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A CERTAIN CLASS OF ANALYTIC FUNCTIONS ASSOCIATED WITH A DIFFERENTIAL OPERATOR

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Abstract

For $0 \leq \mu \leq \lambda,\ 0 \leq \alpha < 1,\ -\pi/2 < \beta < \pi/2$ and $m \in \mathbb{N} \cup \{0\}$, a new class $R^m(\lambda, \mu, \alpha, \beta)$ of analytic functions defined by means of the differential operator $D^m_{\lambda\mu}$ is introduced. Basic properties of the class $R^m(\lambda, \mu, \alpha, \beta)$ are investigated. Connections with previous known results are also pointed out.

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1 Introduction

Let $\mathcal{H}$ be the class of analytic functions in the unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. Denote by $\mathcal{A}$ the class of functions f in $\mathcal{H}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad z \in \mathcal{U}. \tag{1}$$

Let $\mathcal{R}$ denote the family of functions $f \in \mathcal{A}$ which satisfy the condition

$$\Re(f'(z) + zf''(z)) > 0, \quad z \in \mathcal{U}. \tag{2}$$

The class $\mathcal{R}$ was introduced and investigated by P. N. Chichra [4] and R. Sing and S. Sing [12].

Later, H. Silverman [11] investigated the class $\mathcal{R}(\alpha)$ $(0 \leq \alpha < 1)$ of all functions $f \in \mathcal{A}$ which satisfy the inequality

$$\Re(f'(z) + zf''(z)) > \alpha, \quad z \in \mathcal{U}. \tag{3}$$

In [4], [11] and [12] lower bounds for $\Re f'(z)$ and $\Re \dfrac{f(z)}{z}$ were obtained for functions belonging to the classes $\mathcal{R}$ and $\mathcal{R}(\alpha)$ respectively.

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Let $\mathcal{P}_{\alpha,\beta}$ be the class of functions $p \in \mathcal{H}$ with $p(0) = 1$ such that

$$\Re(e^{i\beta}p(z)) > \alpha \cos \beta, \quad z \in \mathcal{U}. \tag{4}$$

Here and through the rest of the paper we suppose that α, β are real numbers with $0 \leq \alpha < 1$ and $|\beta| < \frac{\pi}{2}$.

Note that for $\alpha = \beta = 0$ the class $\mathcal{P}_{\alpha,\beta}$ reduces to the well known Carathéodory class of functions

$$\mathcal{P} = \{p \in \mathcal{H},\ p(0) = 1 \text{ and } \Re p(z) > 0\}.$$

It is easy to see that a function $p \in \mathcal{H}$ belongs to the class $\mathcal{P}_{\alpha,\beta}$ if and only if

$$\frac{e^{i\beta}p(z) - (\alpha \cos \beta + i \sin \beta)}{(1-\alpha)\cos \beta} \in \mathcal{P}. \tag{5}$$

The function

$$p_{\alpha,\beta}(z) = \frac{1 + e^{-i\beta}(e^{-i\beta} - 2\alpha \cos \beta)z}{1 - z}, \quad z \in \mathcal{U}. \tag{6}$$

maps the open unit disk onto the half-plane $H_{\alpha,\beta} = \left\{z \in \mathbb{C} : \Re(e^{i\beta}z) > \alpha \cos \beta)\right\}$. If

$$p_{\alpha,\beta}(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \tag{7}$$

then

$$p_n = 2e^{-i\beta}(1-\alpha)\cos \beta, \ \ n \geq 1. \tag{8}$$

Herglotz' representation formula for the class $\mathcal{P}$ (see [6]) together with (5) shows that a function $p \in \mathcal{H}$ belongs to the class $\mathcal{P}_{\alpha,\beta}$ if and only if there exists a Borel probability measure μ on the unit circle $T = \{x \in \mathbb{C} : |x| = 1\}$ such that

$$p(z) = \int_{|x|=1} \frac{1 + e^{-i\beta}(e^{-i\beta} - 2\alpha \cos \beta)xz}{1 - xz} d\mu(x), \quad z \in \mathcal{U}. \tag{9}$$

If $f \in \mathcal{A}$ is given by (1.1) and $g \in \mathcal{A}$ is given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

then the Hadamard product (or convolution) of the functions f and g is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z), \quad z \in \mathcal{U}.$$

For a function $f \in \mathcal{A}$ we consider the following differential operator introduced by Răducanu and Orhan in [8]:

$$\begin{aligned} D^0_{\lambda\mu}f(z) &= f(z) \\ D^1_{\lambda\mu}f(z) &= D_{\lambda\mu}f(z) = \lambda\mu z^2 f''(z) + (\lambda-\mu)zf'(z) + (1-\lambda+\mu)f(z) \\ D^m_{\lambda\mu}f(z) &= D_{\lambda\mu}\left(D^{m-1}_{\lambda\mu}f(z)\right) \end{aligned} \tag{10}$$

where $0 \leq \mu \leq \lambda$ and $m \in \mathbb{N} := \{1, 2, \ldots\}$.

Note that, if $f \in \mathcal{A}$ is given by (1), then

$$D^m_{\lambda\mu}f(z) = z + \sum_{n=2}^{\infty} A_n(\lambda, \mu, m)a_n z^n \tag{11}$$

where

$$A_n(\lambda, \mu, m) = [1 + (\lambda\mu n + \lambda - \mu)(n-1)]^m \quad n \geq 2. \tag{12}$$

It should be remarked that the operator $D^m_{\lambda\mu}$ generalizes two other differential operators considered earlier:

(i) $D^m_{10}f(z) = D^m f(z)$, the operator introduced by Sălăgean in [10]

(ii) $D^m_{\lambda 0}f(z) = D^m_{\lambda}f(z)$, the operator studied by Al-Oboudi in [1].

In view of (11) the operator $D^m_{\lambda\mu}f(z)$ can be written in terms of convolution as

$$D^m_{\lambda\mu}f(z) = (f * g_{\lambda\mu})(z), \quad z \in \mathcal{U} \tag{13}$$

where

$$g_{\lambda\mu}(z) = z + \sum_{n=2}^{\infty} A_n(\lambda, \mu, m)z^n, \quad z \in \mathcal{U}. \tag{14}$$

Define the function $g^{(-1)}_{\lambda\mu}(z)$ such that

$$\left(g^{(-1)}_{\lambda\mu} * g_{\lambda\mu}\right)(z) = \frac{z}{1-z}, \quad z \in \mathcal{U}. \tag{15}$$

It is easy to observe that

$$f(z) = \left(g^{(-1)}_{\lambda\mu} * D^m_{\lambda\mu}f\right)(z), \quad z \in \mathcal{U}. \tag{16}$$

Making use of the differential operator $D^m_{\lambda\mu}f$, we define the following class of functions.

Definition 1. *We say that a function $f \in \mathcal{A}$ is in the class $R^m(\lambda, \mu, \alpha, \beta)$ if $(D^m_{\lambda\mu}f(z))' + z(D^m_{\lambda\mu}f(z))'' \in \mathcal{P}_{\alpha,\beta}$, that is*

$$\Re\left\{e^{i\beta}\left[(D^m_{\lambda\mu}f(z))' + z(D^m_{\lambda\mu}f(z))''\right]\right\} > \alpha\cos\beta \tag{17}$$

for $0 \leq \alpha < 1$, $\beta \in \mathbb{R}$ with $|\beta| < \dfrac{\pi}{2}$, $0 \leq \mu \leq \lambda$ and $m \in \mathbb{N} \cup \{0\}$.

The class $R^m(\lambda,\mu,\alpha,\beta)$ contains as particular cases the following classes of functions:

(i) $R^0(\lambda,\mu,0,0)=\mathcal{R}$, the class investigated by P. N. Chichra in [4] and R. Sing and S. Sing in [12].

(ii) $R^0(\lambda,\mu,\alpha,0)=\mathcal{R}(\alpha)$, the class studied by Silverman in [11].

In this paper we investigate some properties of the class $R^m(\lambda,\mu,\alpha,\beta)$. In particular, for this class, we derive inclusion results, membership characterization, integral formula, coefficient estimates and also convolution property. Connections with previous known results are also pointed out.

2 Inclusion results

In order to obtain our results, we shall need the following two lemmas.

Lemma 1. *([5], [9]) Let $\{c_n\}_{n=1}^{\infty}$ be a convex decreasing sequence, i.e*

$$c_n-2c_{n+1}+c_{n+2}\geq 0 \ \textit{and} \ c_{n+1}-c_{n+2}\geq 0, \quad n\in\mathbb{N}.$$

Then

$$\Re\left\{\sum_{n=1}^{\infty}c_n z^{n-1}\right\}>\frac{1}{2}, \quad z\in\mathcal{U}.$$

The next lemma follows from Herglotz' representation formula for the class $\mathcal{P}$ (see [6]).

Lemma 2. *Let $P(z)$ be analytic in $\mathcal{U}$ with $P(0)=1$ and $\Re P(z)>\frac{1}{2}$ in $\mathcal{U}$. Then, for any analytic function F in $\mathcal{U}$, the function $F*P$ takes values in the convex hull of $F(\mathcal{U})$.*

Theorem 1. *Let $\lambda\geq 0$ and $\mu\geq 0$ such that $\lambda\geq\mu+1$. Then*

$$R^{m+1}(\lambda,\mu,\alpha,\beta)\subset R^m(\lambda,\mu,\alpha,\beta), \quad m\in\mathbb{N}\cup\{0\}.$$

Proof. Let f given by (1) be in $R^{m+1}(\lambda,\mu,\alpha,\beta)$. It follows that

$$\Re\left\{e^{i\beta}\left[(D_{\lambda\mu}^{m+1}f(z))'+z(D_{\lambda\mu}^{m+1}f(z))''\right]\right\}>\alpha\cos\beta$$

or, making use of (11) and (12)

$$\Re\left\{e^{i\beta}\left[1+\sum_{n=2}^{\infty}n^2[1+(\lambda\mu n+\lambda-\mu)(n-1)]^{m+1}a_n z^{n-1}\right]\right\}>\alpha\cos\beta, \quad z\in\mathcal{U}.$$

We have

$$\begin{aligned}(D^m_{\lambda\mu}f(z))' + z(D^m_{\lambda\mu}f(z))'' &= 1 + \sum_{n=2}^{\infty} n^2[1 + (\lambda\mu n + \lambda - \mu)(n-1)]^m a_n z^{n-1} \\ &= \left\{1 + \sum_{n=2}^{\infty} n^2[1 + (\lambda\mu n + \lambda - \mu)(n-1)]^{m+1} a_n z^{n-1}\right\} \\ &\quad * \left\{1 + \sum_{n=2}^{\infty} \frac{z^{n-1}}{1 + (\lambda\mu n + \lambda - \mu)(n-1)}\right\}.\end{aligned}$$

Let

$$P(z) = 1 + \sum_{n=2}^{\infty} \frac{1}{1 + (\lambda\mu n + \lambda - \mu)(n-1)} z^{n-1}$$

and consider the sequence

$$c_1 = 1 \text{ and } c_n = \frac{1}{1 + (\lambda\mu n + \lambda - \mu)(n-1)}, \quad n \geq 2.$$

After lengthy but elementary calculations, we obtain that for $\lambda \geq \mu + 1$, the sequence $\{c_n\}_{n=1}^{\infty}$ is convex decreasing. Therefore, from Lemma 1 we have $\Re P(z) > \dfrac{1}{2}$ for all $z \in \mathcal{U}$. Now, our result follows as an application of Lemma 2. □

Making use of Lemma 1 and Lemma 2 we obtain the next result.

Theorem 2. *Let $f \in R^m(\lambda, \mu, \alpha, \beta)$. Then*

(i) $\Re\left\{e^{i\beta}(D^m_{\lambda\mu}f(z))'\right\} > \alpha\cos\beta, \quad z \in \mathcal{U};$

(ii) $\Re\left\{e^{i\beta}\left(\dfrac{D^m_{\lambda\mu}f(z)}{z}\right)\right\} > \alpha\cos\beta, \quad z \in \mathcal{U}.$

Proof. Let $f \in R^m(\lambda, \mu, \alpha, \beta)$. It follows that

$$\Re\left\{e^{i\beta}\left[(D^m_{\lambda\mu}f(z))' + z(D^m_{\lambda\mu}f(z))''\right]\right\} > \alpha\cos\beta$$

or equivalently

$$\Re\left\{e^{i\beta}\left[1 + \sum_{n=2}^{\infty} n^2[1 + (\lambda\mu n + \lambda - \mu)(n-1)]^m a_n z^{n-1}\right]\right\} > \alpha\cos\beta.$$

(i) The sequence $\{c_n\}_{n=1}^{\infty}$ defined by $c_1 = 1$ and $c_n = \dfrac{1}{n}$, $n \geq 2$ is a convex decreasing sequence and in view of Lemma 1, we have

$$\Re\left\{1+\sum_{n=2}^{\infty}\frac{1}{n}z^{n-1}\right\}>\frac{1}{2},\quad z\in\mathcal{U}.$$

Writing $(D_{\lambda\mu}^{m}f(z))'$ as

$$(D_{\lambda\mu}^{m}f(z))'=\left\{1+\sum_{n=2}^{\infty}n^2[1+(\lambda\mu n+\lambda-\mu)(n-1)]^m a_n z^{n-1}\right\}*\left\{1+\sum_{n=2}^{\infty}\frac{1}{n}z^{n-1}\right\}$$

and making use of Lemma 2, we conclude that $\Re\left\{e^{i\beta}(D_{\lambda\mu}^{m}f(z))'\right\}>\alpha\cos\beta,\ \ z\in\mathcal{U}$.

(ii) We observe that the sequence $\{c_n\}_{n=1}^{\infty}$ given by $c_1=1$ and $c_n=\dfrac{1}{n^2},\ n\geq 2$ is a convex decreasing sequence. It follows, from Lemma 1 that

$$\Re\left\{1+\sum_{n=2}^{\infty}\frac{1}{n^2}z^{n-1}\right\}>\frac{1}{2},\quad z\in\mathcal{U}.$$

Since

$$\frac{D_{\lambda\mu}^{m}f(z)}{z}=\left\{1+\sum_{n=2}^{\infty}n^2[1+(\lambda\mu n+\lambda-\mu)(n-1)]^m a_n z^{n-1}\right\}*\left\{1+\sum_{n=2}^{\infty}\frac{1}{n^2}z^{n-1}\right\},$$

we obtain our result as an application of Lemma 2. □

Letting $m=0$ and $\alpha=\beta=0$ in Theorem 2 (i), we have the next result due to Chichra [4].

Corollary 1. *If* $\Re\{f'(z)+zf''(z)\}>0,\ z\in\mathcal{U}$, *then* $\Re f'(z)>0,\ z\in\mathcal{U}$ *and thus, the function* f *is univalent in* $\mathcal{U}$.

Letting $m=0$ in Theorem 2, we obtain the following result.

Corollary 2. *If* $\Re\left\{e^{i\beta}(f'(z)+zf''(z))\right\}>\alpha\cos\beta,\ z\in\mathcal{U}$, *then*

(i) $\Re e^{i\beta}f'(z)>\alpha\cos\beta,\ z\in\mathcal{U}$;

(ii) $\Re\left\{e^{i\beta}\left[\dfrac{f(z)}{z}\right]\right\}>\alpha\cos\beta,\ z\in\mathcal{U}$.

3 Membership characterization

A necessary and sufficient condition for a function $f \in \mathcal{A}$ to be in the class $R^m(\lambda,\mu,\alpha,\beta)$, in terms of convolution, is given in the following theorem.

Theorem 3. *Let $0 \le \alpha < 1, |\beta| < \dfrac{\pi}{2}$ and $0 \le \mu \le \lambda, m \in \mathbb{N}$. Then, $f \in \mathcal{A}$ belongs to the class $R^m(\lambda,\mu,\alpha,\beta)$ if and only if $(f * H_{\lambda\mu\theta})(z)/z \neq 0$ in $\mathcal{U}$, where*

$$H_{\lambda\mu\theta}(z) = (h_{\lambda\mu} * h_\theta)(z) \tag{18}$$

with $h_{\lambda\mu}(z)$ and $h_\theta(z)$ defined by

$$h_{\lambda\mu}(z) = z + \sum_{n=2}^{\infty} n^2 A_n(\lambda,\mu,m) z^n \tag{19}$$

and

$$h_\theta(z) = \frac{z}{1-z}\left\{1 - \frac{1+e^{-i\beta}(e^{-i\beta}-2\alpha\cos\beta)e^{i\theta}}{e^{i\theta}[1+e^{-i\beta}(e^{-i\beta}-2\alpha\cos\beta)]}z\right\},\ \ 0<\theta<2\pi,\ z\in\mathcal{U}. \tag{20}$$

Proof. Let $p(z) = (D^m_{\lambda\mu}f(z))' + z(D^m_{\lambda\mu}f(z))'' = [z(D^m_{\lambda\mu}f(z))']'$. Since $p \in \mathcal{P}_\alpha, \beta$ if and only if $p \prec p_{\alpha,\beta}$ and noting that the function $p_{\alpha,\beta}$ given by (6) is univalent, we have that $p(z) \in \mathcal{P}_{\alpha,\beta}$ if and only if

$$p(z) \neq \frac{1+e^{-i\beta}(e^{-i\beta}-2\alpha\cos\beta)e^{i\theta}}{1-e^{i\theta}},\ \ 0<\theta<2\pi,\ z\in\mathcal{U}$$

or

$$(1-e^{i\theta})p(z) - \left\{1+e^{-i\beta}(e^{-i\beta}-2\alpha\cos\beta)e^{i\theta}\right\} \neq 0,\ \ 0<\theta<2\pi,\ z\in\mathcal{U}.$$

Further, using the convolution, we obtain

$$\begin{aligned}&(1-e^{i\theta})p(z) - \left\{1+e^{-i\beta}(e^{-i\beta}-2\alpha\cos\beta)e^{i\theta}\right\}\\ &= (1-e^{i\theta})\left[\frac{1}{1-z} * p(z)\right] - \left\{1+e^{-i\beta}(e^{-i\beta}-2\alpha\cos\beta)e^{i\theta}\right\} * p(z)\\ &= \left\{\frac{1-e^{i\theta}}{1-z} - \left[1+e^{-i\beta}(e^{-i\beta}-2\alpha\cos\beta)e^{i\theta}\right]\right\} * p(z) \neq 0.\end{aligned}$$

Consider the function $q_\theta(z)$ defined by

$$q_\theta(z) = \frac{\dfrac{1-e^{i\theta}}{1-z} - \left[1+e^{-i\beta}(e^{-i\beta}-2\alpha\cos\beta)e^{i\theta}\right]}{-e^{i\theta}\left[1+e^{-i\beta}(e^{-i\beta}-2\alpha\cos\beta)\right]}$$

or

$$q_\theta(z) = \frac{1}{1-z}\left\{1 - \frac{1+e^{-i\beta}(e^{-i\beta}-2\alpha\cos\beta)e^{i\theta}}{e^{i\theta}\left[1+e^{-i\beta}(e^{-i\beta}-2\alpha\cos\beta)\right]}z\right\},\ 0<\theta<2\pi,\ z\in\mathcal{U}. \tag{21}$$

It follows that $p(z) \in \mathcal{P}_{\alpha,\beta}$ if and only if $(q_\theta * p)(z) \neq 0$. Since

$$zp(z) = z[z(D^m_{\lambda\mu} f(z))']' = (f * h_{\lambda\mu})(z)$$

and $zq_\theta(z) = h_\theta(z)$, we obtain that $p(z) \in \mathcal{P}_{\alpha,\beta}$ if and only if $(f * h_{\lambda\mu} * h_\theta)(z)/z \neq 0$.

Consequently, we have that $f \in R^m(\lambda, \mu, \alpha, \beta)$ if and only if $(f * H_{\lambda\mu\theta})(z)/z \neq 0$ in $\mathcal{U}$, where $H_{\lambda\mu\theta}$ is given by (18). □

Theorem 4. *The coefficients H_n of the function $H_{\lambda\mu\theta}(z)$ defined by (3.1) satisfy the inequality*

$$|H_n| \leq \frac{n^2 A_n(\lambda, \mu, m)}{(1-\alpha)\cos\beta}\ , \quad n \geq 2$$

where $A_n(\lambda, \mu, m)$ is given by (12).

Proof. In view of (18), (19) and (20) we have

$$H_{\lambda\mu\theta}(z) = z + \sum_{n=2}^{\infty} \frac{e^{i\theta} - 1}{[1 + e^{-i\beta}(e^{-i\beta} - 2\alpha\cos\beta)]e^{i\theta}} n^2 A_n(\lambda, \mu, m) z^n$$

or

$$H_{\lambda\mu\theta}(z) = z + \sum_{n=2}^{\infty} H_n z^n$$

where

$$H_n = \frac{e^{i\theta} - 1}{2e^{i(\theta-\beta)}(1-\alpha)\cos\beta} n^2 A_n(\lambda, \mu, m)\ , \quad n \geq 2.$$

It is easy to check that

$$|H_n| \leq \frac{n^2 A_n(\lambda, \mu, m)}{(1-\alpha)\cos\beta}\ , \quad n \geq 2$$

and thus, our theorem is proved. □

Theorem 4 enables us to show that the function class $R^m(\lambda, \mu, \alpha, \beta)$ is non-empty.

Corollary 3. *Let $f(z) = z + az^n$. If*

$$|a| \leq \frac{(1-\alpha)\cos\beta}{n^2 A_n(\lambda, \mu, m)}$$

then, $f \in R^m(\lambda, \mu, \alpha, \beta)$.

Proof. Since

$$\left| \frac{(f * H_{\lambda\mu\theta})(z)}{z} \right| = |1 + aH_n z^{n-1}| \geq 1 - |a||H_n||z| \geq 1 - |z| > 0\ , \quad z \in \mathcal{U}$$

it follows that $f \in R^m(\lambda, \mu, \alpha, \beta)$. □

4 Integral representation

Making use of the integral representation of the functions in $\mathcal{P}_{\alpha,\beta}$, given by (9), we obtain an integral representation for the class $R^m(\lambda,\mu,\alpha,\beta)$.

Theorem 5. *A function $f \in \mathcal{A}$ is in the class $R^m(\lambda,\mu,\alpha,\beta)$ if and only if it can be expressed as*

$$f(z) = g_{\alpha\beta}^{(-1)}(z) * \int_{|x|=1} \left[z + 2(1-\alpha)e^{-i\beta}\cos\beta\bar{x}\sum_{n=2}^{\infty}\frac{(xz)^n}{n^2} \right] d\mu(x) \tag{22}$$

where $\mu(x)$ is a Borel probability measure on $T = \{x \in \mathbb{C} : |x| = 1\}$ and $g_{\alpha\beta}^{(-1)}(z)$ is given by (15).

Proof. In view of the definition of the class $R^m(\lambda,\mu,\alpha,\beta)$, we have that $f \in R^m(\lambda,\mu,\alpha,\beta)$ if and only if

$$(D_{\lambda\mu}^m f(z))' + z(D_{\lambda\mu}^m f(z))'' \in \mathcal{P}_{\alpha,\beta}.$$

Making use of (9), we obtain

$$(D_{\lambda\mu}^m f(z))' + z(D_{\lambda\mu}^m f(z))'' = \int_{|x|=1} \frac{1 + e^{-i\beta}(e^{-i\beta} - 2\alpha\cos\beta)xz}{1-xz} d\mu(x)$$

or

$$\left[z(D_{\lambda,\mu}^m f(z))' \right]' = \int_{|x|=1} \frac{1 + e^{-i\beta}(e^{-i\beta} - 2\alpha\cos\beta)xz}{1-xz} d\mu(x).$$

Integrating the above equality, we have

$$z(D_{\lambda,\mu}^m f(z))' = \int_{|x|=1} \left[\int_0^z \frac{1 + e^{-i\beta}(e^{-i\beta} - 2\alpha\cos\beta)x\zeta}{1-x\zeta} d\zeta \right] d\mu(x)$$

which is equivalent to

$$(D_{\lambda,\mu}^m f(z))' = \int_{|x|=1} \left[1 + 2(1-\alpha)e^{-i\beta}\cos\beta\sum_{n=1}^{\infty}\frac{(xz)^n}{n+1} \right] d\mu(x).$$

Integrating again this equality, we obtain

$$D_{\lambda,\mu}^m f(z) = \int_{|x|=1} \left[z + 2(1-\alpha)e^{-i\beta}\cos\beta\bar{x}\sum_{n=2}^{\infty}\frac{(xz)^n}{n^2} \right] d\mu(x). \tag{23}$$

Equality (22) follows easily from (16) and (23).

Since this deductive process can be converse, we have proved our theorem. □

5 Coefficient estimates

The first result on coefficient estimates for the class $R^m(\lambda,\mu,\alpha,\beta)$ is the following.

Theorem 6. *If $f \in R^m(\lambda,\mu,\alpha,\beta)$ is given by (1), then*

$$|a_n| \le \frac{2(1-\alpha)\cos\beta}{n^2 A_n(\lambda,\mu,m)} \quad , \quad n \ge 2 \tag{24}$$

where $A_n(\lambda,\mu,m)$ is given by (12).

Proof. Let $f \in R^m(\lambda,\mu,\alpha,\beta)$. Then

$$p(z) = (D^m_{\lambda\mu} f(z))' + z(D^m_{\lambda\mu} f(z))'' \in \mathcal{P}_{\alpha,\beta}.$$

Since

$$p(z) = 1 + \sum_{n=2}^{\infty} n^2 A_n(\lambda,\mu,m) a_n z^{n-1},$$

in view of (8), we have

$$|n^2 A_n(\lambda,\mu,m) a_n| \le 2(1-\alpha)\cos\beta,\ n \ge 2,$$

that is

$$|a_n| \le \frac{2(1-\alpha)\cos\beta}{n^2 A_n(\lambda,\mu,m)} \quad , \quad n \ge 2.$$

□

In order to obtain our next result on coefficient estimates, we need the following lemma.

Lemma 3. *([7]) Let $w(z) = c_1 z + c_2 z^2 + \dots$ be an analytic function with $|w(z)| < 1$ in $\mathcal{U}$. Then, for any complex number ν*

$$|c_2 - \nu c_1^2| \le \max\{1, |\nu|\}. \tag{25}$$

The equality is attained for $w(z) = z^2$ and $w(z) = z$.

Theorem 7. *Let $f \in R^m(\lambda,\mu,\alpha,\beta)$ be given by (1) and let δ be a complex number. Then*

$$|a_3 - \delta a_2^2| \le \frac{2(1-\alpha)\cos\beta}{9A_3(\lambda,\mu,m)} \max\{1, |\nu|\}, \tag{26}$$

where

$$\nu = \frac{9(1-\alpha)e^{-i\beta}\cos\beta A_3(\lambda,\mu,m)\delta - 8A_2(\lambda,\mu,m)^2}{8A_2(\lambda,\mu,m)^2}$$

and

$$A_2(\lambda,\mu,m) = (2\lambda\mu + \lambda - \mu + 1)^m \quad , \quad A_3(\lambda,\mu,m) = (6\lambda\mu + 2(\lambda-\mu) + 1)^m.$$

The result is sharp.

Proof. Suppose $f \in R^m(\lambda,\mu,\alpha,\beta)$. Then $(D^m_{\lambda\mu}f(z))' + z(D^m_{\lambda\mu}f(z))'' \in \mathcal{P}_{\alpha,\beta}$. It follows from (6), that there exists an analytic function $w(z) = \sum_{n=1}^{\infty} c_n z^n$, with $|w(z)| < 1$ in $\mathcal{U}$ such that

$$(D^m_{\lambda\mu}f(z))' + z(D^m_{\lambda\mu}f(z))'' = \frac{1 + e^{-i\beta}(e^{-i\beta} - 2\alpha\cos\beta)w(z)}{1 - w(z)}$$

which is equivalent to

$$(1 - w(z))\left[(D^m_{\lambda\mu}f(z))' + z(D^m_{\lambda\mu}f(z))''\right] = 1 + e^{-i\beta}(e^{-i\beta} - 2\alpha\cos\beta)w(z). \quad (27)$$

Equating the coefficients of z and z^2 on both sides of (27), we obtain

$$a_2 = \frac{(1-\alpha)e^{-i\beta}\cos\beta}{2A_2(\lambda,\mu,m)}c_1 \quad (28)$$

and

$$a_3 = \frac{2(1-\alpha)e^{-i\beta}\cos\beta}{9A_3(\lambda,\mu,m)}(c_2 + c_1^2). \quad (29)$$

From (28) and (29), it follows that

$$a_3 - \delta a_2^2 = \frac{2(1-\alpha)e^{-i\beta}\cos\beta}{9A_3(\lambda,\mu,m)}[c_2 - \nu c_1^2]$$

where

$$\nu = \frac{9(1-\alpha)e^{-i\beta}\cos\beta A_3(\lambda,\mu,m)\delta - 8A_2(\lambda,\mu,m)^2}{8A_2(\lambda,\mu,m)^2}.$$

Applying Lemma 3, we get

$$|a_3 - \delta a_2^2| = \frac{2(1-\alpha)\cos\beta}{9A_3(\lambda,\mu,m)}|c_2 - \nu c_1^2|$$

$$\leq \frac{2(1-\alpha)\cos\beta}{9A_3(\lambda,\mu,m)}\max\{1, |\nu|\}.$$

The sharpness of (26) follows from the sharpness of inequality (25). □

6 Convolution property

Making use of Lemma 2, we obtain a convolution property for the class $R^m(\lambda,\mu,\alpha,\beta)$.

Theorem 8. *The class $R^m(\lambda,\mu,\alpha,\beta)$ is closed under the convolution with a convex function. That is, if $f \in R^m(\lambda,\mu,\alpha,\beta)$ and g is convex in $\mathcal{U}$, then $f * g \in R^m(\lambda,\mu,\alpha,\beta)$.*

Proof. It is known that, if g is a convex function in $\mathcal{U}$, then

$$\Re\frac{g(z)}{z} > \frac{1}{2}. \tag{30}$$

Suppose $f \in R^m(\lambda,\mu,\alpha,\beta)$. Making use of the convolution properties, we have

$$z[D^m_{\lambda\mu}(f*g)(z)]' = z(D^m_{\lambda\mu}f(z))' * g(z)$$

and thus

$$\begin{aligned}&(D^m_{\lambda\mu}(f*g)(z))' + z(D^m_{\lambda\mu}(f*g)(z))''\\ &= \left[(D^m_{\lambda\mu}f(z))' + z(D^m_{\lambda\mu}f(z))''\right] * \frac{g(z)}{z}.\end{aligned} \tag{31}$$

Since

$$\Re\left\{e^{i\beta}[(D^m_{\lambda\mu}f(z))' + z(D^m_{\lambda\mu}f(z))'']\right\} > \alpha\cos\beta,$$

the desired result follows immediately from (30), (31) and Lemma 2. □

Corollary 4. *The class $R^m(\lambda,\mu,\alpha,\beta)$ is invariant under Bernardi integral operator (see [3]) defined by*

$$F_c(f)(z) = \frac{1+c}{z^c}\int_0^z t^{c-1}f(t)dt\ ,\ \Re c > 0$$

that is, if $f \in R^m(\lambda,\mu,\alpha,\beta)$, then $F_c(f) \in R^m(\lambda,\mu,\alpha,\beta)$.

Proof. Assume $f \in R^m(\lambda,\mu,\alpha,\beta)$. It is easy to check that $F_c(f)(z) = (f*g)(z)$, where

$$g(z) = \sum_{n=1}^{\infty}\frac{1+c}{n+c}z^n.$$

Since the function g is convex (see [2]), by applying Theorem 8, the result follows. □

References

[1] Al-Oboudi, F. M., *On univalent functions defined by a generalized Sălăgean operator*, Internat. J. Math. Math. Sci., **27** (2004), 1429-1436.

[2] Barnard, R. W., Kellogg, C., *Applications of convolution operators to problems in univalent function theory*, Michigan Math. J., **27** (1980), no. 1, 81-94.

[3] Bernardi,S. D., *Convex and starlike functions*, Trans. Amer. Math. Soc., **135** (1969), 429-446.

[4] Chichra,P. N., *New subclasses of the class of close-to-convex functions*, Proc. Amer. Math. Soc., **62** (1977), no. 1, 37-43.

[5] Fejér, L., *Untersuchungen über Potenzreichen mit mehrfach monotoner koeffizientenfolge*, Acta Lit. Sci., **8** (1936), 89-115.

[6] Goodman, A. W., *Univalent functions*, vol. I, Mariner Publishing Company, Inc., 1983.

[7] Keogh, F. R., Merkes, E. P., *A coefficient inequality for certain classes of analytic functions*, Proc. Amer. Math. Soc., **20** (1969), 8-12.

[8] Răducanu, D., Orhan, H., *Subclasses of analytic functions defined by a generalized differential operator*, Int. Journ. Math. Anal., **4** (2010), no. 1-4, 1-16.

[9] Ruscheweyh, St., *Convolutions in geometric function theory*, Les Presses de l'Univ. de Montreal, 1982.

[10] Sălăgean, G. S., *Subclasses of univalent functions*, Complex Analysis 5th Romanian-Finnish Seminar, Part. I (Bucharest, 1981), Lect. Notes Math., **1013**, Springer-Verlag, (1983), 362-372.

[11] Silverman, H., *A class of bounded starlike functions*, Internat. J. Math. Math. Sci., **17** (1994), no. 2, 249-252.

[12] Singh, R., Singh, S., *Convolution properties of a class of starlike functions*, Proc. Amer. Math. Soc., **106** (1989), no. 1, 145-152.