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A NEW MIXED δ-SHOCK MODEL

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Abstract

In this paper, we introduce and study a realistic shock model. In this new type of *mixed δ-shock model* it is assumed that the failure of a system subjected to a random sequence of shocks occurs if a shock exceeds a given tolerance level or two strong shocks, not necessarily consecutive, occur in an interval of length δ. We provide the distribution of the number of shocks until the failure of the system. Special attention is given to the Poisson case. A lower bound for the expected index of the fatal shock is also obtained.

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1 Introduction

Shock models are extensively studied in literature, especially in reliability theory. General properties of the distribution of the lifetime of a system subjected to a sequence of shocks occurring randomly in time as events in a renewal process have been studied by Esary et al. (1973), Shaked and Shantikumar (1991), Nakagawa (2007), among many others. Let μ be the index of the fatal shock. If the sequence of occurrence of shocks follows a Poisson process with parameter λ, then the survival function of the lifetime T_μ of the system is simply represented in relation to the decreasing sequence of probabilities $\overline{p}_n$ of surviving the first n shocks (i.e. $\overline{p}_n = \mathbb{P}\{\mu > n\}$). So, we have (see Esary et al. (1973))

$$\mathbb{P}\{T_\mu > t\} = \sum_{n=0}^{\infty} \overline{p}_n \frac{(\lambda t)^n}{n!} e^{-\lambda t}, \forall t \geq 0. \tag{1}$$

The above formula holds if μ and the Poisson renewal process are independent. The probabilities $\overline{p}_n$ depend on the particular situations causing the failure of the system.

There are four typical shock models studied in literature: the extreme shock model, the cumulative model, the run shock model, and the δ-shock model. In

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the *extreme shock model*, a system is considered to fail as soon as the magnitude of any shock exceeds a given threshold (see Gut and Hüsler (1999)). In the *cumulative shock model* (see Gut (1990)) the system will break down if the cumulative magnitudes of shocks cross over a critical level. The run shock model (see Mallor and Omey (2001)) assumes that a system will be operational unless k-consecutive shocks with critical magnitudes occur. In the *δ-shock model* (see Li and Kong (2007)), the system will fail when the interval between two successive shocks is less that a positive constant δ. There are many generalizations of these shock models (see, for example, Cirillo and Hüsler (2011)). A combination of some of the above described models is commonly called a "mixed shock model".

The δ-shock model was extended in many directions. In a mixed δ-shock model it is assumed that the failure of the system occurs when the time between two successive shocks is less than a critical threshold δ, or the magnitude of a shock (alternatively, the cumulative magnitude of shocks) is larger than another critical threshold γ. This model, introduced by Wang and Zang (2005), were recently developed by Parvardeh and Balakrishnan (2015). Also, we mention that Eryilmaz (2012) studied a mixed δ-shock model in which the system fails if the interval between k-consecutive shocks is less than δ or m-consecutive shocks have the magnitude larger than a tolerance level γ.

In this paper, we propose a new realistic mixed δ-shock model. Given three positive constants (levels) α, β and δ, with $\beta < \alpha$, we shall suppose that the failure of the system appears in the following two situations: 1) the magnitude of a shock exceeds the tolerance level α, or 2) the time between two shocks whose magnitudes exceed the critical level β is at most equal to the threshold δ. In the second situation, it is not relevant that the two stronger than β shocks (causing the failure of the system) are consecutive.

Let $\{(X_n, Y_n), n \geq 1\}$ be a sequence of independent identically distributed (i.i.d.) two-dimensional random vectors. X_n means the magnitude of the nth shock and Y_n is the time between the $(n-1)$th and the nth shock, for $n \geq 1$. X_n and Y_n are assumed independent positive random variables, with the common distribution functions F and G, and common survival functions $\overline{F}$ and $\overline{G}$, respectively. In what follows, we assume $0 < F(\beta) < F(\alpha) < 1$ and $G(\delta) \in (0, 1)$. Also, we denote by X_0 a real number of the interval $(\beta, \alpha]$. Let $T_n = \sum_{i=1}^{n} Y_i$ be the moment of the nth shock, $n = 1, 2, \cdots$, and denote $T_0 = 0$. Let us consider the renewal process $(N(t))_{t>0}$ associated to the sequence $(Y_n, T_n)_{n\geq 1}$. That is $N(t) = \max\{n \in \mathbb{N} : \ T_n \leq t\}$, for $t > 0$. If μ is the index of the fatal shock, then T_μ is the lifetime (time to failure) of the system and X_μ is the magnitude of the fatal shock. In the particular case of a Poisson process of parameter (intensity) λ, Y_n is an exponential random variable with the hazard rate λ and $\mathbb{P}\{N(t) = n\} = e^{-\lambda t}(\lambda t)^n/n!, \ n = 0, 1, 2, \cdots$.

Our study focuses on the evaluation of the random variable μ. Let us define the random variable $W(n) = \max\{i \in \{0, 1, \cdots, n\} : \ X_i \in (\beta, \alpha]\}$. Clearly, $W(0) = 0$. The description of our mixed δ-shock model leads to the following definition of μ

$$\mu = \min\left\{n \in \mathbb{N}^* : (X_n > \alpha) \text{ or } (X_n \in (\beta, \alpha] \text{ and } T_n - T_{W(n-1)} \leq \delta)\right\}. \qquad (2)$$

2 Main results

Let us denote by G_n the distribution function of T_n. Also, let $\overline{G}_n = 1 - G_n$ be the survival function of T_n. We have $G_n = \underbrace{G * G * \cdots * G}_{n\ -\ \text{times}}$, where

$$G_{i+1}(t) = (G_i * G)(t) = \int_0^\infty G(t-x) dG_i(x),\ t \geq 0,\ i \geq 1.$$

Firstly, we point out the survival function of μ.

Lemma 1. *The survival function of the arithmetic random variable μ is given by*

$$\overline{p}_n = P\{\mu > n\} = \sum_{k=0}^{n} \psi_{n,k}(\delta) F^{n-k}(\beta)[F(\alpha) - F(\beta)]^k,\ n \geq 1, \qquad (3)$$

where

$$\psi_{n,k}(t) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \overline{G}_{i_1}(t) \overline{G}_{i_2 - i_1}(t) \cdots \overline{G}_{i_k - i_{k-1}}(t),\ t \geq 0,\ k \in \{1, \cdots, n\},$$

and $\psi_{n,0}(t) = 1,\ t \geq 0$.

Proof. Let n be a positive integer. Observe that $\{\mu > n\} \subset \{X_1 \leq \alpha, \cdots, X_n \leq \alpha\}$. Denote $I_n = \{i \in \{1, \cdots, n\} : X_i \in (\beta, \alpha]\}$. Since $\{\mu > n\} = \bigcup_{k=0}^{n} \{\mu > n,\ |I_n| = k\}$, we get $\mathbb{P}\{\mu > n\} = \mathbb{P}\{\mu > n,\ I_n = \emptyset\} + \sum_{k=1}^{n} \mathbb{P}\{\mu > n,\ |I_n| = k\}$. We have

$$\mathbb{P}\{\mu > n,\ I_n = \emptyset\} = \mathbb{P}\{X_1 \leq \beta, \cdots, X_n \leq \beta\} = F^n(\beta).$$

Then, for $k \in \{1, \cdots, n\}$, we obtain

$$\mathbb{P}\{\mu > n,\ |I_n| = k\} = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \mathbb{P}\{\mu > n,\ I_n = \{i_1, \cdots, i_k\}\},$$

with

$$\begin{aligned} &\mathbb{P}\{\mu > n,\ I_n = \{i_1, \cdots, i_k\}\} \\ &= \mathbb{P}\left[\left(\bigcap_{i \notin I_n} \{X_i \leq \beta\}\right) \cap \left(\bigcap_{i \in I_n} \{X_i \in (\beta, \alpha]\}\right) \cap \left(\bigcap_{j=1}^{k} \left\{\sum_{s=i_{j-1}}^{i_j} Y_s > \delta\right\}\right)\right], \end{aligned}$$

where $i_0 = 0$. By taking into account the independence of the random variables $X_i, Y_i,\ i = 1, \cdots, n$, we find

$$\mathbb{P}\left\{\mu > n,\ I_n = \{i_1, \cdots, i_k\}\right\} = F^{n-k}(\beta)[F(\alpha) - F(\beta)]^k \prod_{j=1}^{k} \overline{G}_{i_j - i_{j-1}}(\delta).$$

Thus, we get the conclusion. □

Thus, we can find the exact distribution of μ.

Theorem 1. *The random variable μ has the following distribution:*

$$\mathbb{P}\{\mu = 1\} = \overline{F}(\alpha) + G(\delta)[F(\alpha) - F(\beta)],$$

and, for $n > 1$,

$$\mathbb{P}\{\mu = n\} = \sum_{k=0}^{n-1} \left[\psi_{n-1,k}(\delta) - \psi_{n,k}(\delta) F(\beta)\right] F^{n-k-1}(\beta)[F(\alpha) - F(\beta)]^k - \overline{G}^n(\delta)[F(\alpha) - F(\beta)]^n. \tag{4}$$

Proof. We have
$\mathbb{P}\{\mu = 1\} = \mathbb{P}\left(\{X_1 > \alpha\} \cup \{X_1 \in (\beta, \alpha],\ Y_1 \le \delta\}\right) = \overline{F}(\alpha) + G(\delta)[F(\alpha) - F(\beta)]$.
For $n > 1$, we observe that $\mathbb{P}\{\mu = n\} = \overline{p}_{n-1} - \overline{p}_n$. Thus, (4) follows from (3) and the relation $\psi_{n,n}(\delta) = \overline{G}^n(\delta)$. □

Theorem 1 shows that it is crucial to evaluate the functions $\psi_{n,k}$. Let us consider the generating function of the sequence $(\overline{G}_n(t))_{n \ge 1}$:

$$u(t, z) = \overline{G}_1(t) z + \overline{G}_2(t) z^2 + \cdots + \overline{G}_n(t) z^n + \cdots,$$

for $t \ge 0$ and $z \in \mathbb{C}$. Since $\overline{G}_n(t) \in [0, 1]$, for all $t \ge 0$ and $n \in \mathbb{N} \setminus \{0\}$, the above power series is convergent for $|z| < 1$. Then, for $k \in \mathbb{N}^*$, the function $u^k(t, z)$ can be expressed as a convergent power series in $z \in \mathbb{C}$, with $|z| < 1$. Assume that

$$u^k(t, z) = \sum_{i=k}^{\infty} c_i^{(k)}(t) z^i,\ t \ge 0,\ |z| < 1.$$

Lemma 2. *For a positive integer n, we have*

$$\psi_{n,k}(t) = \sum_{i=k}^{n} c_i^{(k)}(t),\ k \in \{1, 2, \cdots, n\},\ t \in [0, \infty).$$

Proof. For $k \in \{1, \cdots, n\}$, $t \ge 0$ and $z \in \mathbb{C}$, with $|z| < 1$, we obtain

$$u^k(t, z) = \sum_{s_1, s_2, \cdots, s_k \ge 1} \overline{G}_{s_1}(t) \overline{G}_{s_2}(t) \cdots \overline{G}_{s_k}(t) z^{s_1 + s_2 + \cdots + s_k}$$

$$= \sum_{i=k}^{\infty} \sum_{s_1,\cdots,s_k \geq 1;\ \sum_{j=1}^{k} s_j = i} \overline{G}_{s_1}(t)\overline{G}_{s_2}(t)\cdots\overline{G}_{s_k}(t)z^i.$$

Thus, for $i \geq k$, we find that the coefficient $c_i^{(k)}(t)$ has the form:

$$\sum_{s_1,\cdots,s_k \geq 1;\ \sum_{j=1}^{k} s_j = i} \overline{G}_{s_1}(t)\cdots\overline{G}_{s_k}(t) = \sum_{1 \leq i_1 < \cdots < i_k = i} \overline{G}_{i_1}(t)\overline{G}_{i_2 - i_1}(t)\cdots\overline{G}_{i_k - i_{k-1}}(t),$$

where $i_j = s_1 + s_2 + \cdots + s_j$, for $j = 1, 2, \cdots, k$. Therefore,

$$\sum_{i=k}^{n} c_i^{(k)}(t) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \overline{G}_{i_1}(t)\overline{G}_{i_2 - i_1}(t)\cdots\overline{G}_{i_k - i_{k-1}}(t) = \psi_{n,k}(t).$$

□

The above lemma leads to a nice result in the Poisson case.

Lemma 3. *Assume that the sequence $(Y_n, T_n)_{n \geq 1}$ generates a Poisson process with parameter λ, i.e. $\mathbb{P}\{Y_n > t\} = e^{-\lambda t},\ t \geq 0,\ n \geq 1$. Then we have:*

$$u^k(t,z) = \frac{z^k e^{\lambda k t(z-1)}}{(1-z)^k},\ |z| < 1,\ t \geq 0,\ k \in \mathbb{N}^*.$$

and

$$\psi_{n,k}(t) = e^{-\lambda k t} \sum_{i=0}^{n-k} \frac{(\lambda k t)^i}{i!} \binom{n-i}{k},\ n \geq k.$$

Proof. We have

$$u(t,z) = \sum_{n=1}^{\infty} \left(e^{-\lambda t} \sum_{k=0}^{n-1} \frac{\lambda^k t^k}{k!} \right) z^n = e^{-\lambda t} \sum_{k=0}^{\infty} \left(\frac{\lambda^k t^k}{k!} \sum_{n=k+1}^{\infty} z^n \right)$$

$$= \frac{z e^{-\lambda t}}{1-z} \sum_{k=0}^{\infty} \frac{\lambda^k t^k z^k}{k!} = \frac{z e^{-\lambda t(1-z)}}{1-z}.$$

Hence

$$u^k(t,z) = \frac{z^k e^{\lambda k t(z-1)}}{(1-z)^k},\ |z| < 1,\ t \geq 0,\ k \in \mathbb{N}^*.$$

For fixed $t \in [0, \infty)$ and $k \in \mathbb{N}^*$, let us denote

$$f(z) = (1-z)^{-k} e^{\lambda k t(z-1)} = \frac{u^k(t,z)}{z^k} = \sum_{p=0}^{\infty} c_{p+k}^{(k)}(t) z^p,\ z \in \mathbb{C},\ |z| < 1.$$

Therefore, $c_k^{(k)}(t) = f(0) = e^{-\lambda k t}$ and $c_{p+k}^{(k)}(t) = \dfrac{f^{(p)}(0)}{p!},\ p \in \mathbb{N}^*$. By using the general Leibniz rule, we obtain

$$f^{(p)}(z) = e^{\lambda k t(z-1)} \sum_{i=0}^{p} \left[\binom{p}{i} \frac{(k+p-i-1)!}{(k-1)!} (\lambda k t)^i (1-z)^{-(k+p-i)} \right],\ p \geq 1.$$

Thus,

$$c_{p+k}^{(k)}(t) = e^{-\lambda kt} \sum_{i=0}^{p} \left[\binom{k+p-i-1}{k-1} \frac{(\lambda kt)^i}{i!} \right].$$

From Lemma 2, it results

$$\psi_{n,k}(t) = \sum_{p=0}^{n-k} c_{p+k}^{(k)}(t) = e^{-\lambda kt} \sum_{p=0}^{n-k} \sum_{i=0}^{p} \left[\binom{k+p-i-1}{k-1} \frac{(\lambda kt)^i}{i!} \right]$$

$$= e^{-\lambda kt} \sum_{i=0}^{n-k} \frac{(\lambda kt)^i}{i!} \sum_{p=i}^{n-k} \binom{k+p-i-1}{k-1} = e^{-\lambda kt} \sum_{i=0}^{n-k} \frac{(\lambda kt)^i}{i!} \binom{n-i}{k}.$$

□

Therefore, we obtain the explicit formula of the probability $\overline{p}_n$ of surviving the first n shocks.

Theorem 2. *If the sequence of occurrence of shocks follows a Poisson process with parameter λ, we have*

$$\overline{p}_n = \mathbb{P}\{\mu > n\} = \sum_{k=0}^{n} \sum_{i=0}^{n-k} \frac{(\lambda k\delta)^i}{i!} \binom{n-i}{k} e^{-\lambda k\delta} F^{n-k}(\beta)[F(\alpha) - F(\beta)]^k, \ n \geq 1.$$

Proof. We apply Lemma 1 and Lemma 3. □

Remark that, since μ depends on the sequence $(Y_n)_{n\geq 1}$, the classical relation (1) is inappropriate to express the reliability of our model.

Finally, we propose a nice lower bound for the mean of the random variable μ. We refer here to the general case.

Theorem 3. *The following inequality holds*

$$\mathbb{E}(\mu) \geq \frac{1}{1 - F(\alpha) + F(\alpha)G(\delta) - F(\beta)G(\delta)}.$$

Proof. We start from the elementary inequality

$$\overline{G}_i(\delta) = \mathbb{P}\{Y_1 + \cdots + Y_i > \delta\} \geq \mathbb{P}\{Y_1 > \delta\} = \overline{G}(\delta), \ i = 1, 2, \cdots.$$

Hence

$$\psi_{n,k}(\delta) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \overline{G}_{i_1}(\delta) \overline{G}_{i_2 - i_1}(\delta) \cdots \overline{G}_{i_k - i_{k-1}}(\delta) \geq \binom{n}{k} \overline{G}^k(\delta).$$

From Lemma 1, we find

$$\overline{p}_n = \mathbb{P}\{\mu > n\} \geq \sum_{k=0}^{n} \binom{n}{k} \overline{G}^k(\delta) F^{n-k}(\beta)[F(\alpha) - F(\beta)]^k$$

$$= \{F(\beta) + \overline{G}(\delta)[F(\alpha) - F(\beta)]\}^n = [F(\alpha) - F(\alpha)G(\delta) + F(\beta)G(\delta)]^n .$$

Remark that $\overline{p}_0 = \mathbb{P}\{\mu > 0\} = 1$. The mean of μ can be expressed as

$$\mathbb{E}(\mu) = \sum_{n=1}^{\infty} n\mathbb{P}\{\mu = n\} = \sum_{n=1}^{\infty} n\left(\overline{p}_{n-1} - \overline{p}_n\right) = \overline{p}_0 + \sum_{n=1}^{\infty} \overline{p}_n[(n+1) - n] = \sum_{n=0}^{\infty} \overline{p}_n.$$

Since

$$0 < \overline{G}(\delta)F(\alpha) + F(\beta)G(\delta) = F(\alpha) - F(\alpha)G(\delta) + F(\beta)G(\delta) < F(\alpha) < 1,$$

we obtain

$$\mathbb{E}(\mu) \geq \sum_{n=0}^{\infty} [F(\alpha) - F(\alpha)G(\delta) + F(\beta)G(\delta)]^n = \frac{1}{1 - F(\alpha) + F(\alpha)G(\delta) - F(\beta)G(\delta)}.$$

□

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