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MORE ON (ω)HYPERCONNECTEDNESS

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Abstract

In this article, some results are established on hyperconnectedness in an (ω)topological space. The notion of (ω)dense spaces is introduced. The notion of (ω)mildly-separated sets which is somewhat unique to an (ω)topological space is used in the paper.

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1 Introduction

A set X equipped with a countable number of topologies $\mathcal{J}_n$ satisfying $\mathcal{J}_n \subset \mathcal{J}_{n+1}$, $n \in N$ is called an (ω)topological space (Bose and Tiwari [1], [2]) and is denoted by $(X, \{\mathcal{J}_n\})$, or simply by X. The notion of an (ω)topological space could have possible applications in infinite neural networks and other infinite networks as elucidated in Bose and Tiwari [3]. In [3], we have studied the notions of connectedness and hyperconnectedness in an (ω)topological space. In this paper, we establish some more results on (ω)hyperconnectedness analogous to the results obtained by Sharma [5] for hyperconnectedness in a topological space. We also introduce and study the notion of (ω)dense spaces, analogous to the notion of dense spaces introduced by Levine [4].

In Bose and Tiwari [3], we have studied the notions of connectedness (Definition 2) and hyperconnectedness (Definition 3) in an (ω)topological space. In this paper, some more results on (ω)hyperconnectedness analogous to the results obtained by Sharma [5] for hyperconnectedness in a topological space are established. The notion of (ω)dense spaces (Definition 13), analogous to the notion of dense spaces introduced by Levine [4] is defined. The notion of (ω)mildly-separated sets (Definition 12) has been used which is somewhat unique to an (ω)topological space.

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2 Preliminaries

Throughout the paper, unless mentioned otherwise, X denotes the (ω)topological space $(X, \{\mathfrak{J}_n\})$. For ready reference definitions of some terms which have been used in the sequel are provided.

Definition 1. *(Bose and Tiwari [1]) For an (ω)topological space $(X, \{\mathfrak{J}_n\})$, a set $G \in \cup_n \mathfrak{J}_n$ is called an (ω)open set. A set F is (ω)closed if its complement $F^c = X - F$ is (ω)open. For any set $E \subset X$, $(\omega)clE$ denotes the intersection of all (ω)closed sets which contain E.*

Definition 2. *(Bose and Tiwari [3]) An (ω)topological space $(X, \{\mathfrak{J}_n\})$ is said to be (ω)connected, if X cannot be expressed as the union of two disjoint nonempty (ω)open sets.*

Definition 3. *(Bose and Tiwari [3]) X is said to be (ω)hyperconnected if for any two nonempty (ω)open sets U and V, $U \cap V \neq \emptyset$.*

Definition 4. *(Bose and Tiwari [3]) A set $A \subset X$ is said to be (ω)semiopen if there exists an n such that for some $U \in \mathfrak{J}_n$, we have $U \subset A \subset (\omega)clU$. In particular we write A is $(\mathfrak{J}_n - \omega)$semiopen, since $U \in \mathfrak{J}_n$.*

Definition 5. *(Bose and Tiwari [1]) A set $E \subset X$ is said to be (ω)dense if $(\omega)clE = X$.*

Definition 6. *(Bose and Tiwari [3]) X is said to be submaximal if every (ω)dense subset of X is (ω)open.*

Definition 7. *(Singal and Arya [6]) An subset A of a topological space is said to be regularly-open if it is the interior of its own closure or, equivalently, if it is the interior of some closed set. A set F is said to be regularly-closed if it is the closure of its own interior or, equivalently, it is the closure of some open set.*

Definition 8. *(Tiwari and Bose [7]) If an (ω)open set G is regularly $(\mathfrak{J}_n)$open for some n, then it is said to be a* regularly (ω)open *set. An (ω)closed set F is said to be* regularly (ω)closed *if it is regularly $(\mathfrak{J}_n)$closed for some n.*

3 Some results on (ω)hyperconnectedness

The following definitions are being introduced.

Definition 9. *The complement of a (ω)semiopen set is said to be (ω)semiclosed. The set of all (ω)semiclosed sets is denoted by $SF_\omega(X, \{\mathfrak{J}_n\})$ or simply, $SF_\omega(X)$. A set whose complement is $(\mathfrak{J}_n - \omega)$semiopen is said to be $(\mathfrak{J}_n - \omega)$semiclosed. The set of all $(\mathfrak{J}_n - \omega)$semiclosed sets is denoted by $(\mathfrak{J}_n)SF_\omega(X)$.*

Definition 10. *For any $n \in N$, the smallest $(\mathfrak{J}_n - \omega)$semiclosed set containing a set A is called the $(\mathfrak{J}_n - \omega)$semiclosure of A and is denoted by $(\mathfrak{J}_n)A^c$. $(\omega)A^c$ denotes the intersection of all (ω)semiclosed sets containing A.*

It may or may not be an (ω)semiclosed set.

Definition 11. *Two subsets A and B of X, are said to be (ω)semi-separated if for some $n \in N$, we have $(\mathfrak{J}_n)A^c \cap B = \emptyset$ and $A \cap (\mathfrak{J}_n)B^c = \emptyset$.*

Definition 12. *Two subsets A and B of X, are said to be (ω)mildly-separated if for some $n \in N$, we have $(\mathfrak{J}_n)A^c \cap B = \emptyset$ or $A \cap (\mathfrak{J}_n)B^c = \emptyset$.*

It is obvious that a (ω)semi-separated space is (ω)mildly-separated, but the converse may not be true.

Definition 13. *An (ω)topological space X is said to be an (ω)dense space, or simply an $(\omega)D-$space if every non-empty (ω)open set is (ω)dense in X.*

Example 1. *Let us consider the (ω)topological space $(X, \{\mathfrak{J}_n\})$ where $X = N$ and the topological space $\mathfrak{J}_n$ is defined as follows:*

$$\begin{aligned} \mathfrak{J}_1 &= \{\emptyset\} \cup \{A | X - A \subset \{1\}\} \\ \mathfrak{J}_2 &= \{\emptyset\} \cup \{A | X - A \subset \{1, 2\}\} \text{ and in general,} \\ \mathfrak{J}_n &= \{\emptyset\} \cup \{A | X - A \subset \{1, 2, ..., n\}\} \text{ for all } n. \end{aligned}$$

Then X is an $(\omega)D-$space but it is not (ω)submaximal.

The following example shows that unlike in case of a topological space, the subspace of an $(\omega)D-$space is not always an $(\omega)D-$space.

Example 2. *Let us consider the (ω)topological space defined in Example 1. Considering the subspace $Y = \{1, 2\}$ of the (ω)topological space X and the set $B = \{1\}$, we see that B is open in Y, as $B = Y \cap U$, where $U = X - \{2\}$ is (ω)open in X. Similarly the set $\{2\}$ is (ω)open in Y. Then B is closed in Y and so the $(\omega|Y)clB = B$, $(\omega|Y)clB$ denotes the intersection of all closed subsets of subspace Y which contain the set B. Thus Y is not an $(\omega)D-$space.*

Theorem 1. *If X is an $(\omega)D-$space and Y is an (ω)open subset of X, then the subspace Y is also an $(\omega)D-$space.*

The proof is omitted.

Theorem 2. *The following statements are equivalent:*

- (i) *X is an $(\omega)D-$space.*
- (ii) *X is (ω)hyperconnected.*
- (iii) *Every (ω)open set in X is (ω)connected.*
- (iv) *X does not contain two non-empty disjoint (ω)semiopen sets.*
- (v) *The only subsets of X which are both (ω)semiopen and (ω)semiclosed in X are empty set and X itself.*

(vi) X cannot be expressed as the union of two non-empty disjoint sets where one is (ω)semiopen and the other is (ω)semiclosed.

(vii) X cannot be expressed as the union of two non-empty (ω)mildly-separated sets.

Proof. $(i) \Leftrightarrow (ii)$: Obvious

$(ii) \Rightarrow (iii)$: Let E be an (ω)open set which is not connected then there exist two (ω)open sets A', B' such that $A' = A \cap E$ and $B' = B \cap E$, for some (ω)open sets A, B, such that $E = A' \cup B'$ with $A' \cap B' = \emptyset$. This contradicts (ii).

$(iii) \Rightarrow (ii)$: Obvious

$(ii) \Rightarrow (iv)$: Let if possible, S_1, S_2 are two non-empty disjoint (ω)semiopen sets. Then there exist for some n, two sets $U_1, U_2 \in \mathfrak{J}_n$, such that $U_1 \subset S_1 \subset (\omega)clU_1$, and $U_2 \subset S_2 \subset (\omega)clU_2$. Now as X is (ω)hyperconnected, $U_1 \cap U_2 \neq \emptyset$. Hence we get $S_1 \cap S_2 \neq \emptyset$.

$(iv) \Rightarrow (v)$: Let if possible, S is a set which is both (ω)semiopen and (ω)semiclosed. Then $X - S$ is also both (ω)semiopen and (ω)semiclosed, contradicting (iv).

$(v) \Rightarrow (vi)$: Let if possible $X = A \cup B$, with $A \cap B = \emptyset$ where A is a non-empty (ω)semiopen and B is a non-empty (ω)semiclosed. Then obviously A and B are both (ω)semiopen and (ω)semiclosed sets, contradicting (v).

$(vi) \Rightarrow (vii)$: Let if possible, X can be expressed as the union of two (ω)mildly-separated sets, say $X = A \cup B$, with $(\mathfrak{J}_n)A^c \cap B = \emptyset$, for some n. Then clearly $X = (\mathfrak{J}_n)A^c \cup B$, where $(\mathfrak{J}_n)A^c$ is (ω)semiclosed and $B = X - (\mathfrak{J}_n)A^c$ is (ω)semiopen. This contradicts (vi).

$(vii) \Rightarrow (i)$: Let if possible O be an (ω)open set such that $(\omega)clO \neq X$. Then $(\omega)O^c \neq X$. Now $(\omega)O^c$ is (ω)semiopen as $O \subset (\omega)O^c \subset (\omega)clO$. Hence, $X - (\omega)O^c$ is (ω)semiclosed. Hence it is easy to see that X is (ω)mildly-separated, contradicting (vii). The proof is now complete. □

Theorem 3. *If X is an $(\omega)D-$space, then X cannot be expressed as the union of two non-empty (ω)semi-separated sets.*

Proof. From the previous theorem we know that in an $(\omega)D-$space, the only subsets of X which are both (ω)semiopen and (ω)semiclosed are empty set and X itself. Let if possible, $X = A \cup B$, where A and B are non-empty (ω)semi-separated sets, then for some $n \in N$, we have $(\mathfrak{J}_n)A^c \cap B = \emptyset$ and $A \cap (\mathfrak{J}_n)B^c = \emptyset$. It is then obvious that $(\mathfrak{J}_n)A^c = A$. Then A is (ω)semiclosed and B is (ω)semiopen. Similarly, we get B is (ω)semiclosed and A is (ω)semiopen, thus we arrive at a contradiction and so the result follows. □

The converse of the theorem fails to hold as we see by the following example. We recall that every topological space can be considered to be a (ω)topological space in which all constituent topologies are equal.

Example 3. *Let us consider the topological space $(N, \mathfrak{J})$ defined as follows:*

$$\mathfrak{J} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, N\}.$$

Then $\mathfrak{J}$ is not (ω)hyperconnected, however it is easy to see that N cannot be expressed as the union of two (ω)semi-separated sets.

Lemma 1. *If A is a (ω)semiopen set in X, and $A \subset B \subset (\omega)clA$, then B is also (ω)semiopen.*

The proof is omitted.

Theorem 4. *Let X be (ω)hyperconnected and let A be a non-empty (ω)semiopen subset of X, then $(\omega)A^c = X$.*

Proof. Since X is (ω)hyperconnected so by Theorem 2, $(\omega)clA = X$. For any n, we have $A \subset (\mathfrak{J}_n)A^c \subset (\omega)clA$. By Lemma 1 we see that $(\mathfrak{J}_n)A^c$ is (ω)semiopen. Then by Theorem 2(v) again it follows that $(\mathfrak{J}_n)A^c = X$. Since $(\mathfrak{J}_n)A^c = X$, for all n, $(\omega)A^c = X$. □

Theorem 5. *If X is (ω)hyperconnected, then every non-empty (ω)open subset of X, is also (ω)hyperconnected.*

The proof is obvious.

Theorem 6. *If X is (ω)hyperconnected, then X cannot be expressed as the union of a finite number of (ω)closed sets.*

Proof. Let F_1, F_2 be two (ω)closed sets, then $X - F_1, X - F_2$ are two (ω)open sets. As X is (ω)hyperconnected we have,

$$\emptyset \neq (X - F_1) \cap (X - F_2) = X - F_1 \cup F_2.$$

Then clearly $F_1 \cup F_2 \neq X$. The proof for a finite number of sets now follows by induction. □

Theorem 7. *If A is a non-empty subset of an (ω)hyperconnected space X, then for any n, $(\mathfrak{J}_n)int((\mathfrak{J}_n)clA)$ and $(\mathfrak{J}_n)cl((\mathfrak{J}_n)intA) = X$ or $\emptyset$.*

Proof. Since X is (ω)hyperconnected, so for any n we have

$$(\mathfrak{J}_n)intA \subset (\mathfrak{J}_n)cl((\mathfrak{J}_n)intA)) \subset (\omega)cl((\mathfrak{J}_n)intA))$$

and hence $(\mathfrak{J}_n)cl((\mathfrak{J}_n)intA))$ is (ω)semiopen. Also as $(\mathfrak{J}_n)cl((\mathfrak{J}_n)intA))$ is (ω)closed, it is also (ω)semiclosed. Then by Theorem 2, we see that $(\mathfrak{J}_n)cl((\mathfrak{J}_n)intA)) = X$ or $\emptyset$. Similarly for $(\mathfrak{J}_n)int((\mathfrak{J}_n)clA)$. □

Corollary 1. *If A is any regularly (ω)open or regularly (ω)closed set in a (ω)hyperconnected space X, then $A = X$ or $\emptyset$.*

Proof. The proof follows from Theorem 7. □

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