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SOME GEOMETRIC PROPERTIES OF CERTAIN FAMILIES OF q-BESSEL FUNCTIONS

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Abstract

In this paper, we are mainly interested in finding sufficient conditions for the q-close-to-convexity of certain families of q-Bessel functions with respect to certain functions in the open unit disk. The strong convexity and strong starlikeness of the same functions are also the part of our investigation.

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1 Introduction

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$ and $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$. Let $\mathcal{S}^*(\alpha)$, $\mathcal{C}(\alpha)$, $\mathcal{K}(\alpha)$, $\widetilde{\mathcal{S}^*}(\alpha)$ and $\widetilde{\mathcal{C}}(\alpha)$ denote the classes of starlike, convex, close-to-convex, strongly starlike and strongly convex functions of order α, respectively, and are defined as:

$$\mathcal{S}^*(\alpha) = \left\{ f : f \in \mathcal{A} \text{ and } \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in \mathcal{U}, \alpha \in [0,1) \right\},$$

$$\mathcal{C}(\alpha) = \left\{ f : f \in \mathcal{A} \text{ and } \Re\left(\frac{(zf'(z))'}{f'(z)}\right) > \alpha, z \in \mathcal{U}, \alpha \in [0,1) \right\},$$

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$$\mathcal{K}(\alpha)=\left\{f: f\in\mathcal{A} \text{ and } \Re\left(\frac{zf'(z)}{g(z)}\right)>\alpha,\quad z\in\mathcal{U},\, \alpha\in[0,1),\ g\in\mathcal{S}^*(0):\equiv\mathcal{S}^*\right\},$$

$$\widetilde{\mathcal{S}^*}(\alpha)=\left\{f: f\in\mathcal{A} \text{ and } \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right|<\frac{\alpha\pi}{2},\quad z\in\mathcal{U},\, \alpha\in(0,1]\right\}$$

and

$$\widetilde{\mathcal{C}}(\alpha)=\left\{f: f\in\mathcal{A} \text{ and } \left|\arg\left(1+\frac{zf''(z)}{f'(z)}\right)\right|<\frac{\alpha\pi}{2},\quad z\in\mathcal{U},\, \alpha\in(0,1]\right\}.$$

It is clear that

$$\widetilde{\mathcal{S}^*}(1)=\mathcal{S}^*(0)=\mathcal{S}^*,\ \widetilde{\mathcal{C}}(1)=\mathcal{C}(0)=\mathcal{C} \text{ and } \mathcal{K}(0)=\mathcal{K},$$

where $\mathcal{S}^*$, $\mathcal{C}$ and $\mathcal{K}$ are the classes of starlike, convex and close-to-convex functions, respectively. If f and g are two analytic functions, then the function f is said to be subordinate to g, written as $f(z)\prec g(z)$, if there exists a Schwarz function w with $w(0)=0$ and $|w|<1$ such that $f(z)=g(w(z))$. Furthermore, if the function g is univalent in $\mathcal{U}$, then we have the following equivalent relation:

$$f(z)\prec g(z)\Longleftrightarrow f(0)=g(0)\quad\text{and}\quad f(\mathcal{U})\subset g(\mathcal{U}).$$

The q-analogue of the normalized starlike functions was first introduced by Ismail *et al.* [7] with the help of q-difference operator D_q. The operator D_q applying on the analytic functions is defined by

$$(D_q f)(z)=\begin{cases}\frac{f(z)-f(qz)}{z(1-q)}, & z\in\mathcal{U}\backslash\{0\},\ q\in(0,1),\\ f'(0), & z=0,\ q\in(0,1).\end{cases}\tag{2}$$

We can easily observe from the definition of (2) that

$$\lim_{q\to1^-}\{(D_q f)(z)\}=f'(z),\qquad z\in\mathcal{U}.$$

By using q-difference operator D_q given by (2) the classes of q-starlike functions and q-close-to-convex functions are defined as follows:

A function $f\in\mathcal{A}$ is said to be in the class q-starlike functions $\mathcal{S}_q^*$ if

$$\left|\frac{z}{f(z)}(D_q f)(z)-\frac{1}{1-q}\right|\le\frac{1}{1-q},\qquad z\in\mathcal{U},\ q\in(0,1).\tag{3}$$

It is observed that, when $q\to1^-$, the function class $\mathcal{S}_q^*$ defined by (3) reduces to the normalized starlike functions class $\mathcal{S}^*$.

A function $f\in\mathcal{A}$ is said to be in the class q-close-to-convex functions $\mathcal{K}_q$ if there exists a function $h\in\mathcal{S}^*$ such that

$$\left|\frac{z}{h(z)}(D_q f)(z)-\frac{1}{1-q}\right|\le\frac{1}{1-q},\qquad z\in\mathcal{U},\ q\in(0,1).\tag{4}$$

It is observed that when $q \to 1^-$, the function class $\mathcal{K}_q$ defined by (4) reduces to the normalized close-to-convex functions class $\mathcal{K}$.

For functions $f, g \in \mathcal{A}$ given by (1) and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, respectively, the Hadamard product (or convolution) of these functions is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \qquad z \in \mathcal{U}.$$

After the solution of famous Bieberbach conjecture with the help of hypergeometric functions, some special functions have become very attractive for many mathematicians. The basic q-hypergeometric function was first introduced by Srivastava [14]. In the last three decades, some different geometric properties like univalence, starlikeness, convexity and close-to-convexity of many special functions were discussed by many authors. These geometric properties of different special functions can be found in the papers [2, 3, 4, 8, 11, 15, 12] and references therein. Also, for comprehensive informations about q-calculus we refer [5, 13].

Our main objective in the present paper is to investigate q-close-to-convexity of certain family of q-Bessel functions. Moreover, we deal with strong starlikeness and strong convexity of the mentioned functions. The motivation of this paper is due to the work of Srivastava and Bansal [16] and Raza and Din [10].

This paper is organized as follows: in the rest of this section we remember the definitions of Jackson's q-Bessel functions and define some families of q-Bessel functions by using Hadamard product. Also, we give the set of lemmas which will be needed in the proofs. In subsection 2.1 we deal with the q-close-to-convexity of some families of q-Bessel functions with respect to certain functions. In subsection 2.2 we investigate strong convexity and strong starlikeness of the mentioned functions.

The Jackson's second and third q-Bessel functions are defined by [1]

$$J_v^{(2)}(z;q) = \frac{\left(q^{v+1};q\right)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n+v}}{(q;q)_n \left(q^{v+1};q\right)_n} q^{n(n+v)} \tag{5}$$

and

$$J_v^{(3)}(z;q) = \frac{\left(q^{v+1};q\right)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+v}}{(q;q)_n \left(q^{v+1};q\right)_n} q^{\frac{1}{2}n(n+1)}, \tag{6}$$

where $q \in (0,1)$, $v > -1$, $z \in \mathbb{C}$ with conditions

$$(b;q)_0 = 1, \ \ (b;q)_n = \prod_{k=1}^{n} \left(1 - bq^{k-1}\right), \ \ (b;q)_\infty = \prod_{k=1}^{\infty} \left(1 - bq^{k-1}\right).$$

The functions defined by (5) and (6) do not belong to the class $\mathcal{A}$. We consider the following normalized forms of the Jackson's second and third q-Bessel functions.

$$\varphi_v^{(2)}(z;q) = 2^v c_v(q) z^{1-\frac{v}{2}} J_v^{(2)}(\sqrt{z};q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+v)}}{4^n (q;q)_n \left(q^{v+1};q\right)_n} z^{n+1} \tag{7}$$

and

$$\varphi_v^{(3)}(z;q) = c_v(q) z^{1-\frac{v}{2}} J_v^{(3)}(\sqrt{z};q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{\frac{1}{2}n(n+1)}}{(q;q)_n (q^{v+1};q)_n} z^{n+1}, \tag{8}$$

where $c_v(q) = \frac{(q;q)_\infty}{(q^{v+1};q)_\infty}$. As a result of the normalizations defined by (7) and (8), these functions belong to the class $\mathcal{A}$.

To discuss the q-close-to-convexity of certain families of q-Bessel functions with respect to the functions $\frac{z}{1-z}$ and $\frac{z}{1-z^2}$, we consider the following Hadamard products:

$$\mathcal{H}_v^{(2)}(z;q) = \frac{z}{1+z} * \varphi_v^{(2)}(z;q) = z + \sum_{n=2}^{\infty} \frac{q^{(n-1)(n-1+v)}}{4^{n-1} (q;q)_{n-1} (q^{v+1};q)_{n-1}} z^n$$

and

$$\mathcal{H}_v^{(3)}(z;q) = \frac{z}{1+z} * \varphi_v^{(3)}(z;q) = z + \sum_{n=2}^{\infty} \frac{q^{\frac{n(n-1)}{2}}}{(q;q)_{n-1} (q^{v+1};q)_{n-1}} z^n,$$

where $\frac{z}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^{n+1}$.

The following lemmas will be used in order to prove our main results.

Lemma 1. *[16] Let (A_n) be a sequence of real numbers such that*

$$B_n = \frac{A_n(1-q^n)}{1-q}, \quad \forall n \in \mathbb{N}, q \in (0,1).$$

Let

$$1 \geq B_1 \geq B_2 \geq B_3 \geq ... \geq B_n \geq ... \geq 0$$

or

$$1 \leq B_1 \leq B_2 \leq B_3 \leq ... \leq B_n \leq ... \leq 2.$$

Then

$$f(z) = z + \sum_{n=2}^{\infty} A_n z^n \in \mathcal{K}_q$$

with respect to $g(z) = \frac{z}{1-z}$.

Lemma 2. *[9] Let (A_n) be a sequence of real numbers such that*

$$B_n = \frac{A_n(1-q^n)}{1-q}, \quad \forall n \in \mathbb{N}, q \in (0,1).$$

Let

$$1 \geq B_3 \geq B_5 \geq B_5 \geq ... \geq B_{2n-1} \geq ... \geq 0$$

or

$$1 \leq B_3 \leq B_5 \leq B_5 \leq ... \leq B_{2n-1} \leq ... \leq 2.$$

Then

$$f(z) = z + \sum_{n=2}^{\infty} A_{2n-1} z^{2n-1} \in \mathcal{K}_q$$

with respect to $h(z) = \frac{z}{1-z^2}$.

Lemma 3. *[6] Let $M(z)$ be convex and univalent in the open unit disk with condition $M(0) = 1$. Let $F(z)$ be analytic in the open unit disk with condition $F(0) = 1$ and $F \prec M$ in the open unit disk. Then $\forall\ n \in \mathbb{N} \cup \{0\}$, we obtain*

$$(n+1)z^{-1-n}\int_0^z t^n F(t)dt \prec (n+1)z^{-1-n}\int_0^z t^n M(t)dt.$$

2 Main results

In this section we present our main results. This section is divided into two subsections.

2.1 q-Close-to-convexity of the functions $\mathcal{H}_v^{(2)}(z;q)$ and $\mathcal{H}_v^{(3)}(z;q)$

In this part, we present some sufficient conditions for the q-close-to-convexity of the functions $\mathcal{H}_v^{(2)}(z;q)$ and $\mathcal{H}_v^{(3)}(z;q)$.

Theorem 1. *Let $v \geq 0$ and $n \in \mathbb{N} = \{1, 2, \dots\}$.*

a. *If $q \in \left(0, \frac{1}{2}\right]$ and*

$$4(1-q^n)^2(1-q^{v+n}) \geq (1-q^{n+1})q^{2n+v-1}, \tag{9}$$

then the function $z \mapsto \mathcal{H}_v^{(2)}(z;q)$ is q-close-to-convex in the open unit disk with respect to $p(z) = \frac{z}{1-z}$.

b. *If $q \in \left(0, \frac{3}{10}\right]$ and*

$$(1-q^n)^2(1-q^{v+n}) \geq (1-q^{n+1})q^n, \tag{10}$$

then the function $z \mapsto \mathcal{H}_v^{(3)}(z;q)$ is q-close-to-convex in the open unit disk with respect to $p(z) = \frac{z}{1-z}$.

Proof.

a. Consider the function $z \to \mathcal{H}_v^{(2)}(z;q)$ as follows:

$$\mathcal{H}_v^{(2)}(z;q) = z + \sum_{n=2}^{\infty} H_n z^n,$$

where $H_n = \frac{q^{(n-1)(n-1+v)}}{4^{n-1}(q;q)_{n-1}(q^{v+1};q)_{n-1}}$. By using Lemma 1, the q-close-to-convexity of function $\mathcal{H}_v^{(2)}(z;q)$ can be process. If we take $n = 1$ in the inequality (9), we have

$$4(1-q)(1-q^{v+1}) \geq (1+q)q^{v+1}. \tag{11}$$

Now, we construct

$$B_n = \frac{(1-q^n)}{1-q}H_n, \quad \forall n \in \mathbb{N}, q \in \left(0, \frac{1}{2}\right].$$

It is easily observed that $B_1 = 1$ and all the values of B_n are positive for $n \in \mathbb{N}$. On the other hand, it is easily seen that

$$B_2 = \frac{(1+q)q^{v+1}}{4\left(1-q\right)\left(1-q^{v+1}\right)}$$

and from the inequality (11) we have $B_2 \leq 1$. Now, we would like to show that $B_{n+1} \leq B_n$ for all $n \geq 2$. For this purpose, consider

$$\frac{B_{n+1}}{B_n} = \frac{\frac{(1-q^{n+1})}{1-q}\frac{q^{n(n+v)}}{4^n(q;q)_n(q^{v+1};q)_n}}{\frac{(1-q^n)}{1-q}\frac{q^{(n-1)(n-1+v)}}{4^{n-1}(q;q)_{n-1}(q^{v+1};q)_{n-1}}} = \frac{1-q^{n+1}}{1-q^n}\frac{\frac{q^{n(n+v)}}{4\prod\limits_{k=1}^{n}(1-q^k)\prod\limits_{k=1}^{n}(1-q^{v+k})}}{\frac{q^{(n-1)(n-1+v)}}{\prod\limits_{k=1}^{n-1}(1-q^k)\prod\limits_{k=1}^{n-1}(1-q^{v+k})}}$$

$$= \frac{1-q^{n+1}}{1-q^n}\frac{q^{2n+v-1}}{4(1-q^n)(1-q^{v+n})} = \frac{\left(1-q^{n+1}\right)q^{2n+v-1}}{4(1-q^n)^2(1-q^{v+n})}.$$

The inequality (9) implies that $\frac{B_{n+1}}{B_n} \leq 1$, so the proof is completed.

b. In order to prove the q-close-to-convexity of function $\mathcal{H}_v^{(3)}(z;q)$, consider the function $z \to \mathcal{H}_v^{(3)}(z;q)$ as follows:

$$\mathcal{H}^{(3)}(z;q) = z + \sum_{n=2}^{\infty} K_n z^n,$$

where $K_n = \frac{q^{\frac{n(n-1)}{2}}}{(q;q)_{n-1}(q^{v+1};q)_{n-1}}$.

By making use of the Lemma 1, the q-close-to-convexity of the function $\mathcal{H}_v^{(3)}(z;q)$ can be shown. By taking $n = 1$ in the inequality (10) , we get

$$(1-q)(1-q^{v+1}) \geq (1+q)q. \tag{12}$$

Now, we construct

$$B_n = \frac{(1-q^n)}{1-q}K_n, \quad \forall n \in \mathbb{N}, q \in \left(0, \frac{3}{10}\right].$$

It is easily seen that $B_1 = 1$ and all the values of B_n are positive for $n \in \mathbb{N}$. On the other hand, it is easily obtained that

$$B_2 = \frac{(1+q)q}{\left(1-q\right)\left(1-q^{v+1}\right)}$$

and from the inequality (12) we have $B_2 \leq 1$. Now, we would like to show that

$B_{n+1} \leq B_n$ for all $n \geq 2$. For this purpose, consider

$$\frac{B_{n+1}}{B_n} = \frac{\frac{(1-q^{n+1})}{1-q} \frac{q^{\frac{n(n+1)}{2}}}{(q;q)_n (q^{v+1};q)_n}}{\frac{(1-q^n)}{1-q} \frac{q^{\frac{n(n-1)}{2}}}{(q;q)_{n-1} (q^{v+1};q)_{n-1}}} = \frac{1-q^{n+1}}{1-q^n} \frac{\frac{q^{\frac{n(n+1)}{2}}}{\prod_{k=1}^{n}(1-q^k) \prod_{k=1}^{n}(1-q^{v+k})}}{\frac{q^{\frac{n(n-1)}{2}}}{\prod_{k=1}^{n-1}(1-q^k) \prod_{k=1}^{n-1}(1-q^{v+k})}}$$

$$= \frac{1-q^{n+1}}{1-q^n} \frac{q^n}{(1-q^n)(1-q^{v+n})} = \frac{\left(1-q^{n+1}\right) q^n}{(1-q^n)^2 (1-q^{v+n})}.$$

The inequality (10) implies that $\frac{B_{n+1}}{B_n} \leq 1$, which is desired. □

Theorem 2. *Let $v \geq 0$ and $n \in \mathbb{N} = \{1, 2, \dots\}$.*

a. *If $q \in \left(0, \frac{3}{5}\right]$ and*

$$16(1-q^{2n})(1-q^{v+2n-1})(1-q^{v+2n})\left(1-q^{2n-1}\right)^2 \geq (1-q^{2n+1}) q^{2v+8n-4}, \quad (13)$$

then the function $z \mapsto \mathcal{H}_v^{(2)}(z;q)$ is q-close-to-convex in the open unit disk with respect to $k(z) = \frac{z}{1-z^2}$.

b. *If $q \in \left(0, \frac{2}{5}\right]$ and*

$$(1-q^{2n-1})^2 (1-q^{2n})(1-q^{v+2n-1})(1-q^{v+2n}) \geq (1-q^{2n+1}) q^{5n-1}, \quad (14)$$

then the function $z \mapsto \mathcal{H}_v^{(3)}(z;q)$ is q-close-to-convex in the open unit disk with respect to $k(z) = \frac{z}{1-z^2}$.

Proof.

a. By using Lemma 2 the q-close-to-convexity of the function $\mathcal{H}_v^{(2)}(z;q)$ with respect to $k(z) = \frac{z}{1-z^2}$ can be proven. If we take $n = 1$ in the inequality (13) we have

$$16(1-q)^2(1+q)(1-q^{v+1})(1-q^{v+2}) \geq (1+q+q^2) q^{2(v+2)}. \quad (15)$$

If we consider

$$B_n = \frac{(1-q^n)}{1-q} H_n, \quad \forall n \in \mathbb{N}, q \in \left(0, \frac{3}{5}\right],$$

then it is easily observed that $B_1 = 1$ and all the values of B_n are positive for $n \in \mathbb{N}$. On the other hand, it is easily seen that

$$B_3 = \frac{q^{2(v+2)}(1+q+q^2)}{16\,(1-q)\,(1-q^2)\,(1-q^{v+1})\,(1-q^{v+2})}$$

and from the inequality (15) we have $B_3 \leq 1$. Now, we would like to show that $B_{2n+1} \leq B_{2n-1}$ for all $n \geq 2$. For this purpose, consider

$$\frac{B_{2n+1}}{B_{2n-1}} = \frac{\frac{(1-q^{2n+1})}{1-q}\frac{q^{2n(2n+v)}}{4^{2n}(q;q)_{2n}(q^{v+1};q)_{2n}}}{\frac{(1-q^{2n-1})}{1-q}\frac{q^{(2n-2)(2n-2+v)}}{4^{2n-2}(q;q)_{2n-2}(q^{v+1};q)_{2n-2}}} = \frac{1-q^{2n+1}}{1-q^{2n-1}}\frac{\frac{q^{2n(2n+v)}}{16\prod_{k=1}^{2n}(1-q^k)\prod_{k=1}^{2n}(1-q^{v+k})}}{\frac{q^{(2n-2)(2n-2+v)}}{\prod_{k=1}^{2n-2}(1-q^k)\prod_{k=1}^{2n-2}(1-q^{v+k})}}$$
$$= \frac{1-q^{2n+1}}{1-q^{2n-1}}\frac{q^{2v+8n-4}}{16(1-q^{2n})(1-q^{v+2n-1})(1-q^{v+2n})\left(1-q^{2n-1}\right)}$$
$$= \frac{\left(1-q^{2n+1}\right)q^{2v+8n-4}}{16(1-q^{2n})(1-q^{v+2n-1})(1-q^{v+2n})\left(1-q^{2n-1}\right)^2}.$$

The inequality (13) implies that $\frac{B_{2n+1}}{B_{2n-1}} \leq 1$, so the proof is completed.

b. With the help of Lemma 2 we show that the function $\mathcal{H}_v^{(3)}(z;q)$ is q-close-to-convex in the open unit disk with respect to $k(z) = \frac{z}{1-z^2}$. If we take $n = 1$ in the inequality (13), we have

$$(1-q)^2(1+q)(1-q^{v+1})(1-q^{v+2}) \geq (1+q+q^2)q^3. \tag{16}$$

Now, we construct

$$B_n = \frac{(1-q^n)}{1-q}K_n, \quad \forall n \in \mathbb{N}, q \in \left(0, \frac{2}{5}\right].$$

It is easily observed that $B_1 = 1$ and all the values of B_n are positive for $n \in \mathbb{N}$. On the other hand, it is easily seen that

$$B_3 = \frac{(1+q+q^2)q^3}{(1-q)^2(1+q)(1-q^{v+1})(1-q^{v+2})}$$

and from the inequality (16) we have $B_3 \leq 1$. Now, we would like to show that $B_{2n+1} \leq B_{2n-1}$ for all $n \geq 2$. For this purpose, consider

$$\frac{B_{2n+1}}{B_{2n-1}} = \frac{\frac{(1-q^{2n+1})}{1-q}\frac{q^{n(2n+1)}}{(q;q)_{2n}(q^{v+1};q)_{2n}}}{\frac{(1-q^{2n-1})}{1-q}\frac{q^{(n-1)(2n-1)}}{(q;q)_{2n-2}(q^{v+1};q)_{2n-2}}} = \frac{1-q^{2n+1}}{1-q^{2n-1}}\frac{\frac{q^{n(2n+1)}}{\prod_{k=1}^{2n}(1-q^k)\prod_{k=1}^{2n}(1-q^{v+k})}}{\frac{q^{(n-1)(2n-1)}}{\prod_{k=1}^{2n-2}(1-q^k)\prod_{k=1}^{2n-2}(1-q^{v+k})}}$$
$$= \frac{1-q^{2n+1}}{1-q^{2n-1}}\frac{q^{5n-1}}{(1-q^{2n})(1-q^{v+2n-1})(1-q^{v+2n})\left(1-q^{2n-1}\right)}$$
$$= \frac{\left(1-q^{2n+1}\right)q^{5n-1}}{(1-q^{2n})(1-q^{v+2n-1})(1-q^{v+2n})\left(1-q^{2n-1}\right)^2}.$$

The inequality (14) implies that $\frac{B_{2n+1}}{B_{2n-1}} \leq 1$, which is desired. □

2.2 Strong convexity and strong starlikeness of the functions $\mathcal{H}_v^{(2)}(z;q)$ and $\mathcal{H}_v^{(3)}(z;q)$

In this section, we are mainly interested in finding some sufficient conditions for the functions $\mathcal{H}_v^{(2)}(z;q)$ and $\mathcal{H}_v^{(3)}(z;q)$ to belong to the function classes of strongly convex functions of order α and strongly starlike functions of order α, respectively.

Theorem 3. *Let $v>-1$ and $q\in(0,1)$.*

a. *If $(1-q)(1-q^v)-2q^v>0$, then the function $z\mapsto \mathcal{H}_v^{(2)}(z;q)\in \widetilde{\mathcal{C}}(\alpha)$, where*

$$\alpha=\frac{2}{\pi}\arcsin\left(\varkappa\sqrt{1-\frac{\varkappa^2}{4}}+\frac{\varkappa}{2}\sqrt{1-\varkappa^2}\right)\ \text{and}\ \varkappa=\frac{q^v}{(1-q)(1-q^v)-q^v}.$$

b. *If $\frac{q^v}{2\{(1-q)(1-q^v)-q^v\}}<1$, then the function $z\mapsto \mathcal{H}_v^{(2)}(z;q)\in \widetilde{\mathcal{S}^*}(\alpha)$, where*

$$\alpha=\frac{2}{\pi}\arcsin\left(\psi\sqrt{1-\frac{\psi^2}{4}}+\frac{\psi}{2}\sqrt{1-\psi^2}\right)\ \text{and}\ \psi=\frac{q^v}{2\{(1-q)(1-q^v)-q^v\}}.$$

Proof.

a. By using the well-known triangle inequality

$$|z_1+z_2|\le|z_1|+|z_2|,\qquad (z_1,z_2\in\mathbb{C}) \tag{17}$$

with the inequalities

$$(n+1)^2\le 4^n,\quad q^{n(n+v)}\le q^{nv},\quad (1-q)^n\le(q;q)_n,\quad (1-q^v)^n\le\left(q^{v+1};q\right)_n,$$

for $n\in\mathbb{N}$, we obtain

$$\begin{aligned}\left|\left(z\mathcal{H}_v'^{(2)}(z;q)\right)'-1\right| &\le \sum_{n=1}^{\infty}\frac{q^{n(n+v)}(n+1)^2}{4^n(q;q)_n(q^{v+1};q)_n}\\ &\le \frac{q^v}{(1-q)(1-q^v)}\sum_{n=1}^{\infty}\left(\frac{q^v}{(1-q)(1-q^v)}\right)^{n-1}\\ &=\frac{q^v}{(1-q)(1-q^v)-q^v}=\varkappa.\end{aligned} \tag{18}$$

From (18), we concluded that

$$\left(z\mathcal{H}_v'^{(2)}(z;q)\right)'\prec 1+\varkappa z\quad\Rightarrow\quad \left|\arg\left(z\mathcal{H}_v'^{(2)}(z;q)\right)'\right|<\arcsin\varkappa. \tag{19}$$

With the help of Lemma 3, take $n=0$ with $F(z)=\left(z\mathcal{H}_v'^{(2)}(z;q)\right)'$ and $M(z)=1+\varkappa z$, we get

$$\frac{z\mathcal{H}_v'^{(2)}(z;q)}{z}\prec 1+\frac{\varkappa}{2}z.$$

This implies that

$$\mathcal{H}_v^{\prime(2)}(z;q) \prec 1+\frac{\varkappa}{2}z.$$

As a result

$$\left|\arg \mathcal{H}_v^{\prime(2)}(z;q)\right| < \arcsin\frac{\varkappa}{2}. \tag{20}$$

By using (19) and (20), we obtain

$$\begin{aligned}\left|\arg\left(\frac{\left(z\mathcal{H}_v^{\prime(2)}(z;q)\right)^{\prime}}{\mathcal{H}_v^{\prime(2)}(z;q)}\right)\right| &= \left|\arg\left(z\mathcal{H}_v^{\prime(2)}(z;q)\right)^{\prime} - \arg\left(\mathcal{H}_v^{\prime(2)}(z;q)\right)\right| \\ &\le \left|\arg\left(z\mathcal{H}_v^{\prime(2)}(z;q)\right)^{\prime}\right| + \left|\arg\left(\mathcal{H}_v^{\prime(2)}(z;q)\right)\right| \\ &< \arcsin\frac{\varkappa}{2} + \arcsin\varkappa \\ &= \arcsin\left(\varkappa\sqrt{1-\frac{\varkappa^2}{4}} + \frac{\varkappa}{2}\sqrt{1-\varkappa^2}\right),\end{aligned}$$

which implies that $\mathcal{H}_v^{(2)} \in \widetilde{\mathcal{C}}(\alpha)$ for $\alpha = \frac{2}{\pi}\arcsin\left(\varkappa\sqrt{1-\frac{\varkappa^2}{4}} + \frac{\varkappa}{2}\sqrt{1-\varkappa^2}\right)$.

b. By using the well-known triangle inequality given by (17) with the inequalities

$$2(n+1) \le 4^n, \quad q^{n(n+v)} \le q^{nv}, \quad (1-q)^n \le (q;q)_n, \quad (1-q^v)^n \le \left(q^{v+1};q\right)_n,$$

for $n \in \mathbb{N}$, we can write that

$$\begin{aligned}\left|\mathcal{H}_v^{\prime(2)}(z;q) - 1\right| &\le \sum_{n=1}^{\infty} \frac{q^{n(n+v)}(n+1)^2}{4^n (q;q)_n \left(q^{v+1};q\right)_n} \\ &\le \frac{1}{2}\frac{q^v}{(1-q)(1-q^v)}\sum_{n=1}^{\infty}\left(\frac{q^v}{(1-q)(1-q^v)}\right)^{n-1} \\ &= \frac{q^v}{2\left\{(1-q)(1-q^v) - q^v\right\}} = \psi.\end{aligned} \tag{21}$$

From (21), we concluded that

$$\mathcal{H}_v^{\prime(2)}(z;q) \prec 1+\psi z \quad \Rightarrow \quad \left|\arg\left(\mathcal{H}_v^{\prime(2)}(z;q)\right)\right| < \arcsin\psi. \tag{22}$$

With the help of Lemma 3, take $n = 0$ with $F(z) = \mathcal{H}_v^{\prime(2)}(z;q)$ and $M(z) = 1+\psi z$, we get

$$\frac{\mathcal{H}_v^{(2)}(z;q)}{z} \prec 1+\frac{\psi}{2}z.$$

As a result

$$\left|\arg\left(\frac{\mathcal{H}_v^{(2)}(z;q)}{z}\right)\right| < \arcsin\frac{\psi}{2}. \tag{23}$$

By using (22) and (23), we obtain

$$\begin{aligned}\left|\arg\left(\frac{z\mathcal{H}_v'^{(2)}(z;q)}{\mathcal{H}_v^{(2)}(z;q)}\right)\right| &= \left|\arg\left(\frac{z}{\mathcal{H}_v^{(2)}(z;q)}\right) - \arg\left(\mathcal{H}_v'^{(2)}(z;q)\right)\right| \\ &\leq \left|\arg\left(\frac{z}{\mathcal{H}_v^{(2)}(z;q)}\right)\right| + \left|\arg\left(\mathcal{H}_v'^{(2)}(z;q)\right)\right| \\ &< \arcsin\frac{\psi}{2} + \arcsin\psi \\ &= \arcsin\left(\psi\sqrt{1-\frac{\psi^2}{4}} + \frac{\psi}{2}\sqrt{1-\psi^2}\right),\end{aligned}$$

which implies that $\mathcal{H}_v^{(2)} \in \widetilde{\mathcal{S}}^*(\alpha)$ for $\alpha = \frac{2}{\pi}\arcsin\left(\psi\sqrt{1-\frac{\psi^2}{4}} + \frac{\psi}{2}\sqrt{1-\psi^2}\right)$. □

Theorem 4. *Let $v > -1$, $q \in (0,1)$.*

a. *If $(1-q)(1-q^v) - 8q^{\frac{1}{2}} > 0$, then $\mathcal{H}_v^{(3)}(z;q) \in \widetilde{\mathcal{C}}(\alpha)$, where*

$$\alpha = \frac{2}{\pi}\arcsin\left(\kappa\sqrt{1-\frac{\kappa^2}{4}} + \frac{\kappa}{2}\sqrt{1-\kappa^2}\right), \tag{24}$$

and $\kappa = \frac{4q^{\frac{1}{2}}}{(1-q)(1-q^v)-4q^{\frac{1}{2}}}$.

b. *If $(1-q)(1-q^v) - 4q^{\frac{1}{2}} > 0$, then $\mathcal{H}_v^{(3)}(z;q) \in \widetilde{\mathcal{S}}^*(\alpha)$, where*

$$\alpha = \frac{2}{\pi}\arcsin\left(\mu\sqrt{1-\frac{\mu^2}{4}} + \frac{\psi_1}{2}\sqrt{1-\psi_1^2}\right), \tag{25}$$

and $\mu = \frac{2q^{\frac{1}{2}}}{(1-q)(1-q^v)-2q^{\frac{1}{2}}}$.

Proof.

a. By using the well-known triangle inequality given by (17) with the inequalities

$$(n+1)^2 \leq 4^n, \quad q^{\frac{n(n+1)}{2}} \leq q^{\frac{n}{2}}, \quad (1-q)^n \leq (q;q)_n, \quad (1-q^v)^n \leq \left(q^{v+1};q\right)_n,$$

for $n \in \mathbb{N}$, we obtain

$$\begin{aligned}\left|\left(z\mathcal{H}_v'^{(3)}(z;q)\right)' - 1\right| &\leq \sum_{n=1}^{\infty}\frac{q^{\frac{n(n+1)}{2}}(n+1)^2}{(q;q)_n\,(q^{v+1};q)_n} \\ &\leq \frac{4q^{\frac{1}{2}}}{(1-q)(1-q^v)}\sum_{n=1}^{\infty}\left(\frac{4q^{\frac{1}{2}}}{(1-q)(1-q^v)}\right)^{n-1} \\ &= \frac{4q^{\frac{1}{2}}}{(1-q)(1-q^v)-4q^{\frac{1}{2}}} = \kappa.\end{aligned} \tag{26}$$

From (18), we concluded that

$$\left(z\mathcal{H}_v'^{(3)}(z;q)\right)' \prec 1+\kappa z \Rightarrow \left|\arg\left(z\mathcal{H}_v'^{(3)}(z;q)\right)'\right| < \arcsin\kappa. \tag{27}$$

With the help of Lemma 3, take $n=0$ with $F(z)=\left(z\mathcal{H}_v'^{(3)}(z;q)\right)'$ and $M(z)=1+\kappa z$, we get

$$\frac{z\mathcal{H}_v'^{(3)}(z;q)}{z} \prec 1+\frac{\kappa}{2}z.$$

This implies that

$$\mathcal{H}_v'^{(3)}(z;q) \prec 1+\frac{\kappa}{2}z.$$

As a result

$$\left|\arg \mathcal{H}_v'^{(3)}(z;q)\right| < \arcsin\frac{\kappa}{2}. \tag{28}$$

By using (27) and (28), we obtain

$$\begin{aligned}\left|\arg\left(\frac{\left(z\mathcal{H}_v'^{(3)}(z;q)\right)'}{\mathcal{H}_v'^{(3)}(z;q)}\right)\right| &= \left|\arg\left(z\mathcal{H}_v'^{(3)}(z;q)\right)' - \arg\left(\mathcal{H}_v'^{(3)}(z;q)\right)\right| \\ &\leq \left|\arg\left(z\mathcal{H}_v'^{(3)}(z;q)\right)'\right| + \left|\arg\left(\mathcal{H}_v'^{(3)}(z;q)\right)\right| \\ &< \arcsin\frac{\kappa}{2} + \arcsin\kappa \\ &= \arcsin\left(\kappa\sqrt{1-\frac{\kappa^2}{4}}+\frac{\kappa}{2}\sqrt{1-\kappa^2}\right),\end{aligned}$$

which implies that $\mathcal{H}_v^{(3)} \in \widetilde{\mathcal{C}}(\alpha)$ for $\alpha = \frac{2}{\pi}\arcsin\left(\kappa\sqrt{1-\frac{\kappa^2}{4}}+\frac{\kappa}{2}\sqrt{1-\kappa^2}\right)$.

b. By using the well-known triangle inequality given by (17) with the inequalities

$$(n+1)\leq 2^n,\quad q^{\frac{n(n+1)}{2}} \leq q^{\frac{n}{2}},\quad (1-q)^n \leq (q;q)_n,\quad (1-q^v)^n \leq \left(q^{v+1};q\right)_n,$$

for $n\in\mathbb{N}$, we obtain

$$\begin{aligned}\left|\mathcal{H}_v'^{(3)}(z;q)-1\right| &\leq \sum_{n=1}^{\infty}\frac{q^{\frac{n(n+1)}{2}}(n+1)}{(q;q)_n\left(q^{v+1};q\right)_n} \\ &\leq \frac{2q^{\frac{1}{2}}}{(1-q)(1-q^v)}\sum_{n=1}^{\infty}\left(\frac{2q^{\frac{1}{2}}}{(1-q)(1-q^v)}\right)^{n-1} \\ &= \frac{2q^{\frac{1}{2}}}{(1-q)(1-q^v)-2q^{\frac{1}{2}}} = \mu.\end{aligned} \tag{29}$$

From (29) we conclude that

$$\mathcal{H}_v'^{(3)}(z;q) \prec 1+\mu z \Rightarrow \left|\arg\left(\mathcal{H}_v'^{(3)}(z;q)\right)\right| < \arcsin\mu. \tag{30}$$

With the help of Lemma 3, take $n=0$ with $F(z)=\mathcal{H}_v'^{(3)}(z;q)$ and $M(z)=1+\mu z$, we get

$$\frac{\mathcal{H}_v^{(3)}(z;q)}{z} \prec 1+\frac{\mu}{2}z.$$

As a result

$$\left|\arg\left(\frac{\mathcal{H}_v^{(3)}(z;q)}{z}\right)\right| < \arcsin\frac{\mu}{2}. \tag{31}$$

By using (30) and (31), we obtain

$$\begin{aligned}\left|\arg\left(\frac{z\mathcal{H}_v'^{(3)}(z;q)}{\mathcal{H}_v^{(3)}(z;q)}\right)\right| &= \left|\arg\left(\frac{z}{\mathcal{H}_v^{(3)}(z;q)}\right) - \arg\left(\mathcal{H}_v'^{(3)}(z;q)\right)\right| \\ &\le \left|\arg\left(\frac{z}{\mathcal{H}_v^{(3)}(z;q)}\right)\right| + \left|\arg\left(\mathcal{H}_v'^{(3)}(z;q)\right)\right| \\ &< \arcsin\frac{\mu}{2} + \arcsin\mu \\ &= \arcsin\left(\mu\sqrt{1-\frac{\mu^2}{4}} + \frac{\mu}{2}\sqrt{1-\mu^2}\right),\end{aligned}$$

which implies that $\mathcal{H}_v^{(3)} \in \widetilde{\mathcal{S}^*}(\alpha)$ for $\alpha = \frac{2}{\pi}\arcsin\left(\mu\sqrt{1-\frac{\mu^2}{4}}+\frac{\mu}{2}\sqrt{1-\mu^2}\right)$. □

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