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A GEOMETRIZATION ON DUAL 1-JET SPACES OF THE TIME-DEPENDENT HAMILTONIAN OF ELECTRODYNAMICS

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Abstract

In this paper we develop the distinguished Riemannian differential geometry (in the sense of d-connections, d-torsions, d-curvatures and the geometrical Maxwell-like and Einstein-like equations) for the time-dependent Hamiltonian of momenta which governs the electrodynamics phenomena.

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1 Time-dependent Hamiltonian models of electrodynamics

An extension of classical mechanics for a non-relativistic particle with a fixed mass m in the presence of the external non-autonomous electromagnetic field $A_i(t, x^i)$ is physically studied by Landau and Lifshitz in [5]. In the same direction, in the classical Lagrange geometry developed on the tangent bundle TM, the Lagrangian $L : TM \to \mathbb{R}$ that governs the electrodynamics phenomena is given by (see Miron [6])

$$L(x, y) = mc\gamma_{ij}(x)y^i y^j + \frac{2e}{m}A_i(x)y^i + U(x),$$

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where $\gamma_{ij}(x)$ is a pseudo-Riemannian metric tensor on M representing the *gravitational potentials*, $A_i(x)$ is a covector field on M representing *electromagnetic potentials*, $U(x)$ is a function and $m \neq 0$, c and e are well-known constants of the physics as the *mass, speed of light* or *electric charge.* In this way, we recall that a jet extension of the Lagrangian function of electrodynamics $L : J^1(\mathbb{R}, M) \to \mathbb{R}$ is set by (see Neagu [10])

$$L(t, x^k, y_1^k) = mch^{11}(t)\varphi_{ij}(x)y_1^i y_1^j + \frac{2e}{m}A_{(i)}^{(1)}(t, x)y_1^i + \mathsf{P}(t, x), \tag{1}$$

where $h_{11}(t)$ (respectively $\varphi_{ij}(x)$) is a pseudo-Riemannian metric on the time manifold $\mathbb{R}$ (respectively spatial manifold M), $A_{(i)}^{(1)}(t, x)$ is a distinguished tensor on $J^1(\mathbb{R}, M)$ and $\mathsf{P}(t, x)$ is a smooth function on the product manifold $\mathbb{R} \times M$.

Via the Legendre transformation, the jet time-dependent Lagrangian function of electrodynamics (1) leads us to the Hamiltonian function of momenta (see [1] and [11])

$$H(t, x^k, p_k^1) = \frac{1}{4mc}h_{11}\varphi^{ij}p_i^1 p_j^1 - \frac{e}{m^2c}h_{11}\varphi^{ij}A_{(j)}^{(1)}p_i^1 + \frac{e^2}{m^3c}\left\|A\right\|^2 - \mathsf{P}, \tag{2}$$

where $H : J^{1*}(\mathbb{R}, M) \to \mathbb{R}$, and $\left\|A\right\|^2(t, x) = h_{11}\varphi^{ij}A_{(i)}^{(1)}A_{(j)}^{(1)}$. In other words, we have $p_i^1 = \partial L/\partial y_1^i$ and $H = p_i^1 y_1^i - L$. The pair $\mathcal{EDH}^n = (J^{1*}(\mathbb{R}, M), H)$, where H is given by (2), is called the *autonomous time-dependent Hamilton space of electrodynamics*. Now, using as a pattern the Miron's geometrical ideas from the works [8] on TM and [7], [9] on T^*M, which were extended on 1-jet spaces and their duals in the works [10] and [1], the distinguished Riemannian geometry for the particular momentum Hamiltonian function (2) (which governs the *time-dependent momentum electrodynamics*) can be constructed on the dual 1-jet space $J^{1*}(\mathbb{R}, M)$ (see the paper [11]).

2 The time-dependent Hamilton space of electrodynamics $\mathcal{EDH}^n$

To start our Hamiltonian geometrical development for time-dependent electrodynamics, let us consider the dual 1-jet space $E^* = J^{1*}(\mathbb{R}, M)$ the *fundamental vertical metrical d-tensor*

$$\mathcal{G}_{(1)(1)}^{(i)(j)} = \frac{1}{2}\frac{\partial^2 H}{\partial p_i^1 \partial p_j^1} = \widetilde{h}_{11}(t)\varphi^{ij}(x^k),$$

where $\widetilde{h}_{11}(t) := (4mc)^{-1} \cdot h_{11}(t)$. Let $H_{11}^1(t) = (h^{11}/2)(dh_{11}/dt)$ (respectively $\gamma_{ij}^k(x)$) be the Christoffel symbols of the metric $h_{11}(t)$ (respectively $\varphi_{ij}(x)$). Obviously, if $\widetilde{H}_{11}^1$ is the Christoffel symbol of the pseudo-Riemannian metric $\widetilde{h}_{11}(t)$, then we have $\widetilde{H}_{11}^1 = H_{11}^1$. In this context, by direct computations, we find (see general formulas from papers [11] and [1])

Theorem 1. *The pair of local functions* $N = \left(\underset{1}{N}{}^{(1)}_{(i)1},\ \underset{2}{N}{}^{(1)}_{(i)j}\right)$ *on the dual* 1*-jet space* E^**, which are given by*

$$\underset{1}{N}{}^{(1)}_{(i)1} = H^1_{11}p^1_i, \quad \underset{2}{N}{}^{(1)}_{(i)j} = \gamma^r_{ij}\left[\frac{2e}{m}A^{(1)}_{(r)} - p^1_r\right] - \frac{e}{m}\left[\frac{\partial A^{(1)}_{(i)}}{\partial x^j} + \frac{\partial A^{(1)}_{(j)}}{\partial x^i}\right], \tag{3}$$

represents a nonlinear connection on E^**. This nonlinear connection is called the* ***canonical nonlinear connection of the time-dependent Hamilton space of electrodynamics*** $\mathcal{EDH}^n$.

Now, let $\{\delta/\delta t,\ \delta/\delta x^i,\ \partial/\partial p^1_i\} \subset \mathcal{X}(E^*)$ and $\{dt, dx^i, \delta p^1_i\} \subset \mathcal{X}^*(E^*)$ be the adapted bases produced by the nonlinear connection (3), where

$$\frac{\delta}{\delta t} = \frac{\partial}{\partial t} - \underset{1}{N}{}^{(1)}_{(r)1}\frac{\partial}{\partial p^1_r}, \qquad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \underset{2}{N}{}^{(1)}_{(r)i}\frac{\partial}{\partial p^1_r},$$
$$\delta p^1_i = dp^1_i + \underset{1}{N}{}^{(1)}_{(i)1}dt + \underset{2}{N}{}^{(1)}_{(i)r}dx^r. \tag{4}$$

Using the above adapted bases, by direct local computations, we can determine the adapted components of the Cartan canonical connection of the space $\mathcal{EDH}^n$, together with its local d-torsions and d-curvatures (see the general formulas from papers [11], [13] and [3]).

Theorem 2. *(i) The canonical Cartan connection of the autonomous time-dependent Hamilton space of electrodynamics* $\mathcal{EDH}^n$ *is defined by the adapted components*

$$C\Gamma(N) = \left(H^1_{11} = H^1_{11},\ A^i_{j1} = 0,\ H^i_{jk} = \gamma^i_{jk},\ C^{i(k)}_{j(1)} = 0\right).$$

(ii) The torsion $\mathbf{T}$ *of the canonical Cartan connection of the space* $\mathcal{EDH}^n$ *is determined by* ***two*** *effective adapted components:*

$$R^{(1)}_{(r)1j} = -\frac{2e}{m}\gamma^s_{rj}A^{(1)}_{(s);1} + \frac{e}{m}\left[\frac{\partial A^{(1)}_{(r)}}{\partial x^j} + \frac{\partial A^{(1)}_{(j)}}{\partial x^r}\right]_{;1},$$
$$R^{(1)}_{(r)ij} = \mathfrak{R}^s_{rij}\left[\frac{2e}{m}A^{(1)}_{(s)} - p^1_s\right] - \frac{e}{m}\left[\frac{\partial A^{(1)}_{(i)}}{\partial x^j} - \frac{\partial A^{(1)}_{(j)}}{\partial x^i}\right]_{:r}, \tag{5}$$

where $\mathfrak{R}^k_{rij}(x)$ *are the local curvature tensors of the pseudo-Riemannian metric* $\varphi_{ij}(x)$*, and* "$_{;1}$" *and* "$_{:k}$" *represent the following* ***generalized Levi-Civita covariant derivatives****:*

- *the* $\mathbb{R}$*-generalized Levi-Civita covariant derivative:*

$$T^{1i(1)(r)...}_{1j(l)(1)...;1} \overset{def}{=} \frac{\partial T^{1i(1)(r)...}_{1j(l)(1)...}}{\partial t} + T^{1i(1)(r)...}_{1j(l)(1)...}H^1_{11} + T^{1i(1)(r)...}_{1j(l)(1)...}H^1_{11} + ...$$
$$... - T^{1i(1)(r)...}_{1j(l)(1)...}H^1_{11} - T^{1i(1)(r)...}_{1j(l)(1)...}H^1_{11} - ...$$

- *the M-generalized Levi-Civita covariant derivative:*

$$T^{1i(1)(r)...}_{1j(l)(1)...:k} \stackrel{def}{=} \frac{\partial T^{1i(1)(r)...}_{1j(l)(1)...}}{\partial x^k} + T^{1s(1)(r)...}_{1j(l)(1)...}\gamma^i_{sk} + T^{1i(1)(s)...}_{1j(l)(1)...}\gamma^r_{sk} + ...$$
$$... - T^{1i(1)(r)...}_{1s(l)(1)...}\gamma^s_{jk} - T^{1i(1)(r)...}_{1j(s)(1)...}\gamma^s_{lk} - ... \;.$$

(iii) The curvature $\mathbf{R}$ *of the Cartan connection of the space* $\mathcal{EDH}^n$ *is given by* ***two*** *adapted components:* $R^{(1)(l)}_{(i)(1)jk} = R^l_{ijk} = \mathfrak{R}^l_{ijk}$.

2.1 The electromagnetic-like geometrical model

To expose our geometrical electromagnetic-like theory on the time-dependent Hamilton space of electrodynamics $\mathcal{EDH}^n$, we emphasize that, by simple direct calculations, we obtain

Proposition 1. *The* ***metrical deflection d-tensors*** *of the space* $\mathcal{EDH}^n$ *are given by the formulas:*

$$\begin{aligned} &\Delta^{(i)}_{(1)j} = [\widetilde{h}_{11}\varphi^{ir}p^1_r]_{|j} = \frac{e}{4m^2c}h_{11}\varphi^{ir}\left[A^{(1)}_{(r):j} + A^{(1)}_{(j):r}\right], \\ &\Delta^{(i)}_{(1)1} = \left[\widetilde{h}_{11}\varphi^{ir}p^1_r\right]_{/1} = 0, \quad \vartheta^{(i)(j)}_{(1)(1)} = [\widetilde{h}_{11}\varphi^{ir}p^1_r]|^{(j)}_{(1)} = \frac{1}{4mc}h_{11}\varphi^{ij}, \end{aligned} \tag{6}$$

where "$_{/1}$", "$_{|j}$" *and* "$|^{(1)}_{(j)}$" *are the local covariant derivatives induced by the Cartan canonical connection* $C\Gamma(N)$.

Moreover, following some general formulas from [11] and [2], we introduce

Definition 1. *The distinguished* 2*-form on the* 1*-jet space* E^**, locally defined by*

$$\mathbb{F} = F^{(i)}_{(1)j}\delta p^1_i \wedge dx^j + f^{(i)(j)}_{(1)(1)}\delta p^1_i \wedge \delta p^1_j,$$

where

$$\begin{aligned} F^{(i)}_{(1)j} &= \frac{1}{2}\left[\Delta^{(i)}_{(1)j} - \Delta^{(j)}_{(1)i}\right] = \frac{e}{8m^2c}\cdot \underset{\{i,j\}}{\mathcal{A}}\left\{h_{11}\varphi^{ir}\left[A^{(1)}_{(r):j} + A^{(1)}_{(j):r}\right]\right\}, \\ f^{(i)(j)}_{(1)(1)} &= \frac{1}{2}\left[\vartheta^{(i)(j)}_{(1)(1)} - \vartheta^{(j)(i)}_{(1)(1)}\right] = 0, \end{aligned} \tag{7}$$

is called the ***momentum electromagnetic field associated with the autonomous time-dependent Hamilton space of electrodynamics*** $\mathcal{EDH}^n$.

Particularizing on the space $\mathcal{EDH}^n$ the geometrical Maxwell-like equations of the momentum electromagnetic field that governs a general time-dependent Hamilton space H^n (see [11] and [1]), we get:

Theorem 3. *The momentum electromagnetic components (7) of the autonomous time-dependent Hamilton space of electrodynamics $\mathcal{EDH}^n$ are governed by the following **geometrical Maxwell-like equations**:*

$$\begin{cases} F^{(i)}_{(1)j/1} = F^{(i)}_{(1)j;1} = \dfrac{e \cdot h_{11}}{8m^2c} \cdot \underset{\{i,j\}}{\mathcal{A}} \varphi^{ir} \left\{ \left[\dfrac{\partial A^{(1)}_{(r)}}{\partial x^j} + \dfrac{\partial A^{(1)}_{(j)}}{\partial x^r} \right]_{;1} - 2\gamma^s_{rj} A^{(1)}_{(s);1} \right\}, \\ \sum\limits_{\{i,j,k\}} F^{(i)}_{(1)j|k} = \sum\limits_{\{i,j,k\}} F^{(i)}_{(1)j:k} = -\dfrac{h_{11}}{8mc} \cdot \sum\limits_{\{i,j,k\}} \left\{ \left[\varphi^{sr} \mathfrak{R}^i_{rjk} - \varphi^{ir} \mathfrak{R}^s_{rjk} \right] p^1_s + \right. \\ \qquad \left. + \dfrac{e}{m} \varphi^{ir} \left[2\mathfrak{R}^s_{rjk} A^{(1)}_{(s)} - \left(\dfrac{\partial A^{(1)}_{(j)}}{\partial x^k} - \dfrac{\partial A^{(1)}_{(k)}}{\partial x^j} \right)_{:r} \right] \right\}, \end{cases}$$

where $\mathcal{A}_{\{i,j\}}$ represents an alternate sum and $\sum_{\{i,j,k\}}$ represents a cyclic sum.

2.2 The gravitational-like geometrical model

To describe our geometrical Hamiltonian momentum gravitational theory on the autonomous time-dependent Hamilton space of electrodynamics $\mathcal{EDH}^n$, we recall that the metrical d-tensor $\mathcal{G}^{(i)(j)}_{(1)(1)} = \widetilde{h}_{11}(t)\varphi^{ij}(x)$ and the canonical nonlinear connection (3) produce a momentum gravitational $\widetilde{h}$-potential $\mathbb{G}$ on the 1-jet space E^*, locally defined by

$$\mathbb{G} = \widetilde{h}_{11} dt \otimes dt + \varphi_{ij} dx^i \otimes dx^j + \widetilde{h}_{11} \varphi^{ij} \delta p^1_i \otimes \delta p^1_j.$$

To analyze the corresponding local geometrical Einstein-like equations (together with their momentum conservation laws) in the adapted basis

$$\{X_A\} = \left\{ \delta/\delta t,\ \delta/\delta x^i,\ \partial/\partial p^1_i \right\},$$

let $C\Gamma(N) = (H^1_{11}, 0, \gamma^i_{jk}, 0)$ be the Cartan canonical connection of the space $\mathcal{EDH}^n$. Taking into account the expressions of its adapted curvature d-tensors on the space $\mathcal{EDH}^n$, we find

Theorem 4. *The Ricci tensor* $\mathrm{Ric}(C\Gamma(N))$ *of the space $\mathcal{EDH}^n$ is characterized only by one effective local adapted Ricci d-tensor: $\mathfrak{R}_{ij} = \mathfrak{R}^r_{ijr}$.*

The scalar curvature $Sc(C\Gamma(N))$ of the Cartan connection of the space $\mathcal{EDH}^n$ is given by $Sc(C\Gamma(N)) = \mathfrak{R}$, where $\mathfrak{R} = \varphi^{ij}\mathfrak{R}_{ij}$ is the scalar curvature of the pseudo-Riemannian metric $\varphi_{ij}(x)$. Particularizing on the space $\mathcal{EDH}^n$ the geometrical Einstein-like equations and the momentum conservation laws that govern an arbitrary time-dependent Hamilton space H^n (see [11] and [1]), we get:

Theorem 5. *The local* ***geometrical Einstein-like equations****, that govern the momentum gravitational potential of the space* $\mathcal{EDH}^n$*, have the form*

$$\begin{cases} \mathfrak{R}_{ij} - \dfrac{\mathfrak{R}}{2}\varphi_{ij} = \mathcal{K}\mathbb{T}_{ij}, \\ 0 = \mathbb{T}_{1i}, \quad 0 = \mathbb{T}_{i1}, \quad 0 = \mathbb{T}^{(i)}_{(1)1}, \quad -\mathfrak{R}h_{11} = 8mc \cdot \mathcal{K}\mathbb{T}_{11}, \\ 0 = \mathbb{T}_{1\stackrel{(j)}{(1)}}, \quad 0 = \mathbb{T}_{i\stackrel{(j)}{(1)}}, \quad 0 = \mathbb{T}^{(i)}_{(1)j}, \quad -\mathfrak{R}h_{11}\varphi^{ij} = 8mc \cdot \mathcal{K}\mathbb{T}^{(i)(j)}_{(1)(1)}, \end{cases} \tag{8}$$

where $\mathbb{T}_{AB}$*,* $A, B \in \left\{1, i, \stackrel{(i)}{(1)}\right\}$*, are the adapted components of the momentum stress-energy d-tensor of matter* $\mathbb{T}$*, and* $\mathcal{K}$ *is the Einstein constant.*

As a consequence, setting $\mathfrak{R}^r_j = \varphi^{rs}\mathfrak{R}_{sj}$, then the *momentum conservation laws* of the geometrical Einstein-like equations (8) take the form (see the papers [11] and [1])

$$\left[\mathfrak{R}^r_j - \frac{\mathfrak{R}}{2}\delta^r_j\right]_{|r} = 0.$$

Open problem. From a physical point of view, an open problem is to describe the properties of such mechanical models which correspond to the momenta-depending geometrical objects introduced above.

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