A GEOMETRIZATION ON DUAL 1-JET SPACES OF THE TIME-DEPENDENT HAMILTONIAN OF ELECTRODYNAMICS

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Abstract

In this paper we develop the distinguished Riemannian differential geometry (in the sense of d-connections, d-torsions, d-curvatures and the geometrical Maxwell-like and Einstein-like equations) for the time-dependent Hamiltonian of momenta which governs the electrodynamics phenomena.

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1 Time-dependent Hamiltonian models of electrodynamics

An extension of classical mechanics for a non-relativistic particle with a fixed mass \( m \) in the presence of the external non-autonomous electromagnetic field \( A_i(t, x^j) \) is physically studied by Landau and Lifshitz in [5]. In the same direction, in the classical Lagrange geometry developed on the tangent bundle \( TM \), the Lagrangian \( L : TM \rightarrow \mathbb{R} \) that governs the electrodynamics phenomena is given by (see Miron [6])

\[
L(x, y) = mc\gamma_{ij}(x)y^i y^j + \frac{2e}{m} A_i(x)y^i + U(x),
\]

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where $\gamma_{ij}(x)$ is a pseudo-Riemannian metric tensor on $M$ representing the gravitational potentials, $A_i(x)$ is a covector field on $M$ representing electromagnetic potentials, $U(x)$ is a function and $m \neq 0$, $c$ and $e$ are well-known constants of the physics as the mass, speed of light or electric charge. In this way, we recall that a jet extension of the Lagrangian function of electrodynamics $L : J^1(\mathbb{R}, M) \rightarrow \mathbb{R}$ is set by (see Neagu [10])

$$L(t, x^k, y_1^i) = m e h^{11}(t) \varphi_{ij}(x) y_1^i y_1^j + \frac{2e}{m} A^{(1)}_{ij}(t, x) y_1^i + P(t, x),$$  \hspace{1cm} (1)

where $h_{11}(t)$ (respectively $\varphi_{ij}(x)$) is a pseudo-Riemannian metric on the time manifold $\mathbb{R}$ (respectively spatial manifold $M$), $A^{(1)}_{ij}(t, x)$ is a distinguished tensor on $J^1(\mathbb{R}, M)$ and $P(t, x)$ is a smooth function on the product manifold $\mathbb{R} \times M$.

Via the Legendre transformation, the jet time-dependent Lagrangian function of electrodynamics (1) leads us to the Hamiltonian function of momenta (see [1] and [11])

$$H(t, x^k, p_k^i) = \frac{1}{4mc} h_{11}^{ij} p_i^1 p_j^1 - \frac{e}{m^2 c} h_{11}^{ij} \varphi_{ij}^{(1)} p_i^1 + \frac{e^2}{m^3 c} \|A\|^2 - P,$$  \hspace{1cm} (2)

where $H : J^{1*}(\mathbb{R}, M) \rightarrow \mathbb{R}$, and $\|A\|^2(t, x) = h_{11}^{ij} \varphi_{ij}^{(1)} A^{(1)}_{ij}$. In other words, we have $p_i^1 = \partial L / \partial \dot{y}_1^i$ and $H = p_i^1 \dot{y}_1^i - L$. The pair $\mathcal{E}D H^n = (J^{1*}(\mathbb{R}, M), H)$, where $H$ is given by (2), is called the autonomous time-dependent Hamilton space of electrodynamics. Now, using as a pattern the Miron’s geometrical ideas from the works [8] on $TM$ and [7], [9] on $T^*M$, which were extended on 1-jet spaces and their duals in the works [10] and [1], the distinguished Riemannian geometry for the particular momentum Hamiltonian function (2) (which governs the time-dependent momentum electrodynamics) can be constructed on the dual 1-jet space $J^{1*}(\mathbb{R}, M)$ (see the paper [11]).

## 2 The time-dependent Hamilton space of electrodynamics $\mathcal{E}D H^n$

To start our Hamiltonian geometrical development for time-dependent electrodynamics, let us consider the dual 1-jet space $E^* = J^{1*}(\mathbb{R}, M)$ the fundamental vertical metrical $d$-tensor

$$g^{(i)(j)} = \frac{1}{2} \frac{\partial^2 H}{\partial p^i \partial p^j} = h_{11}(t) \varphi_{ij}(x^k),$$

where $\tilde{h}_{11}(t) := (4mc)^{-1} \cdot h_{11}(t)$. Let $H_{11}^1(t) = (h_{11}^1 / 2) (dh_{11}^1 / dt)$ (respectively $\gamma_{ij}^1(t)$) be the Christoffel symbols of the metric $h_{11}(t)$ (respectively $\varphi_{ij}(x)$). Obviously, if $\tilde{H}_{11}^1$ is the Christoffel symbol of the pseudo-Riemannian metric $\tilde{h}_{11}(t)$, then we have $\tilde{H}_{11}^1 = H_{11}^1$. In this context, by direct computations, we find (see general formulas from papers [11] and [1])
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**Theorem 1.** The pair of local functions $N = \left(N^{(1)}_{1(i)}, N^{(1)}_{2(i)}\right)$ on the dual 1-jet space $E^*$, which are given by

$$N^{(1)}_{1(i)} = H^{1}_{11} p^{i}_1, \quad N^{(1)}_{2(i)} = \gamma^{i}_{ij} \left[\frac{2e}{m} A^{(1)}(r) - p^{i}_1\right] - \frac{e}{m} \left[\frac{\partial A^{(1)}(i)}{\partial x^j} + \frac{\partial A^{(1)}(j)}{\partial x^i}\right], \quad (3)$$

represents a nonlinear connection on $E^*$. This nonlinear connection is called the canonical nonlinear connection of the time-dependent Hamilton space of electrodynamics $\mathcal{ED}H^n$.

Now, let $\{\delta/\delta t, \delta/\delta x^i, \partial/\partial p^{(1)}_1\} \subset \mathfrak{X}(E^*)$ and $\{dt, dx^i, \delta p^{(1)}_1\} \subset \mathfrak{X}^*(E^*)$ be the adapted bases produced by the nonlinear connection (3), where

$$\frac{\delta}{\delta t} = \frac{\partial}{\partial t} - N^{(1)}_{1(i)} \frac{\partial}{\partial p^{(1)}_1}, \quad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^{(1)}_{2(i)} \frac{\partial}{\partial p^{(1)}_1}, \quad (4)$$

$$\delta p^{(1)}_i = dp^{(1)}_i + N^{(1)}_{1(i)} dt + N^{(1)}_{2(i)} dx^i.$$

Using the above adapted bases, by direct local computations, we can determine the adapted components of the Cartan canonical connection of the space $\mathcal{ED}H^n$, together with its local d-torsions and d-curvatures (see the general formulas from papers [11], [13] and [3]).

**Theorem 2.**

(i) The canonical Cartan connection of the autonomous time-dependent Hamilton space of electrodynamics $\mathcal{ED}H^n$ is defined by the adapted components

$$C\Gamma (N) = \left(H^{1}_{11} = H^{1}_{11}, \quad A^{i}_{j1} = 0, \quad H^{i}_{jk} = \gamma^{i}_{jk}, \quad C^{i(k)}_{j(1)} = 0\right).$$

(ii) The torsion $\mathbf{T}$ of the canonical Cartan connection of the space $\mathcal{ED}H^n$ is determined by two effective adapted components:

$$R^{(1)}_{(r)1j} = -\frac{2e}{m} \gamma^{s}_{jk} A^{(1)}_{(s);i} + \frac{e}{m} \left[\frac{\partial A^{(1)}(r)}{\partial x^j} + \frac{\partial A^{(1)}(j)}{\partial x^i}\right], \quad (5)$$

$$R^{(1)}_{(r)ij} = \mathfrak{R}^{s}_{rij} \left[\frac{2e}{m} A^{(1)}(s) - p^{1}_s\right] - \frac{e}{m} \left[\frac{\partial A^{(1)}(i)}{\partial x^j} - \frac{\partial A^{(1)}(j)}{\partial x^i}\right],$$

where $\mathfrak{R}^{k}_{rij}(x)$ are the local curvature tensors of the pseudo-Riemannian metric $\varphi_{ij}(x)$, and $",1"," and "$k" represent the following generalized Levi-Civita covariant derivatives:

- the $\mathbb{R}$-generalized Levi-Civita covariant derivative:

$$T^{1i(1)(r)}_{lj}(0)(1)... \stackrel{\text{def}}{=} \frac{\partial T^{1i(1)(r)}_{lj}(0)(1)...}{\partial t} + T^{1i(1)(r)}_{lj}(0)(1)... H^{1}_{11} + T^{1i(1)(r)}_{lj}(0)(1)... H^{1}_{11} + ...$$

$$... - T^{1i(1)(r)}_{lj}(0)(1)... H^{1}_{11} - T^{1i(1)(r)}_{lj}(0)(1)... H^{1}_{11} - ...$$
the M-generalized Levi-Civita covariant derivative:

\[
T_{1j(l)(1)\ldots k}^{1s(l)(1)\ldots j}\left[\frac{\partial T_{1j(l)(1)\ldots k}^{1s(l)(1)\ldots j}}{\partial x^k} + T_{1j(l)(1)\ldots k}^{1s(l)(1)\ldots j}\gamma_{sk}^r + \ldots - T_{1s(l)(1)\ldots j}\gamma_{sk}^r - T_{1j(s)(1)\ldots j}\gamma_{sk}^r - \ldots\right].
\]

(iii) The curvature \( R \) of the Cartan connection of the space \( \mathcal{ED}H^n \) is given by two adapted components: \( R_{(i)(1)jk}^{(1)(l)} = R_{ijk}^l = \mathcal{R}_{ijk}^l. \)

2.1 The electromagnetic-like geometrical model

To expose our geometrical electromagnetic-like theory on the time-dependent Hamilton space of electrodynamics \( \mathcal{ED}H^n \), we emphasize that, by simple direct calculations, we obtain

**Proposition 1.** The metrical deflection d-tensors of the space \( \mathcal{ED}H^n \) are given by the formulas:

\[
\Delta_{(i)(1)j}^{(j)} = \frac{\hbar_{11}\varphi^irp_{1i}}{4m^2c}h_{11}\varphi^ir\left[ A_{(r)(j)}^{(1)} + A_{(j)(r)}^{(1)} \right],
\]

\[
\Delta_{(i)(1)1}^{(j)} = \left[ \hbar_{11}\varphi^irp_{1i} \right]_{/1} = 0, \quad \vartheta_{(i)(1)(1)}^{(j)} = \frac{1}{4mc}h_{11}\varphi^{ij}, \quad (6)
\]

where "\(^{1}/1"", "\(_{ij}"" and "\(_{(1)_{1}}"" are the local covariant derivatives induced by the Cartan canonical connection \( CT(N) \).

Moreover, following some general formulas from \([11]\) and \([2]\), we introduce

**Definition 1.** The distinguished 2-form on the 1-jet space \( E^* \), locally defined by

\[
F = \frac{1}{2} \left[ \Delta_{(i)(1)j}^{(j)} - \Delta_{(j)(1)i}^{(i)} \right] = \frac{1}{8m^2c} \mathcal{A}_{(i,j)} \left\{ h_{11}\varphi^ir\left[ A_{(r)(j)}^{(1)} + A_{(j)(r)}^{(1)} \right] \right\},
\]

\[
f_{(i)(1)(1)}^{(j)} - \frac{1}{2} \left[ \vartheta_{(i)(1)(1)}^{(j)} - \vartheta_{(j)(1)(1)}^{(i)} \right] = 0, \quad (7)
\]

is called the momentum electromagnetic field associated with the autonomous time-dependent Hamilton space of electrodynamics \( \mathcal{ED}H^n \).

Particularizing on the space \( \mathcal{ED}H^n \) the geometrical Maxwell-like equations of the momentum electromagnetic field that governs a general time-dependent Hamilton space \( H^n \) (see \([11]\) and \([1]\)), we get:
Theorem 3. The momentum electromagnetic components (7) of the autonomous time-dependent Hamilton space of electrodynamics $EDH^n$ are governed by the following geometrical Maxwell-like equations:

\[
\begin{aligned}
F^{(i)}(i)/j/1 = F^{(i)}(i)/j/1 &= \frac{e \cdot h_{11}}{8m^2c} \cdot A^{ir} \left\{ \left[ \frac{\partial A^{(1)}(s)}{\partial x^i} + \frac{\partial A^{(1)}(j)}{\partial x^r} \right]_{;1} - 2\gamma^s r A^{(1)}(s);1 \right\}, \\
\sum_{\{i,j,k\}} F^{(i)}(i)/j/k &= \sum_{\{i,j,k\}} F^{(i)}(i)/j/k = -\frac{h_{11}}{8mc} \sum_{\{i,j,k\}} \left\{ \left[ \varphi^{sr} \mathfrak{G}^{(s)}_{rjk} + \varphi^{ir} \mathfrak{R}^{s}_{rjk} \right] p^1_{s} \right. \\
&\quad + \left. \frac{e}{m} \varphi^{ir} \left[ 2\mathfrak{R}^{(1)}_{rjk} A^{(1)}_{(s)} \left( \frac{\partial A^{(1)}(s)}{\partial x^k} - \frac{\partial A^{(1)}(j)}{\partial x^r} \right) \right] \right\},
\end{aligned}
\]

where $A_{\{i,j\}}$ represents an alternate sum and $\sum_{\{i,j,k\}}$ represents a cyclic sum.

2.2 The gravitational-like geometrical model

To describe our geometrical Hamiltonian momentum gravitational theory on the autonomous time-dependent Hamilton space of electrodynamics $EDH^n$, we recall that the metrical d-tensor $G^{(i)(j)}(i)(j) = e^{h_{11}(t)} \varphi^{ij}(x)$ and the canonical nonlinear connection (3) produce a momentum gravitational $\vec{h}$-potential $G$ on the 1-jet space $E^*$, locally defined by

\[
G = \vec{h}_{11} dt \otimes dt + \varphi_{ij} dx^i \otimes dx^j + \vec{h}_{11} \varphi^{ij} \delta p^1_i \otimes \delta p^1_j.
\]

To analyze the corresponding local geometrical Einstein-like equations (together with their momentum conservation laws) in the adapted basis

\[
\{X_A\} = \{\delta/\delta t, \delta/\delta x^i, \partial/\partial p^1_i\},
\]

let $CT(N) = (H_{11}, 0, \gamma^i_{jk}, 0)$ be the Cartan canonical connection of the space $EDH^n$. Taking into account the expressions of its adapted curvature d-tensors on the space $EDH^n$, we find

Theorem 4. The Ricci tensor $Ric(CT(N))$ of the space $EDH^n$ is characterized only by one effective local adapted Ricci d-tensor: $\mathfrak{R}_{ij} = \mathfrak{R}^s_{ijr}$.

The scalar curvature $Sc(CT(N))$ of the Cartan connection of the space $EDH^n$ is given by $Sc(CT(N)) = \mathfrak{R}$, where $\mathfrak{R} = \varphi^{ij} \mathfrak{R}_{ij}$ is the scalar curvature of the pseudo-Riemannian metric $\varphi_{ij}(x)$. Particularizing on the space $EDH^n$ the geometrical Einstein-like equations and the momentum conservation laws that govern an arbitrary time-dependent Hamilton space $H^n$ (see [11] and [1]), we get:
Theorem 5. The local geometrical Einstein-like equations, that govern the momentum gravitational potential of the space $EDH^n$, have the form

\[
\begin{align*}
    &\mathcal{R}_{ij} - \frac{\mathcal{R}}{2} \delta_{ij} = \mathcal{K} T_{ij}, \\
    &0 = T_{1i}, \quad 0 = T_{i1}, \quad 0 = T^{(i)}_{(1)1}, \quad -\mathcal{R}h_{11} = 8mc \cdot \mathcal{K} T_{11}, \quad (8) \\
    &0 = T^{(j)}_{1(1)}, \quad 0 = T^{(j)}_{i(1)}, \quad 0 = T^{(i)}_{(1)j}, \quad -\mathcal{R}h_{11} \varphi^{ij} = 8mc \cdot \mathcal{K} T^{(i)(j)}_{(1)(1)},
\end{align*}
\]

where $T_{AB}$, $A, B \in \{1, i, (i)\}$, are the adapted components of the momentum stress-energy d-tensor of matter $T$, and $\mathcal{K}$ is the Einstein constant.

As a consequence, setting $\mathcal{R}_{ij}^r = \varphi^{rs} \mathcal{R}_{sj}$, then the momentum conservation laws of the geometrical Einstein-like equations (8) take the form (see the papers [11] and [1])

\[
\left[\mathcal{R}_{ij}^r - \frac{\mathcal{R}}{2} \delta_{ij}^r\right]_r = 0.
\]

Open problem. From a physical point of view, an open problem is to describe the properties of such mechanical models which correspond to the momentum-depending geometrical objects introduced above.

References


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