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UNIVALENCE CRITERIA RELATED WITH SĂLĂGEAN AND RUSCHEWEYH OPERATORS

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Abstract

In this paper, by means of Sălăgean and Ruscheweyh operators, we obtain new sufficient conditions for univalence using the method of Loewner chains. In particular, we obtain some well-known univalence conditions due to Lewandowski, Becker, Kanas and Lecko.

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1 Introduction

Let $\mathcal{A}$ be the class of all analytic functions f in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$, of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

Denote by $\mathcal{P}$ the class of analytic functions of the form $p(z) = 1 + p_1 z + p_2 z^2 + \dots,\ z \in U$, such that $\Re p(z) > 0$ in U.

In this paper we will study some new criteria of univalence related to Ruscheweyh and Sălăgean derivatives. Our considerations are based upon the theory of Loewner chains.

Before proving our main results we briefly recall the method of Loewner chains:

A function $L(z,t) : U \times [0,\infty) \to \mathbb{C}$ is said to be a Loewner chain if it satisfies the following conditions:

i) $L(z,t)$ is analytic and univalent in U for all $t \in [0,\infty)$,

ii) $L(z,t) \prec L(z,s)$ for all $0 \le t \le s < \infty$,

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where the symbol $' \prec '$ stands for subordination.

Lemma 1. *[5] Let $r \in (0,1]$ and let*

$$L(z,t) = a_1(t)z + a_2(t)z^2 + \cdots, \ a_1(t) \neq 0 \tag{2}$$

be an analytic function in U_r for all $t \geq 0$. Suppose that:

i) $L(z,t)$ is a locally absolutely continuous function in $[0,\infty)$ and locally uniformly with respect to U_r.

ii) $a_1(t)$ is a complex valued continuous function on $[0,\infty)$ such that $\lim_{t\to\infty} |a_1(t)| = \infty$ and

$$\left\{ \frac{L(z,t)}{a_1(t)} \right\}_{t\in[0,\infty)}$$

is a normal family of functions in U_r.

iii) There exists an analytic function $p : U_r \times [0,\infty) \to \mathbb{C}$ satisfying $\Re p(z,t) > 0$ for all $(z,t) \in U \times [0,\infty)$ and

$$z\frac{\partial L(z,t)}{\partial z} = p(z,t)\frac{\partial L(z,t)}{\partial t}, \ z \in U_r, \ t \in [0,\infty). \tag{3}$$

Then, for each $t \in [0,\infty)$, the function $L(z,t)$ has an analytic and univalent extension to the whole disk U, i.e $L(z,t)$ is a Loewner chain.

2 Main results

2.1 Univalence criteria connected with Sălăgean operator

For a function $f \in \mathcal{A}$ Sălăgean (see [7]) introduced the operator $D^n : \mathcal{A} \to \mathcal{A}$ defined by

$$\begin{aligned} &D^0 f(z) = f(z), \\ &D^1 f(z) = Df(z) = zf'(z), \\ &\vdots \\ &D^n f(z) = D(D^{n-1} f(z)), \ z \in U. \end{aligned} \tag{4}$$

In this section, making use of Lemma 1, we obtain some new univalence criteria related to Sălăgean differential operator.

Theorem 1. *Let $f \in \mathcal{A}$ and p an analytic function with $p(0) = 1$. If the inequalities*

$$\left| \frac{2}{p(z)+1} \cdot \frac{zf'(z)}{D^{n+1}f(z)} - 1 \right| \leq 1 \tag{5}$$

and

$$\left| \left(\frac{2}{p(z)+1} \cdot \frac{zf'(z)}{D^{n+1}f(z)} - 1 \right) |z|^2 + (1-|z|^2) \left(\frac{D^{n+2}f(z)}{D^{n+1}f(z)} - 1 + \frac{zp'(z)}{p(z)+1} \right) \right| \leq 1 \tag{6}$$

holds true for $z \in U$, then the function f is univalent in U.

Proof. Let $f \in \mathcal{A}$ and p an analytic function with $p(0) = 1$. We will prove that there exists a real number $r \in (0, 1]$ such that the function

$$L(z,t) := f(e^{-t}z) + (e^{t}z - e^{-t}z)\frac{p(e^{-t}z)+1}{2}(D^{n}f)'(e^{-t}z) \tag{7}$$

is analytic in U_r for all $t \in [0, \infty)$, where $D^n f$ is Sălăgean differential operator defined in [7].

From (7) we observe that $L(z,t) = a_1(t)z + \cdots$, where

$$a_1(t) = e^{t}, \quad a_1(t) \neq 0, \text{ for } t \in [0,\infty) \text{ and } \lim_{t\to\infty} |a_1(t)| = \infty.$$

From the form of the chain $L(z, t)$ it follows that $L(\cdot, t)$ is regular in U for all $t \in [0, \infty)$ and $L(z, \cdot)$ is locally absolutely continuous on $[0, \infty)$ for all $z \in U$.

The limit function $g(z) = z$ belongs to the family $\{L(z, t)/a_1(t)\}$ then, in every closed disk U_r there exists a constant $K > 0$ such that

$$\left|\frac{L(z,t)}{a_1(t)}\right| < K, \quad \forall z \in U_r, \ t \in [0,\infty)$$

uniformly in this disk, provided that t is sufficiently large. Then, by Montel's Theorem, $\left\{\frac{L(z,t)}{a_1(t)}\right\}$ is a normal family in U_r.

Let $p(z, t)$ be the function defined by

$$p(z,t) = z\frac{\partial L(z,t)}{\partial z} / \frac{\partial L(z,t)}{\partial t}.$$

In order to prove that $p \in \mathcal{P}$, we will show that the function

$$w(z,t) := \frac{p(z,t)-1}{p(z,t)+1}$$

is analytic in U and $|w(z, t)| < 1$ for all $z \in U$ and $t \in [0, \infty)$. After a simple computation we obtain

$$w(z,t) = e^{-2t}A(z,t) + (1 - e^{-2t})B(z,t), \tag{8}$$

where

$$A(z,t) = \frac{2}{p(e^{-t}z)+1} \cdot \frac{zf'(e^{-t}z)}{D^{n+1}f(e^{-t}z)} - 1 \tag{9}$$

and

$$B(z,t) = e^{-t}\left(\frac{D^{n+2}f(e^{-t}z)}{D^{n+1}f(e^{-t}z)} - 1 + \frac{zp'(e^{-t}z)}{p(e^{-t}z)+1}\right), \quad z \in U, \ t \in [0,\infty). \tag{10}$$

From (5) and (6) we deduce that $w(z, t)$ is analytic in U.

In view of (5), (9)and (10), we have

$$|w(z,0)| = |A(z,0)| < 1 \text{ and } |w(0,t)| < 1.$$

Since $|e^{-t}z| \le e^{-t} < 1$ for $t > 0$ fixed and $z \in U,\ z \neq 0$, then $w(z,t)$ is analytic in $\bar{U} = \{z \in \mathbb{C} : |z| \le 1\}$. Making use of the maximum modulus principle we know that there exists $\theta = \theta(t) \in \mathbb{R}$ such that

$$|w(z,t)| = \max_{|\zeta|=1} |w(\zeta,t)| = |w(e^{i\theta},t)|,\ \forall z \in U.$$

Let us denote $u = e^{-t}e^{i\theta}$. Then $|u| = e^{-t}$ and, because $u \in U$, we get that $|w(e^{i\theta},t)| \le 1$.

From the above relations we conclude that $|w(z,t)| < 1$ for all $z \in U$ and $t \in [0,\infty)$ which means that $p(z,t)$ is regular in U and $\Re p(z,t) > 0$ for all $t \in [0,\infty)$ and $z \in U$.

Therefore, in view of Lemma 1, $L(z,t)$ is a Loewner chain and hence the function $L(z,0) = f(z)$ is univalent in U. □

By setting $n = 0$ in Theorem 1 we obtain the following corollary due to Lewandowski [4].

Corollary 1. *[4] Let $f \in \mathcal{A}$ and $p \in \mathcal{P}$. If*

$$\left|\frac{1-p(z)}{1+p(z)}|z|^2 + \left(1-|z|^2\right)\left(\frac{zf''(z)}{f'(z)} + \frac{zp'(z)}{p(z)+1}\right)\right| \le 1,\ z \in U,$$

then the function f is univalent in U..

For $p \equiv 1$ the following criterion reduces to a well-known criterion found by Becker [1] and Duren et al. [2].

Corollary 2. *[1] Let $f \in \mathcal{A}$. If*

$$\left(1-|z|^2\right)\left|\frac{zf''(z)}{f'(z)}\right| \le 1,\ z \in U,$$

then the function f is univalent in U..

For $n = 1$, Theorem 1 yields

Corollary 3. *Let $f \in \mathcal{A}$ and p an analytic function with $p(0) = 1$. If the inequalities*

$$\left|\frac{2}{p(z)+1} \cdot \frac{f'(z)}{f'(z)+zf''(z)} - 1\right| \le 1$$

and

$$\left|\left(\frac{2}{p(z)+1} \cdot \frac{f'(z)}{f'(z)+zf''(z)} - 1\right)|z|^2 \right.$$

$$\left. + \left(1-|z|^2\right)\left(\frac{2zf''(z)+z^2f'''(z)}{f'(z)+zf''(z)} + \frac{zp'(z)}{p(z)+1}\right)\right| \le 1$$

holds true for $z \in U$ then the function f is univalent in U..

For the Loewner chain

$$L(z,t) := f(e^{-t}z) + (e^t z - e^{-t}z) \cdot \frac{p(e^{-t}z)+1}{2} \cdot \frac{D^{n+1}f(e^{-t}z)}{D^n f(e^{-t}z)}, \quad z \in U, t \in [0,\infty),$$

following the same steps as in the proof of Theorem 1, we obtain:

Theorem 2. *Let $f \in \mathcal{A}$ and p an analytic function with $p(0) = 1$. If the inequalities*

$$\left| \frac{2}{p(z)+1} \cdot f'(z) \frac{D^n f(z)}{D^{n+1} f(z)} - 1 \right| \leq 1$$

and

$$\left| \left(\frac{2}{p(z)+1} \cdot f'(z) \frac{D^n f(z)}{D^{n+1} f(z)} - 1 \right) |z|^2 \right.$$

$$\left. + (1-|z|^2) \left(\frac{D^{n+2} f(z)}{D^{n+1} f(z)} - \frac{D^{n+1} f(z)}{D^n f(z)} + \frac{zp'(z)}{p(z)+1} \right) \right| \leq 1$$

holds true for $z \in U$, then the function f is univalent in U.

Setting $n = 0$ in previous theorem we obtain the following result:

Corollary 4. *Let $f \in \mathcal{A}$ and p an analytic function with $p(0) = 1$. If the inequalities*

$$\left| \frac{2}{p(z)+1} \cdot \frac{f(z)}{z} - 1 \right| \leq 1$$

and

$$\left| \left(\frac{2}{p(z)+1} \cdot \frac{f(z)}{z} - 1 \right) |z|^2 \right.$$

$$\left. + (1-|z|^2) \left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} + \frac{zp'(z)}{p(z)+1} \right) \right| \leq 1$$

holds true for $z \in U$, then the function f is univalent in U.

For $p \equiv 1$ in previous corollary we obtain Corollary 3.5 due to Kanas and Lecko [3].

Setting $p(z) = \dfrac{f(z)}{z}$ we obtain:

Corollary 5. *Let $f \in \mathcal{A}$ with $\Re \dfrac{f(z)}{z} > 0$. If the inequality*

$$\left| \left(\frac{f(z)}{z} - 1 \right) |z|^2 + (1-|z|^2) \left[1 + \frac{zf''(z)}{f'(z)} \left(\frac{f(z)}{z} + 1 \right) - \frac{zf'(z)}{f(z)} \right] \right| \leq \left| \frac{f(z)}{z} + 1 \right|$$

holds true for $z \in U$, then the function f is univalent in U.

Now, setting $p(z) = \dfrac{zf'(z)}{f(z)}$ in Corollary 4, we obtain the following result:

Corollary 6. *Let $f \in \mathcal{A}$. If the inequalities*

$$\left|2\frac{f(z)}{z} - \frac{zf'(z)}{f(z)} - 1\right| \leq \left|1 + \frac{zf'(z)}{f(z)}\right|$$

and

$$\left|\left(2\frac{f(z)}{z} - \frac{zf'(z)}{f(z)} - 1\right)|z|^2\right.$$

$$\left.+(1-|z|^2)\left(2\frac{zf'(z)}{f(z)+1}\right)\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right| \leq \left|1 + \frac{zf'(z)}{f(z)}\right|$$

holds true for $z \in U$, then the function f is univalent in U.

2.2 Univalence criteria connected with Ruscheweyh operator

For a function $f \in \mathcal{A}$ Ruscheweyh (see [6]) introduced the operator $R^\lambda : \mathcal{A} \to \mathcal{A}$ defined by

$$R^\lambda f(z) = \frac{1}{(1-z)^{\lambda+1}} * f(z), \quad \lambda > -1,\ z \in U.$$

In particular, for $\lambda = n$, we have

$$R^n(z) = \frac{z}{n!}\frac{\mathrm{d}^n}{\mathrm{d}z^n}\left\{z^{n-1}f(z)\right\}, \quad n \in \ltimes,\ z \in U.$$

Following the same steps as in the proof of Theorem 1 for the Loewner chain:

$$L(z,t) := f(e^{-t}z) + (e^t z - e^{-t}z)\frac{p(e^{-t}z)+1}{2}(R^n f)'(e^{-t}z),\ z \in U,\ t \in [0,\infty), \quad (11)$$

and using the well known condition:

$$z(R^n f(z))' = (n+1)R^{n+1}f(z) - nR^n f(z),$$

we obtain the following theorem:

Theorem 3. *Let $f \in \mathcal{A}$ and $p \in \mathcal{P}$. If the inequalities*

$$\left|\frac{2}{p(z)+1} \cdot \frac{zf'(z)}{(n+1)R^{n+1}f(z) - nR^n f(z)} - 1\right| \leq 1$$

and

$$\left|\left(\frac{2}{p(z)+1} \cdot \frac{zf'(z)}{(n+1)R^{n+1}f(z) - nR^n f(z)} - 1\right)|z|^2\right.$$

$$\left.+(1-|z|^2)\left[(n+1)\left(\frac{(n+2)R^{n+2}f(z) - (n+1)R^{n+1}f(z)}{(n+1)R^{n+1}f(z) - nR^n f(z)} - 1\right) + \frac{zp'(z)}{p(z)+1}\right]\right| \leq 1$$

holds true for $z \in U$, then the function f is univalent in U.

For $n = 0$ in Theorem 3 we obtain the result in Corollary 1 and for $n = 0$ and $p = 1$ we obtain the result in Corollary 2.

By setting $n = 1$ in Theorem 3, we have

Corollary 7. *Let $f \in \mathcal{A}$ and let p be an analytic function with $p(0) = 1$. If the inequalities*

$$\left|\frac{2}{p(z)+1} \cdot \frac{zf'(z)}{zf'(z)+z^2f''(z)} - 1\right| \leq 1$$

and

$$\left|\left(\frac{2}{p(z)+1} \cdot \frac{zf'(z)}{zf'(z)+z^2f''(z)} - 1\right)|z|^2\right.$$

$$\left.+(1-|z|^2)\left[\frac{zp'(z)}{p(z)+1} + \frac{2zf'(z)+4z^2f''(z)+z^3f'''(z)}{zf'(z)+z^2f''(z)} - 2\right]\right| \leq 1$$

holds true for $z \in U$, then the function f is univalent in U.

For the Loewner chain

$$L(z,t) := f(e^{-t}z) + (e^t z - e^{-t}z) \cdot \frac{p(e^{-t}z)+1}{2} \cdot \frac{R^{n+1}f(e^{-t}z)}{R^n f(e^{-t}z)}, \quad z \in U, t \in [0,\infty),$$

we obtain:

Theorem 4. *Let $f \in \mathcal{A}$ and let p be an analytic function with $p(0) = 1$. If the inequalities*

$$\left|\frac{2}{p(z)+1} \cdot f'(z)\frac{R^n f(z)}{R^{n+1}f(z)} - 1\right| \leq 1$$

and

$$\left|\left(\frac{2}{p(z)+1} \cdot f'(z)\frac{R^n f(z)}{R^{n+1}f(z)} - 1\right)|z|^2\right.$$

$$\left.+(1-|z|^2)\left((n+2)\frac{R^{n+2}f(z)}{R^{n+1}f(z)} - (n+1)\frac{R^{n+1}f(z)}{R^n f(z)} - 1 + \frac{zp'(z)}{p(z)+1}\right)\right| \leq 1$$

holds true for $z \in U$, then the function f is univalent in U.

For $n = 0$ we obtain the result in Corollary 4.

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