \\ &\quad +\eta(U_3)g(U_1,U_2)+\eta(U_1)\eta(U_2)\eta(U_3). \end{aligned} \tag{21}$$

Let Tor' be a tensor field of type $(1,2)$ given by

$$g(\mathrm{Tor}'(U_1,U_2),U_3)=g(\mathrm{Tor}(U_3,U_1),U_2). \tag{22}$$

Adding (13) and (22), we obtain

$$\mathrm{Tor}'(U_1,U_2)=g(U_1,\phi U_2)\xi-\eta(U_1)\phi U_2. \tag{23}$$

From (21) and by using (22), (23) we have

$$\begin{aligned} g(H(U_1,U_2),U_3) &= \frac{1}{2}[g(\mathrm{Tor}(U_1,U_2),U_3)+g(\mathrm{Tor}'(U_1,U_2),U_2) \\ &\quad +g(\mathrm{Tor}'(U_2,U_1),U_1)]-\eta(U_1)g(U_2,U_3)-\eta(U_2)g(U_1,U_3) \\ &\quad +\eta(U_3)g(U_1,U_2)+\eta(U_1)\eta(U_2)\eta(U_3) \\ &= -\eta(U_1)g(\phi U_2,U_3)-\eta(U_1)g(U_2,U_3)-\eta(U_2)g(U_1,U_3) \quad (24) \\ &\quad +\eta(U_3)g(U_1,U_2)+\eta(U_1)\eta(U_2)\eta(U_3). \end{aligned}$$

Now contracting U_3 in (24) and using (2), one obtains that

$$H(U_1,U_2)=-\eta(U_1)\phi U_2+g(U_1,U_2)\xi-\eta(U_2)U_1-\eta(U_1)U_2+\eta(U_1)\eta(U_2)\xi. \tag{25}$$

Combining (17) and (25), it follows that

$$\widehat{\nabla}_{U_1}U_2=\nabla_{U_1}U_2-\eta(U_1)\phi U_2+g(U_1,U_2)\xi-\eta(U_2)U_1-\eta(U_1)U_2+\eta(U_1)\eta(U_2)\xi.$$

Finally, we can state following theorem;

Proposition 1. *The linear connection given by* (12) *is a special type of quarter-symmetric non-metric ϕ and η-connection on Kenmotsu manifolds.*

By using (1), (2) and (12), the Riemannian curvature of M with respect to $\widehat{\nabla}$ is expressed by

$$\begin{aligned} \widehat{R}(U_1,U_2)U_3 &= R(U_1,U_2)U_3+\eta(U_1)(\nabla_{U_2}\phi)(U_3)-\eta(U_2)(\nabla_{U_1}\phi)(U_3) \\ &\quad +(\nabla_{U_2}\eta)(U_3)U_1-(\nabla_{U_1}\eta)(U_3)U_2+(\nabla_{U_1}\eta)(U_3)\eta(U_2)\xi \\ &\quad -(\nabla_{U_2}\eta)(U_3)\eta(U_1)\xi+\eta(U_1)g(U_2,\phi U_3)\xi-\eta(U_2)g(U_1,\phi U_3)\xi \\ &\quad +\eta(U_1)g(U_2,U_3)\xi-\eta(U_2)g(U_1,U_3)\xi+\eta(U_1)\eta(U_3)\phi U_2 \\ &\quad -\eta(U_2)\eta(U_3)\phi U_1+\eta(U_2)\eta(U_3)U_1-\eta(U_1)\eta(U_3)U_2, \end{aligned} \tag{26}$$

for all $(U_1, U_2, U_3 \in \Gamma(TM)$, where

$$\widehat{R}(U_1, U_2)U_3 = \widehat{\nabla}_{U_1}\widehat{\nabla}_{U_2}U_3 - \widehat{\nabla}_{U_2}\widehat{\nabla}_{U_1}U_3 - \widehat{\nabla}_{[U_1,U_2]}U_3. \tag{27}$$

By making use of (4) and (6) in (26), finally we get

$$\widehat{R}(U_1, U_2)U_3 = R(U_1, U_2)U_3 + g(U_2, U_3)U_1 - g(U_1, U_3)U_2. \tag{28}$$

So the equation (28) turns into

$$\widehat{R}(U_1, U_2)U_3 = -\widehat{R}(U_2, U_1)U_3, \tag{29}$$

and then

$$\widehat{R}(U_1, U_2)U_3 + \widehat{R}(U_2, U_3)U_1 + \widehat{R}(U_3, U_1)U_2 = 0. \tag{30}$$

Taking the inner product of (28) with U_4, it follows that

$$\begin{aligned}\widehat{R}(U_1, U_2, U_3, U_4) \quad = \quad & R(U_1, U_2, U_3, U_4) + g(U_2, U_3)g(U_1, U_4) \\ & -g(U_1, U_3)g(U_2, U_4),\end{aligned} \tag{31}$$

where $U_4 \in \Gamma(TM)$, $\widehat{R}(U_1, U_2, U_3, U_4) = g(\widehat{R}(U_1, U_2)U_3, U_4)$ and $\widehat{R}(U_1, U_2, U_3, U_4)$ $= g(R(U_1, U_2)U_3, U_4)$. From the equation (31), it follows that

$$\widehat{R}(U_1, U_2, U_3, U_4) = -\widehat{R}(U_1, U_2, U_4, U_3).$$

Let take a local orthonormal basis of $\Gamma(TM)$ as $\{X_1, ..., X_{2n+1}\}$. Then by putting $U_1 = U_4 = X_i$ in (31) and taking summation over i, $1 \leq i \leq 2n+1$ and also using (2), we get

$$\widehat{Ric}(U_2, U_3) = Ric(U_2, U_3) + 2ng(U_2, U_3), \tag{32}$$

where $\widehat{Ric}$ and Ric denote the Ricci tensor of M with respect to $\bar{\nabla}$ and ∇ respectively. The relation (32), implies that

$$\widehat{Ric}(U_2, U_3) = \widehat{Ric}(U_3, U_2).$$

Let $\widehat{scal}$ and $scal$ denote the scalar curvature of M with respect to $\bar{\nabla}$ and ∇ respectively, i.e., $\widehat{scal} = \sum_{i=1}^{2n+1} \widehat{Ric}(X_i, X_i)$ and $r = \sum_{i=1}^{2n+1} Ric(X_i, X_i)$. Then by putting $U_2 = U_3 = X_i$ in (32) and taking summation over i, $1 \leq i \leq 2n+1$ and also using (2), it follows that

$$\widehat{scal} = scal + 2n(2n+1).$$

Summing up all of the above equations we can state the following theorem:

Theorem 2. *Let M be a Kenmotsu manifold M with respect to $\widehat{\nabla}$. Then,*

1. *the curvature tensor* $\widehat{R}$ *is given by*

$$\widehat{R}(U_1,U_2)U_3 = R(U_1,U_2)U_3 + g(U_2,U_3)U_1 - g(U_1,U_3)U_2,$$

and $\widehat{R}$ *has following symmetry properties:*

$$\widehat{R}(U_1,U_2)U_3 = -\widehat{R}(U_2,U_1)U_3$$
$$\widehat{R}(U_1,U_2)U_3 + \widehat{R}(U_2,U_3)U_1 + \widehat{R}(U_3,U_1)U_2 = 0,$$
$$\widehat{R}(U_1,U_2,U_3,U_4) = -\widetilde{\widehat{R}}(U_1,U_2,U_4,U_3),$$

2. *the Ricci tensor* $\widehat{Ric}$ *is symmetric and given by*

$$\widehat{Ric}(U_2,U_3) = \widehat{Ric}(U_2,U_3) + 2ng(U_2,U_3),$$

3. *the scalar curvature* $\widehat{R}$ *is given by*

$$\widehat{scal} = scal + 2n(2n+1).$$

If $\widehat{R} = 0$, then the equation (31) turns into

$$R(U_1,U_2,U_3,U) = g(U_1,U_3)g(U_2,U) - g(U_2,U_3)g(U_1,U). \tag{33}$$

Therefore, $g(R(U_1,U_2)U_3,U) = k[g(U_2,U_3)g(U_1,U) - g(U_1,U_3)g(U_2,U)]$, where $k = -1$. It follows that the Kenmotsu manifolds with respect to the Levi-Civita connection is a space of constant curvature.

In view of above discussions we state the following result:

Corollary 1. *If is M is $\widehat{\nabla}-$flat, i.e. $\widehat{R} = 0$, then it has constant curvature with respect to Levi-Civita connection ∇.*

Putting $U_3 = U_1$, $U = U_2$ in (33), we get

$$R(U_1,U_2,U_1,U_2) = [g(U_1,U_1)g(U_2,U_2) - g(U_1,U_2)g(U_1,U_2)].$$

Then with the help of the above equation, the sectional curvature of M with respect of the Levi-Civita connection is obtained as -1. A Kenmotsu manifold M has constant sectional curvature -1 if and only if M is obtained by a concircular structure transformation from $\mathbb{C}^n \times \mathbb{R}$ endowed with the canonical cosymplectic structure [26]. Thus, we can state the following Corollary:

Corollary 2. *If M is $\widehat{\nabla}-$flat, then M with respect to the Levi-Civita connection is obtained by a concircular structure transformation from $\mathbb{C}^n \times \mathbb{R}$ endowed with the canonical cosymplectic structure.*

4 Z-tensor on a Kenmotsu manifold with respect to the quarter-symmetric non-metric ϕ and η-connection

In this section we examine $\mathcal{Z}$-tensor on a Kenmotsu manifold with respect to $\widehat{\nabla}$. We study some semi-symmetry condition with $\mathcal{Z}$-tensor. We use notations $\mathcal{Z}$ and $\widehat{\mathcal{Z}}$ for $\mathcal{Z}$-curvature tensor admitting ∇ and $\widehat{\nabla}$, respectively.

$\mathcal{Z}$-tensor on a Kenmotsu manifold with respect to $\widehat{\nabla}$ is

$$\widehat{\mathcal{Z}}(U_1, U_2) = \widehat{Ric}(U_1, U_2) + \psi g(U_1, U_2) \tag{34}$$

for all $U_1, U_2 \in \Gamma(TM)$, where ψ is an arbitrary non-vanishing function on M [19].

From (32) we obtain

$$\widehat{\mathcal{Z}}(U_1, U_2) = Ric(U_1, U_2) + (\psi + 2n)g(U_1, U_2). \tag{35}$$

By taking $U_1 = U_2 = \xi$, then we get $\widehat{\mathcal{Z}}(\xi, \xi) = \psi$ which is not possible to be zero. Thus we state:

Corollary 3. *A Kenmotsu manifold with respect to $\widehat{\nabla}$ cannot be $\widehat{\mathcal{Z}}$-flat.*

A Riemannian manifold is called semi-symmetic if $R \cdot R = 0$. This notion is a generalization of symmetric spaces. Also we recall that a manifold is Ricci semi-symmetric if $R \cdot Ric = 0$. Let us take $\mathcal{Z}$ instead of Ricc tensor, then we get $R \cdot \mathcal{Z} = 0$. We call such manifolds $\mathcal{Z}$-semi-symmetric.

Theorem 3. *A Kenmotsu manifold is $\mathcal{Z}$-semi-symmetric if and only if it is Ricci semi-symmetric with respect to ∇.*

Proof. Let $U_1, U_2, U_3, U_4 \in \Gamma(TM)$. Then we have

$$(R(U_1, U_2) \cdot \mathcal{Z})(U_3, U_4) = \mathcal{Z}(R(U_1, U_2)U_3, U_4) + \mathcal{Z}(U_3, R(U_1, U_2)U_4).$$

Then from the definition of the $\mathcal{Z}$-tensor and symmetry properties of R we get

$$(R(U_1, U_2) \cdot \mathcal{Z})(U_3, U_4) = (R(U_1, U_2) \cdot Ric)(U_3, U_4)$$

which gives the proof. □

Since an Einstein manifold is Ricci semi-symmetric, we can state:

Corollary 4. *An Einstein Kenmotsu manifold is $\mathcal{Z}$-semi-symmetric.*

The converse of the above corollary is not true in general.

Lemma 1. *On a Kenmotsu manifold we have*

$$(\widehat{R}(U_1, U_2) \cdot \widehat{\mathcal{Z}})(U_3, U_4) = (R(U_1, U_2) \cdot \mathcal{Z})(U_3, U_4) + \mathcal{T}(U_1, U_2, U_3, U_4)$$

for all $U_1, U_2, U_3, U_4 \in \Gamma(TM)$, where

$$\begin{aligned} \mathcal{T}(U_1, U_2, U_3, U_4) &= g(U_2, U_3)Ric(U_1, U_4) + g(U_2, U_4)Ric(U_1, U_3) \\ &\quad -g(U_1, U_3)Ric(U_2, U_4) - g(U_1, U_4)Ric(U_2, U_3). \end{aligned} \tag{36}$$

Proof. For all $U_1, U_2, U_3, U_4 \in \Gamma(TM)$, we have

$$(\widehat{R}(U_1, U_2) \cdot \widehat{\mathcal{Z}})(U_3, U_4) = \widehat{\mathcal{Z}}(\widehat{R}(U_1, U_2)U_3, U_4) + \widehat{\mathcal{Z}}(U_3, \widehat{R}(U_1, U_2)U_4).$$

Then from (28) and (34) we get

$$\begin{aligned}\widehat{\mathcal{Z}}(\widehat{R}(U_1, U_2)U_3, U_4) \quad &= \mathcal{Z}(R(U_1, U_2)U_3, U_4) + 2n(R(U_1, U_2, U_3, U_4)) \\ &+g(U_2, U_3)Ric(U_1, U_4) - g(U_1, U_3)Ric(U_2, U_4) \\ &(2n + \psi)[g(U_2, U_4)g(U_1, U_3) - g(U_1, U_4)g(U_2, U_3)]\end{aligned}$$

and

$$\begin{aligned}\widehat{\mathcal{Z}}(U_3, \widehat{R}(U_1, U_2)U_4) \quad &= \mathcal{Z}(U_3, R(U_1, U_2)U_4) + 2nR(U_1, U_2, U_4, U_3) \\ &+g(U_2, U_4)Ric(U_1, U_3) - g(U_1, U_4)Ric(U_2, U_3) \\ &+(2n + \psi)[g(U_2, U_4)g(U_1, U_3) - g(U_1, U_4)g(U_2, U_3)].\end{aligned}$$

Taking into consideration the last two equalities we obtain (36).

□

Corollary 5. *On a Kenmotsu manifold endowed with the connection* $\widehat{\nabla}$, $(\widehat{R}(U_1, U_2) \cdot \widehat{\mathcal{Z}}) = 0$ *if and only if* M *is* $\mathcal{Z}-$*semi-symmetric and* $\mathcal{T} = 0$.

As we can see that if the manifold is Einstein then we have $\mathcal{T} = 0$. Hence, from Corollary 4 we conclude:

Theorem 4. *On an Einstein Kenmotsu manifold endowed with the connection* $\widehat{\nabla}$, *we have* $\widehat{R}(U_1, U_2) \cdot \widehat{\mathcal{Z}} = 0$.

In [12] it is proved that a Kenmotsu manifold satisfying $R(\xi, U).Ric = 0$ is an Einstein manifold. Thus, from Corollary 4 and Lemma 1 we have:

Corollary 6. *Let* M *be a Kenmotsu manifold with respect to the quarter-symmetric non-metric* ϕ *and* η*-connection. If* $R(\xi, U) \cdot Ric = 0$, *then we have* $\widehat{R}(\xi, U) \cdot \widehat{\mathcal{Z}} = 0$.

For $U_1, U_2 = \xi, U_3, U_4, U_5$ vector fields on M, $(\mathcal{Z}(U_1, \xi) \cdot R)$ is defined by

$$(\mathcal{Z}(U_1, \xi) \cdot R)(U_3, U_4)U_5 = [(U_1 \wedge_{\mathcal{Z}} \xi) \cdot R]\,(U_3, U_4)U_5$$

where $\wedge_{\mathcal{Z}}$ is an endomorphism on $\Gamma(TM)$ is defined by

$$(U_1 \wedge_{\mathcal{Z}} U_2)U_3 = \mathcal{Z}(U_2, U_3)U_1 - \mathcal{Z}(U_1, U_3)U_2.$$

Thus, we obtain

$$\begin{aligned}((\mathcal{Z}(U_1, \xi) \cdot R))(U_3, U_4)U_5 \quad &= (U_1 \wedge_{\mathcal{Z}} \xi)R)(U_3, U_4)U_5 - R((U_1 \wedge_{\mathcal{Z}} \xi)U_3, U_4)U_5 \\ &-R(U_3, (U_1 \wedge_{\mathcal{Z}} \xi)U_4)U_5 - R(U_3, U_4)(U_1 \wedge_{\mathcal{Z}} \xi)U_5.\end{aligned}$$

Suppose that $((\mathcal{Z}(U_1,\xi)\cdot R))(U_3,U_4)U_5=0$. Then from the definition of $\mathcal{Z}$-tensor and using the above equality we get

$$\begin{aligned}\mathcal{Z}(\xi,R(U_3,U_4)U_5)U_1-\mathcal{Z}(U_1,R(U_3,U_4)U_5)\xi-\mathcal{Z}(\xi,U_3)R(U_1,U_4)U_5\\+\mathcal{Z}(U_1,U_3)R(\xi,U_4)U_5-\mathcal{Z}(\xi,U_4)R(U_3,U_1)U_5+\mathcal{Z}(U_1,U_4)R(U_3,\xi)U_5\\-\mathcal{Z}(\xi,U_5)R(U_3,U_4)U_1+\mathcal{Z}(U_1,U_5)R(U_3,U_4)\xi=0.\end{aligned}\tag{37}$$

By using Ricci and Riemann curvature properties we obtain ;

$$\begin{aligned}\mathcal{Z}(\xi,R(U_3,U_4)U_5)U_1 &= (-2n+\psi)\eta(R(U_3,U_4)U_5)U_1,\\ \mathcal{Z}(U_1,R(U_3,U_4)U_5)\xi &= Ric(U_1,R(U_3,U_4)U_5)\xi+\psi R(U_3,U_4,U_5,U_1)\xi,\\ \mathcal{Z}(\xi,U_3)R(U_1,U_4)U_5 &= (-2n+\psi)\eta(U_3)R(U_1,U_4)U_5,\\ \mathcal{Z}(U_1,U_3)R(\xi,U_4)U_5 &= \eta(U_5)Ric(U_1,U_3)U_4+\psi\eta(U_5)g(U_1,U_3)U_4\\ &\quad -g(U_4,U_5)Ric(U_1,U_3)\xi-\psi g(U_4,U_5)g(U_1,U_3)\xi,\\ \mathcal{Z}(\xi,U_4)R(U_3,U_1)U_5 &= (-2n+\psi)\eta(U_4)R(U_3,U_1)U_5,\\ \mathcal{Z}(U_1,U_4)R(U_3,\xi)U_5 &= -\eta(U_5)Ric(U_1,U_4)U_3-\psi\eta(U_5)g(U_1,U_4)U_3\\ &\quad +g(U_3,U_5)Ric(U_1,U_4)\xi+\psi g(U_3,U_5)g(U_1,U_4)\xi,\\ \mathcal{Z}(\xi,U_5)R(U_3,U_4)U_1 &= (-2n+\psi)\eta(U_5)R(U_3,U_4)U_1,\\ \mathcal{Z}(U_1,U_5)R(U_3,U_4)\xi &= \eta(U_3)Ric(U_1,U_5)U_4+\psi\eta(U_3)g(U_1,U_5)U_4\\ &\quad -\eta(U_4)Ric(U_1,U_5)U_3-\psi\eta(U_4)g(U_1,U_5)U_3.\end{aligned}$$

We put these equalities into (37). Also, by taking inner product with ξ we get

$$\begin{aligned}-2n(-g(U_3,U_1)\eta(U_5)\eta(U_4)+g(U_4,U_1)\eta(U_3)\eta(U_5))\\-\psi(R(U_3,U_4,U_5,U_1)+g(U_4,U_5)g(U_1,U_3)-g(U_3,U_5)g(U_1,U_4)\\+\eta(U_4)\eta(U_5)Ric(U_1,U_3)-\eta(U_3)\eta(U_5)Ric(U_1,U_4)\\-g(U_4,U_5)Ric(U_1,U_3)+g(U_3,U_5)Ric(U_1,U_4)=0.\end{aligned}\tag{38}$$

By choosing $U_3=\xi$ we obtain

$$2n\eta(U_1)g(\phi U_4,\phi U_5)=0$$

which provides that $((\mathcal{Z}(U_1,\xi)\cdot R))(U_3,U_4)U_5$ can not be zero. Thus we state:

Theorem 5. *On a Kenmotsu manifold $\mathcal{Z}(U_1,\xi)\cdot R=0$ cannot satisfy.*

We consider an η-Einstein Kenmotsu manifold. Since $Ric(\xi,\xi)=-2n$, we have $\lambda_1+\lambda_2=-2n$. As we know η-Einstein Kenmotsu manifold is a natural example of $N(k)-$quasi-Einstein manifold. In [22], it is proven that:

Theorem 6. *An $n-$dimensional $N(k)-$quasi-Einstein manifold (M^n,g) satisfies the condition $\mathcal{Z}(U,\xi)\cdot R=0$ if and only if $\lambda_1+\lambda_2=0$ [22].*

By this theorem we can see that an η-Einstein Kenmotsu manifold cannot satisfy the condition $\mathcal{Z}(U,\xi)\cdot R=0$. Thus we state an example for Theorem 5.

Theorem 7. *If a Kenmotsu manifold satisfies $\widehat{\mathcal{Z}}(U_1,\xi)\cdot\widehat{R}=0$ condition then it is $\eta-$Einstein.*

Proof. Suppose that $\widehat{\mathcal{Z}}(U_1,\xi)\cdot\widehat{R}=0$. Then we have

$$\begin{aligned}
&\widehat{\mathcal{Z}}(\xi,\widehat{R}(U_3,U_4)U_5)U_1-\widehat{\mathcal{Z}}(U_1,\widehat{R}(U_3,U_4)U_5)\xi-\widehat{\mathcal{Z}}(\xi,U_3)\widehat{R}(U_1,U_4)U_5\\
&+\widehat{\mathcal{Z}}(U_1,U_3)\widehat{R}(\xi,U_4)U_5-\widehat{\mathcal{Z}}(\xi,U_4)\widehat{R}(U_3,U_1)U_5+\widehat{\mathcal{Z}}(U_1,U_4)\widehat{R}(U_3,\xi)U_5\\
&-\widehat{\mathcal{Z}}(\xi,U_5)\widehat{R}(U_3,U_4)U_1+\widehat{\mathcal{Z}}(U_1,U_5)\widehat{R}(U_3,U_4)\xi=0. \qquad (39)
\end{aligned}$$

By using curvature properties of $(M,\widehat{\nabla})$ we obtain;

$$\begin{aligned}
\widehat{\mathcal{Z}}(\xi,\widehat{R}(U_3,U_4)U_5)U_1 &= \mathcal{Z}(\xi,R(U_3,U_4)U_5)U_1+(2n+\psi)[\eta(U_3)g(U_4,U_5)\\
&\quad-\eta(U_4)g(U_3,U_5)]U_1,\\
\widehat{\mathcal{Z}}(U_1,\widehat{R}(U_3,U_4)U_5)\xi &= \mathcal{Z}(U_1,R(U_3,U_4)U_5)\xi+g(U_3,U_5)Ric(U_1,U_4)\xi\\
&\quad-g(U_4,U_5)Ric(U_1,U_3)\xi+(2n+\psi)[g(U_3,U_5)g(U_1,U_4)\\
&\quad-g(U_4,U_5)g(U_1,U_3)]\xi),\\
\widehat{\mathcal{Z}}(\xi,U_3)\widehat{R}(U_1,U_4)U_5 &= \mathcal{Z}(\xi,U_3)R(U_1,U_4)U_5+\psi\eta(U_3)[g(U_1,U_5)U_4\\
&\quad-g(U_4,U_5)U_1]+2\eta(U_3)R(U_1,U_4)U_5,\\
\widehat{\mathcal{Z}}(U_1,U_3)\widehat{R}(\xi,U_4)U_5 &= \mathcal{Z}(U_1,U_3)R(\xi,U_4)U_5+g(U_3,U_5)Ric(U_1,U_3)\xi\\
&\quad-\eta(U_5)Ric(U_1,U_3)U_4+\psi[g(U_4,U_5)g(U_1,U_3)\xi\\
&\quad-\eta(U_5)g(U_1,U_3)U_4],\\
\widehat{\mathcal{Z}}(\xi,U_5)\widehat{R}(U_3,U_4)U_5 &= \mathcal{Z}(\xi,U_4)R(U_3,U_4)U_5+\psi\eta(U_4)[g(U_1,U_3)U_1\\
&\quad-g(U_1,U_5)U_3]+2n\eta(U_4)R(U_3,U_1)U_5,\\
\widehat{\mathcal{Z}}(U_1,U_4)\widehat{R}(\xi,U_3)U_5 &= -\mathcal{Z}(U_1,U_4)R(U_3,\xi)U_5-g(U_3,U_5)Ric(U_1,U_4)\xi\\
&\quad+\eta(U_5)Ric(U_1,U_4)U_3-\psi[g(U_3,U_5)g(U_1,U_4)\xi\\
&\quad+\eta(U_5)g(U_1,U_4)U_3],\\
\widehat{\mathcal{Z}}(\xi,U_5)\widehat{R}(U_3,U_4)U_1 &= \mathcal{Z}(\xi,U_5)R(U_3,U_4)U_1+\psi\eta(U_5)[g(U_3,U_1)U_4\\
&\quad-g(U_4,U_1)U_3]+2\eta(U_5)R(U_3,U_4)U_1,\\
\widehat{\mathcal{Z}}(U_1,U_5)\widehat{R}(U_3,U_4)\xi &= \mathcal{Z}(U_1,U_5)R(U_3,U_4)\xi+(\psi+4n)[\eta(U_4)g(U_1,U_5)U_3\\
&\quad-\eta(U_3)g(U_1,U_5)U_4]+Ric(U_1,U_5)\eta(U_4)U_3\\
&\quad-Ric(U_1,U_5)\eta(U_3)U_4.
\end{aligned}$$

We introduce these equalities into (39). Then we have

$$\begin{aligned}
&(\psi-2n)[\eta(R(U_3,U_4)U_5)U_1-Ric(U_1,R(U_3,U_4)U_5)\xi+\eta(U_3)g(U_1,U_5)U_4\\
&-\eta(U_4)g(U_1,U_5)U_1]+2(\psi+n)[\eta(U_3)g(U_4,U_5)U_1-\eta(U_4)g(U_3,U_5)U_1]\\
&-\psi[\eta(U_3)R(U_1,U_4)U_5+\eta(U_4)R(U_3,U_1)U_5+\eta(U_5)R(U_3,U_4)U_1\\
&-R(U_3,U_4,U_5,U_1)\xi+\eta(U_5)g(U_1,U_3)U_4+\eta(U_5)g(U_1,U_4)U_3\\
&+g(U_4,U_5)g(U_1,U_3)\xi+g(U_3,U_5)g(U_1,U_4)\xi]-3[g(U_3,U_5)Ric(U_1,U_4)\\
&-g(U_4,U_5)Ric(U_1,U_3)]\xi+\eta(U_5)\{[Ric(U_1,U_3)-Ric(U_1,U_4)]\xi\\
&+Ric(U_1,U_3)U_4+Ric(U_1,U_4)U_3\})=0.
\end{aligned}$$

By taking inner product ξ and by choosing $U_3 = \xi$, and after a long computation, we get

$$0 = \eta(U_5)\{[(-\psi + 2n) - 3]Ric(\phi U_1, \phi U_4) - \psi g(\phi U_1, \phi U_4)\}.$$

From (1) and (10) we obtain

$$Ric(U_1, U_4) = \frac{\psi}{2n - \psi - 3} g(U_1, U_4) - \frac{2n(2n - \psi - 3) + \psi}{2n - \psi - 3} \eta(U_1)\eta(U_4).$$

This proves that M is $\eta-$Einstein. □

Let $\mathcal{P}$ be any curvature tensor of a type $(1, 3)$. Suppose that $\mathcal{P}$ satisfies the following condition:

$$g(\mathcal{P}(U_1, U_2)U_3, U_4) = g(\mathcal{P}(U_1, U_2)U_4, U_3), \tag{40}$$

for all U_1, U_2, U_3, U_4 vector fields on M. Then, by using (35) we get

$$(\mathcal{P}(U_1, U_2) \cdot \widehat{\mathcal{Z}})(U_3, U_4) = (\mathcal{P}(U_1, U_2) \cdot Ric)(U_3, U_4).$$

Therefore, we obtain

Theorem 8. *Let M be Kenmotsu manifold and $\mathcal{P}$ be a $(1, 3)-$type curvature tensor which satisfies (40). If M is $\mathcal{P}-$ Ricci semi-symmetric with respect to Levi-Civita connection ∇, then we have $\mathcal{P} \cdot \widehat{\mathcal{Z}} = 0$.*

We can state following corollary:

Corollary 7. *On a Ricci semi-symmetric Kenmotsu manifold we have $R \cdot \widehat{\mathcal{Z}} = 0$.*

In [30], the authors defined a generalization of curvature tensors with named $\mathcal{T}$-tensors. In same the paper, the authors proved that a $\mathcal{T}-$tensor do not satisfy the condition (40) in general. Also, they state that quasi- conformal ($\tilde{\mathcal{C}}$), conformal ($\mathcal{C}$), conharmonic ($\mathcal{K}$), concircular ($\mathcal{V}$) and M-projective ($\mathcal{W}$) curvature tensors satisfy (40). Thus we can state:

Corollary 8. *Let M be a Kenmotsu manifold with respect to $\widehat{\nabla}$. If M is*

- *quasi-conformal Ricci semi-symmetric with respect to ∇, then $\tilde{\mathcal{C}} \cdot \widehat{\mathcal{Z}} = 0$*
- *conformal Ricci semi-symmetric with respect to ∇, then $\mathcal{C} \cdot \widehat{\mathcal{Z}} = 0$*
- *conharmonic Ricci semi-symmetric with respect to ∇, then $\mathcal{K} \cdot \widehat{\mathcal{Z}} = 0$*
- *concircular Ricci semi-symmetric with respect to ∇, then $\mathcal{V} \cdot \widehat{\mathcal{Z}} = 0$*
- *M-projective Ricci semi-symmetric with respect to ∇, then $\mathcal{W} \cdot \widehat{\mathcal{Z}} = 0$.*

5 An example of Kenmotsu manifolds admitting $\widehat{\nabla}$

In this section we consider a 5−dimensional example of Kenmotsu manifold. In [16] this example was examined by applying the generalized Tanaka-Webster connection. The authors computed the covariant derivatives of basis vector fields with respect to the Levi-Civita connection. By using these equalities we apply our new connection to this example and we verify some results which have been proved in the sections.

Let M be a 5−dimensional manifold defined by

$$M = \{P : P = (z_1, z_2, z_3, z_4, z_5) \in \mathbb{R}^5, z_i \in \mathbb{R}, 1 \leq i \leq 5\}$$

where $(z_1, z_2, z_3, z_4, z_5)$ are the standard coordinates in $\mathbb{R}^5$. Let consider 5 vector fields on M as follow;

$$X_1 = e^{-z_5}\frac{\partial}{\partial z_1},\ X_2 = e^{-z_5}\frac{\partial}{\partial z_2},\ X_3 = e^{-z_5}\frac{\partial}{\partial z_3},\ X_4 = e^{-z_5}\frac{\partial}{\partial z_4},\ X_5 = e^{-z_5}\frac{\partial}{\partial z_5}. \tag{41}$$

Define a Riemannian metric g by $g(X_i, X_j) = 0$ for $i \neq j, 1 \leq i, j \leq 5$ and $g(X_i, X_i) = 1$. Thus, the vector fields (41) have formed an orthonormal vector fields set which is the basis of M.

Let ϕ and η be given by

$$\eta(U_1) = g(U_1, X_5), \quad \text{for all } U_1 \in \Gamma(TM)$$
$$\phi X_1 = X_3,\ \phi X_2 = X_4,\ \phi X_3 = -X_1,\ \phi X_4 = -X_2,\ \phi X_5 = 0.$$

It is obvious that $\eta(X_5) = 1$. We can write $U_1 = \sum_{i=1}^{5} a_i X_i$ for scalars a_i on M. Thus by following easy computations we obtain $\phi^2 U_1 = -U_1 + a_5 X_5$ which is equal to

$$\phi^2 U_1 = -U_1 + \eta(U_1)X_5.$$

We obtain $g(\phi U_1, \phi U_2) = g(U_1, U_2) - \eta(U_1)\eta(U_2)$ for $U_2 = \sum_{j=1}^{5} b_j X_j$. Thus $\mathcal{S} = (\phi, \eta, X_5 = \xi, g)$ defines an almost contact metric structure on M. Moreover, we get the fundamental 2−form Ω as

$$\Omega(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_3}) = g(\frac{\partial}{\partial z_1}, \phi\frac{\partial}{\partial z_3}) = g(\frac{\partial}{\partial z_1}, -\frac{\partial}{\partial z_1}) = -e^{2z_5}$$

and hence, we obtain $\Omega = -e^{2z_5} dz_1 \wedge dz_3$. Thus, $d\Omega = -2e^{2z_5} dz_5 \wedge dz_1 \wedge dz_3 = 2\eta \wedge \Omega$. Also, it can be seen that M is normal. And finally (M, ϕ, ξ, η, g) is a Kenmotsu manifold.

From the definition of Lie derivation on M, we obtain

$$[X_1, X_2] = [X_1, X_3] = [X_1, X_4] = [X_2, X_3] = 0, [X_1, X_5] = X_1,$$
$$[X_4, X_5] = X_4, [X_2, X_4] = [X_3, X_4] = 0, [X_2, X_5] = X_2, [X_3, X_5] = X_3.$$

Recall the classical Kozsul's formula from the Riemannian geometry.

$$2g(\nabla_{U_1}U_2, U_3) = U_1 g(U_2, U_3) + U_2 g(U_1, U_3) - U_3 g(U_1, U_2) - g(U_1, [U_2, U_3]) \\ -g(U_2, [U_1, U_3]) + g(U_3, [U_1, U_2]),$$

for all $U_1, U_2, U_3 \in \Gamma(TM)$. Taking $X_5 = \xi$ and using Koszul's formula we get the following

$$\nabla_{X_1}X_1 = -X_5,\ \nabla_{X_1}X_5 = X_1,\ \nabla_{X_2}X_2 = -X_5,\ \nabla_{X_2}X_5 = X_2, \\ \nabla_{X_3}X_3 = -X_5,\ \nabla_{X_3}X_5 = X_3,\ \nabla_{X_4}X_4 = -X_5,\ \nabla_{X_4}X_5 = X_4,$$

and the others are zero.

Let consider M with the connection $\widehat{\nabla}$ defined in (12). Thus, we obtain the covariant derivatives of vector fields X_1, X_2, X_3, X_4, X_5 as $\widehat{\nabla}_{X_i}X_j = 0$ for $1 \leq i, j \leq 5$. This shows that all the Christoffel symbols of $\widehat{\nabla}$ vanish and from this reason the manifold has become $\widehat{\nabla}-$flat. Moreover, we get the Ricci curvature of M with $\widehat{\nabla}$ as $\widehat{Ric} = 0$ and so the scalar curvature is $\widehat{scal} = 0$.

$\widehat{\mathcal{Z}}$ curvature tensor of M is given by

$$\widehat{\mathcal{Z}}(U_1, U_2) = \psi g(U_1, U_2) = \psi \sum_{i=1}^{5} a_i b_i \neq 0$$

for $U_1 = \sum_{i=1}^{5} a_i X_i$ and $U_2 = \sum_{j=1}^{5} b_j X_j$. Thus M is not $\widehat{\mathcal{Z}}-$flat and then we verified the Theorem 3.

Since $\widehat{R} = 0$ from (28), we get

$$R(U_1, U_2)U_3 = -g(U_2, U_3)U_1 + g(U_1, U_3)U_2 \tag{42}$$

and thus we obtain

$$\begin{aligned}(R(U_1, U_2) \cdot \mathcal{Z})(U_3, U_4) &= \mathcal{Z}(-g(U_2, U_3)U_1 + g(U_1, U_3)U_4, U_4) \\ &\quad +\mathcal{Z}(U_3, -g(U_2, U_4)U_1 + g(U_1, U_3)U_4),\end{aligned}$$

for all $U_1, U_2, U_3 \in \Gamma(TM)$. From the definition of $\mathcal{Z}$ we get $(R(U_1, U_2) \cdot \mathcal{Z})(U_3, U_4) = \mathcal{T}(U_1, U_2, U_3, U_4)$. Since the manifold is flat with respect to $\widehat{\nabla}$, we obtain $\widehat{R} \cdot \widehat{\mathcal{Z}} = 0$ and thus the Lemma 1 is verified. On the other hand, by using the definition of $\mathcal{T}$ and since $Ric = -4$ [16] we get $\mathcal{T}(X_1, X_2, X_3, X_4) = 0$. Thus $R \cdot \mathcal{Z} = 0$ and this verifies the Corollary 5.

From (42), we get

$$\begin{aligned}(\mathcal{Z}(U_1, U_2) \cdot R)(U_3, U_5) &= -4\{\eta(U_3)g(U_4, U_5)U_1 \eta(U_3)g(U_1, U_5)U_4 + \eta(U_5)U_4 \\ &\quad -g(U_4, U_5)\xi + \eta(U_4)g(U_1, U_5, U_3) - \eta(U_5)U_3 \\ &\quad +g(U_3, U_5)\xi + \eta(U_5)g(U_1, U_4)U_3 - \eta(U_5)g(U_1, U_3)U_4 \\ &\quad +\eta(U_3)U_4 - \eta(U_4)U_3\} + \psi\{\eta(U_3)g(U_4, U_5)U_1\}.\end{aligned}$$

Let take $U_i = X_i,\ 1 \leq i \leq 5$. Then we obtain

$$(\mathcal{Z}(X_1, X_2) \cdot R)(X_3, X_4)X_5 = 4\{X_3 - X_4\}.$$

Suppose that $\mathcal{Z} \cdot R = 0$. By taking inner product with X_3 we get $4g(X_3, X_3) = 0$, which is impossible. So, we have a contradiction, $\mathcal{Z} \cdot R = 0$ can not vanish. This verifies the Theorem 5.

References

[1] De, U.C., Mantica, C.A., Suh, Y.J., *On weakly cyclic Z symmetric manifolds*, Acta Math. Hungar. **146** (2015), 153–167.

[2] De, U.C., Pal, P., *On almost pseudo-Z-symmetric manifolds*, Acta Univ. Palacki. Fac. Rer. Nat. Math. **53** (2014), 25–43.

[3] Taşci, A.Y., and Zengin, F.Ö., *Z-symmetric manifold admitting concircular Ricci symmetric tensor*, Afrika Mat. **31** (2020), 1093-1104.

[4] Agashe, N.S. and Chafle, M.R., *A semi-symmetric non-metric connection on a Riemannian Manifold*, Indian J. Pure Appl. Math., **23**(1992), 399-409.

[5] Alghamdi, M. A., and Bahadır, O., *Some Characterizations in Kenmotsu manifolds with a new connection*, BSG Proceedings, **27**, (2020), 1-16.

[6] Andonie, O.C., *On semi-symmetric non-metric connection on a Riemannian manifold*, Ann. Fac. Sci. De Kinshasa, Zaire Sect. Math. Phys., **2** (1976).

[7] Barman, A., *Concircular curvature tensor of a semi-symmetric metric connection in a Kenmotsu manifold*, Thai J. Math. **13** (2015), 245-257.

[8] Barman, A. and De, U. C., *Projective curvature tensor of semi-symmetric metric connection in a Kenmotsu manifold*, Int. Electronic. J. Geom. **6** (2013), 159-169.

[9] Barman, A., *On a type of quarter-symmetric non-metric ϕ-connection on a Kenmotsu manifold*, Bull. Math. Anal. Appl. **4** (2012), 1-11.

[10] Barua, B. and Mukhopadhyay, S., *A sequence of semi-symmetric connections on a Riemannian manifold*, Proceedings of seventh national seminar on Finsler, Lagrange and Hamiltonian spaces, 1992, Braşov, Romania.

[11] Blair, D. E., *Riemannian geometry of contact and symplectic manifolds*, Birkhäuser, Boston, 2002.

[12] Binh T.Q., Tamassy, L., De, U. C. and Tarafdar, M., *Some remarks on almost Kenmotsu manifolds*, Math. Pannonica, **13** (2002), 31-39.

[13] Chaubey, S., and Yadav, S. K., *Study of Kenmotsu manifolds with semi-symmetric metric connection*, Universal Journal of Mathematics and Applications, **1(2)**(2018), 89-97.

[14] Dileo G. and Pastore A. M., *Almost Kenmotsu manifolds and local symmetry*, Bull. Belg. Math. Soc. Simon Stevin, **14** (2007), 343-354.

[15] Friedman, A.A. and Schouten, J.A., *Über die Geometric der halbsymmetrischen Übertragung*, Math., Z. **21** (1924), 211-223.

[16] Ghosh, G., and Chand, D. U., *Kenmotsu manifolds with generalized Tanaka-Webster connection*, Publ. Inst. Math. (Beograd) (N.S.) **102(116)** (2017), 221-230.

[17] Golab, S., *On semi-symmetric and quarter-symmetric linear connections*, Tensor N.S. **29** (1975), 249-254.

[18] Hayden H. A., *Subspaces of space with torsion*, Proc. London Math. Soc. **34** (1932), 27-50.

[19] Kenmotsu K., *A class of almost contact Riemannian manifolds*, Tohoku Math. J. **24** (1972), 93-103.

[20] Kim, T.W. and Pak, H.K., *Canonical foliations of certain classes of almost contact metric structures*, Acta Math. Sin. (Engl. Ser.), **21** (2005), 841-846.

[21] Liang Y., *On semi-symmetric recurrent-metric connection*, Tensor, N. S., **55** (1994), 107-112.

[22] Mallick, S., and De, U.C., *Z Tensor on N (k)-quasi-Einstein manifolds*, Kyungpook Math. J. **56** (2016), no. 3, 979-991.

[23] Mantica C.A. and Molonari, L.G., *Weakly Z symmetric manifolds*, Acta Math. Hunger., **135** (2012), 80–96.

[24] Olszak Z., *Locally conformal almost cosymplectic manifolds*, Colloq. Math. **57** (1989), 73-87.

[25] Prakash, A., and Pandey, V.K., *On a quarter symmetric non-metric connection in a Kenmotsu manifolds*, Int. J. Pure Appl. Math., **83** (2013), no. 2, 271-278.

[26] Pitiş, G., *Geometry of Kenmotsu manifolds*, Braşov, 2007, page-151.

[27] Prvanovic, M., *On pseudo metric semi-symmetric connections*, Publ. Inst. Math. (Beograd) (N.S.) **18** (1975), 157-164.

[28] Sular, S., Özgür, C. and De, U.C., *Quarter-symmetric metric connection in a Kenmotsu manifold*, SUT J. Math. **44** (2008), no. 2, 297-306.

[29] Tripathi, M.M., *A new connection in a Riemannian manifold*, Int. Electronic J. Geom. **1(1)** (2008), 15-24.

[30] Tripathi, M.M. and Gupta, P., *T -curvature tensor on a semi-Riemannian manifold*, J. Adv. Math. Stud. **4** (2011), 117-129.

[31] Ünal, İ., *N(k)-contact metric manifolds admitting Z-tensor*, Karamanoglu Mehmetbey University Journal of Engineering and Natural Sciences, **2** (2020), no. 1, 64-69.

[32] Yano, K., *On semi-symmetric connection*, Rev. Roumaine Math. Pures Appl. **15** (1970), 1570-1586.

[33] Vanli, A. T., and Sari, R., *On semi-invariant submanifolds of a generalized Kenmotsu manifold admitting a semi-symmetric metric connection*, Pure and Applied Mathematics Journal, **4**(2015), 14-18.