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DISTINGUISHED CURVATURES IN EXTENDED RELATIVISTIC DYNAMICS

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Abstract

In this paper we present the distinguished (d-) curvatures for a Lagrangian inspired by relativistic optics in non-uniform media.

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1 Introduction

In geometrical optics [5], a special role is played by the Synge-Beil metric (see [1], [2], [4], [7]–[11])

$$g_{\alpha\beta}(x, y) = \varphi_{\alpha\beta}(x) + \gamma^2 y_\alpha y_\beta, \tag{1}$$

where $\gamma(x) \geq 0$ is a positive smooth function on the space-time M^4, and $\varphi_{\alpha\beta}(x)$ is a pseudo-Riemannian metric on M^4. One assumes that the manifold M^4 (which is connected, simply connected and has $\dim M^4 = 4$) is endowed with the local coordinates $(x^\alpha)_{\alpha=\overline{1,4}} = \left(x^1 = t, x^2, x^3, x^4\right)$; for simplicity we use the system of units where the light velocity is $c = 1$. Obviously, the following rule holds: $y_\alpha = \varphi_{\alpha\mu} y^\mu$. Since the components of $\varphi_{\alpha\beta}(x)$ are dimensionless, the same are γy_α; so we have $[\varphi_{\alpha\beta}(x)] = 1,\ [\gamma y_\alpha] = 1$.

In such a context, let us restrict our geometric-physical study to the Minkowski manifold $\mathcal{M}^4 = \left(\mathbb{R}^4, \eta_{ij}\right)$ which has the local coordinates $(x) := \left(x^i\right)_{i=\overline{1,4}}$. It follows that the dimension of the corresponding tangent bundle $T\mathbb{R}^4$ is equal to eight, and its local coordinates are[3]

$$(x, y) := (x^i, y^i)_{i=\overline{1,4}} = (\underbrace{x^1, x^2, x^3, x^4}_{\text{space-time coordinates}}, \underbrace{y^1, y^2, y^3, y^4}_{\text{tangent vector}}).$$

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[3]In this paper the Latin letters i, j, k, ... run from 1 to 4. The Einstein convention of summation is adopted all over this work.

Emerging from formula (1), we introduce the following metric on $T\mathbb{R}^4$, which is inspired by the J.L. Synge optics framework for the non-uniform medium:

$$\mathfrak{g}_{ij}(x,y) = \eta_{ij} + \gamma^2(x)y_i y_j, \tag{2}$$

where $\eta = (\eta_{ij}) = \text{diag}\ (-1,1,1,1)$ is the Minkowski metric, and $y_i = \eta_{ir}y^r$. Usually, we have $\gamma^2(x) = n^2(x) - 1$, where $n = n(x)$ is the refractive index of the non-uniform medium (see [2], [7]–[10]). Using the metric (2), in what follows we will examine the special case of a possible anisotropic relativistic dynamical model (suggested in private discussions by physicist V.M. Red'kov), which is governed by the Lagrangian (in this model one considers that the particle has the mass $m = 1$) (see also [9])

$$\begin{aligned} L(x,y) &= \frac{1}{2}\mathfrak{g}_{ij}(x,y)y^i y^j \\ &= \frac{1}{2}(\eta_{ij} + \gamma^2 y_i y_j)y^i y^j \\ &= \frac{1}{2}\eta_{ij}y^i y^j + \frac{\gamma^2}{2}\|y\|^4, \end{aligned} \tag{3}$$

where $\|y\|^2 = -(y^1)^2 + (y^2)^2 + (y^3)^2 + (y^4)^2 = \eta_{ij}y^i y^j$.

Remark 1. *Suppose that the refractive index $n(x)$ is invariant with respect to Lorentz transformations. Since the Minkowski metric η_{ij} is invariant with respect to the linear transformations of coordinates induced by the Lorentz group $O(3,1)$, it immediately follows that the Lagrangian (3) has a global geometrical character with respect to these Lorentz transformations.*

Remark 2. *A similar 3-dimensional anisotropic non-relativistic Lagrangian in which the Minkowski metric $(\eta_{ij})_{i,j=\overline{1,4}}$ is replaced with Euclidian metric $(\delta_{ij})_{i,j=\overline{1,3}}$ is studied by Neagu, Oana and Red'kov in paper [10]. That Lagrangian is invariant with respect to the orthogonal group $O(3)$ and governs the **non-relativistic extended dynamics**.*

Following the geometrical ideas from Lagrangian geometry of tangent bundles [6] or jet bundles [3], we further construct the pseudo-Riemann-Lagrange geometrical objects, such as the canonical nonlinear connection, the Cartan canonical linear connection, together with its d-torsions and d-curvatures, naturally associated with the Lagrangian (3).

2 Geometrical objects in relativistic extended dynamics

The Lagrangian (3) produces the fundamental metrical distinguished tensor

$$g_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j} = \sigma(x,y)\eta_{ij} + 2\gamma^2(x)y_i y_j,$$

where $\sigma(x,y) = (1/2) + \gamma^2(x)||y||^2$. Working on the domains of $T\mathbb{R}^4$ in which $\sigma(x,y) \neq 0$ and $\tau(x,y) = (1/2) + 3\gamma^2(x)||y||^2 \neq 0$, then the inverse matrix $[g^{-1}] = (g^{jk})_{j,k=\overline{1,4}}$ has the components

$$g^{jk}(x,y) = \frac{1}{\sigma(x,y)}\eta^{jk} - \frac{2\gamma^2(x)}{\sigma(x,y)\cdot\tau(x,y)}y^j y^k,$$

where $\eta^{jk} = \eta_{jk}$.

Following the Miron and Anastasiei geometrical ideas from the book [6], we deduce that, for the anisotropic Lagrangian (3), the associated *canonical nonlinear connection* $N = \left(N^i_j\right)$ on the tangent bundle $T\mathbb{R}^4$ has the local components (these are computed in [9])

$$\begin{aligned} N^i_j = & \ \frac{2\gamma}{\sigma}y^i y_j \gamma_0 + \frac{\gamma||y||^2}{\sigma}\left(\delta^i_j\gamma_0 + y^i\gamma_j - \gamma^i y_j - \frac{2\gamma^2}{\sigma}y^i y_j\gamma_0 - \frac{6\gamma^2}{\tau}y^i y_j \gamma_0\right) \\ & + \frac{\gamma^3||y||^4}{2\sigma}\left[\frac{1}{\sigma}\gamma^i y_j - \frac{3}{\tau}y^i\gamma_j - \frac{3}{\tau}\delta^i_j\gamma_0 + \frac{6\gamma^2}{\sigma\tau^2}\left(\tau + 3\sigma\right)y^i y_j \gamma_0\right], \end{aligned} \tag{4}$$

where $\gamma_s = \partial\gamma/\partial x^s, \gamma_{rs} = \partial^2\gamma/\partial x^r\partial x^s$ and $\gamma^i = \eta^{ir}\gamma_r, \gamma_0 = \gamma_r y^r$.

Remark 3. *In a uniform medium with the constant refractive index $n(x) = n \in [1,\infty)$, we have $\gamma_s = 0$. Consequently, in this case we have $N^i_j = 0$.*

The nonlinear connection (4) produces the dual *adapted bases* of *d*-vector fields

$$\left\{\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^r_i\frac{\partial}{\partial y^r}\ ;\ \frac{\partial}{\partial y^i}\right\} \subset \mathcal{X}(T\mathbb{R}^4) \tag{5}$$

and of *d*-covector fields

$$\left\{dx^i\ ;\ \delta y^i = dy^i + N^i_r dx^r\right\} \subset \mathcal{X}^*(T\mathbb{R}^4). \tag{6}$$

The description of all geometrical objects on the tangent space $T\mathbb{R}^4$ (e.g., the Cartan canonical linear connection and its torsion and curvature) will be made in local adapted components, with respect to the adapted bases (5) and (6).

For instance, by using the derivative operators (5) and the notations $N_{ij} := N^r_i\eta_{rj}$, $N^i_0 = N^i_r y^r, N_{i0} := N_{ir}y^r$, $N_{0j} := N_{rj}y^r$, $N_{00} := N_{ij}y^i y^j$, by direct local computations, we find the adapted local components of the Cartan canonical connection $C\Gamma(N) = (L^i_{jk},\ C^i_{jk})$ associated with the Lagrangian (3), which are computed in [9]:

$$\begin{aligned} L^i_{jk} \ = \ & -\frac{\gamma}{\sigma}\left[\gamma\left(\delta^i_j N_{k0} + \delta^i_k N_{j0} - \eta^{ir}\eta_{jk}N_{r0}\right) + ||y||^2\left(\eta_{jk}\gamma^i - \delta^i_j\gamma_k - \delta^i_k\gamma_j\right) + \right. \\ & + \gamma\left\{\left(N_{jk} + N_{kj}\right)y^i + \left(N^i_k - \eta^{ir}N_{rk}\right)y_j + \left(N^i_j - \eta^{ir}N_{rj}\right)y_k\right\} + \\ & \left. + 2\left(\gamma^i y_j y_k - y^i y_j\gamma_k - y^i y_k\gamma_j\right)\right] + \frac{2\gamma^3 y^i}{\sigma\tau}\left[\gamma\left(y_j N_{k0} + y_k N_{j0} - \eta_{jk}N_{00}\right) + \right. \\ & + ||y||^2\left(\delta_{jk}\gamma_r y^r - y_j\gamma_k - y_k\gamma_j\right) + 2\left(y_j y_k\gamma_r y^r - y_j\gamma_k||y||^2 - y_k\gamma_j||y||^2\right) + \\ & \left. + \gamma\left\{\left(N_{jk} + N_{kj}\right)||y||^2 + \left(N_{k0} - N_{0k}\right)y_j + \left(N_{j0} - N_{0j}\right)y_k\right\}\right], \end{aligned} \tag{7}$$

$$C^i_{jk} = \frac{\gamma^2}{\sigma}\left(y^i\eta_{jk} + \delta^i_j y_k + \delta^i_k y_j\right) - \frac{2\gamma^4}{\sigma\tau}\left(||y||^2\eta_{jk} + 2y_j y_k\right) y^i. \qquad (8)$$

Moreover, the local components of the torsion tensor of the Cartan canonical N-linear connection produced by the anisotropic optical Lagrangian (3) are given by R^i_{jk}, P^i_{jk}, C^i_{jk}, where (these are also computed in [9])

$$\begin{aligned} R^i_{jk} \;=\; & y^i\left(y_j\gamma_{0k} - y_k\gamma_{0j}\right)\varphi + (\delta^i_j\gamma_{0k} - \delta^i_k\gamma_{0j})\varepsilon + \eta^{is}(\gamma_{sk}y_j - \gamma_{sj}y_k)\omega \\ & + y^i(y_j\gamma_k - \gamma_j y_k)(\gamma_s\gamma^s)[\tfrac{7\varphi}{\gamma} - \tfrac{6}{\tau^2} - 18\varphi\varepsilon + \varphi^2\left\|y\right\|^2 + 2\varphi\omega - \varepsilon\phi\left\|y\right\|^2] \\ & + (\delta^i_j\gamma_k - \delta^i_k\gamma_j)(\gamma_s\gamma^s)[\tfrac{2\varepsilon}{\gamma} - \tfrac{\|y\|^2}{\sigma\tau} - 9\varepsilon^2 - \varphi\varepsilon\left\|y\right\|^2] \\ & + \gamma^i(y_j\gamma_k - \gamma_j y_k)[\tfrac{\omega}{\gamma} + \tfrac{2\|y\|^2}{\sigma^2} + \tfrac{2\omega}{\sigma\tau} + \tfrac{\varepsilon\gamma\|y\|^2}{2\sigma^3} + \omega^2] \\ & - (\delta^i_j y_k - \delta^i_k y_j)(\gamma_s\gamma^s)^2\varphi(\varphi\left\|y\right\|^2 + \omega + \varepsilon) - (\delta^i_j y_k - \delta^i_k y_j)(\gamma_s\gamma^s)\omega\varepsilon, \end{aligned}$$

$$\begin{aligned} P^i_{jk} \;=\; & y^i y_j y_k\gamma_0[12\varphi\gamma^2\tau(\tfrac{1}{\sigma} - \sigma^2) - \tfrac{9\gamma^3}{\sigma^2\tau} - \tfrac{4\gamma^3}{\sigma\tau}] + (\delta^i_j y_k + \delta^i_k y_j)\gamma_0\varphi \\ & + y^i(y_j\gamma_k + \gamma_j y_k)[\varphi - \tfrac{2\gamma}{\sigma} + \tfrac{6\gamma^3\|y\|^2}{\sigma\tau}] + y^i\eta_{jk}\gamma_0[\varphi - \tfrac{2\gamma^3\|y\|^2}{\sigma\tau}] \\ & + (\delta^i_j\gamma_k + \delta^i_k\gamma_j)(-\tfrac{3\gamma^3\|y\|^4}{2\sigma\tau}) + \gamma^i\eta_{jk}\tfrac{\gamma^3\|y\|^4}{2\sigma^2} + \gamma^i y_j y_k\tfrac{\gamma(4\sigma^2-1)}{2\sigma^3} \\ & + (\delta^i_j N_{k0} + \delta^i_k N_{j0} - \eta^{ii}\eta_{jk}N_{i0})\tfrac{\gamma^2}{\sigma} - y^i(N_{kj} + N_{jk})(\tfrac{2\gamma^4}{\sigma\tau} - \tfrac{\gamma^2}{\sigma}) \\ & + [(N^i_j - \eta^{ii}N_{ij})y_k + ((N^i_k - \eta^{ii}N_{ik})y_j)]\tfrac{\gamma^2}{\sigma} \\ & - (y^i N_{j0}y_k + y^i y_j N_{k0})\tfrac{4\gamma^4}{\sigma\tau} + (y^i\eta_{jk}N_{00} + y^i y_j N_{0k} + y^i N_{0j}y_k)\tfrac{2\gamma^4}{\sigma\tau}, \end{aligned}$$

where

$$\alpha = 12\varphi\gamma^2\tau\,||y||^2\left(\tfrac{1}{\sigma} - \sigma^2\right) - \tfrac{9\gamma^3||y||^2}{\sigma^2\tau} + 2\varphi - \tfrac{\gamma}{2\sigma^3},$$

$$\varepsilon = \tfrac{\gamma\|y\|^2(2\tau+1)}{4\sigma\tau}, \varphi = \tfrac{\gamma(12\sigma^2-6\sigma+1)}{4\sigma^2\tau^2},$$

$$\phi = \tfrac{\gamma^3(12\sigma-3)}{\sigma^3\tau^2} - \tfrac{12\sigma^2\gamma^2\varphi}{\tau}, \omega = -\tfrac{\gamma\|y\|^2(2\sigma+1)}{4\sigma^2}.$$

In the sequel, note that the local components of the curvature tensor of a general Cartan canonical N-linear connection, are given by the general formulas (see Miron-Anastasiei's book [6])

$$\begin{aligned} R^i_{jkl} &= \frac{\delta L^i_{jk}}{\delta x^l} - \frac{\delta L^i_{jl}}{\delta x^k} + L^r_{jk}L^i_{rl} - L^r_{jl}L^i_{rk} + C^i_{jr}R^r_{kl} \\ P^i_{jkl} &= \frac{\partial L^i_{jk}}{\partial y^l} - C^i_{jl|k} + C^i_{jr}P^r_{kl}, \\ S^i_{jkl} &= \frac{\partial C^i_{jk}}{\partial y^l} - \frac{\partial C^i_{jl}}{\partial y^k} + C^r_{jk}C^i_{rl} - C^r_{jl}C^i_{rk}, \end{aligned}$$

where

$$C^i_{jl|k} \stackrel{def}{=} \frac{\delta C^i_{jl}}{\delta x^k} + C^r_{jl}L^i_{rk} - C^i_{rl}L^r_{jk} - C^i_{jr}L^r_{lk}.$$

Consequently, as a novelty of this paper, using the formulas (4), (7), (8) and the derivative operators (5), after very laborious and complicated calculations, we get that the Cartan canonical N-linear connection produced by the anisotropic optical Lagrangian (3) is characterized by *three* effective local curvature d-tensors, namely $R^i_{jkl},\ P^i_{jkl},\ S^i_{jkl}$, where

$$\begin{aligned} R^i_{jkl} = {} & y^iy_j\left(y_k\gamma_l - \gamma_ky_l\right)\left\{(\gamma_0)^2\left[\left(\omega+\varepsilon+\varphi\left\|y\right\|^2\right)\mathcal{T}_2 - \varepsilon\mathcal{T}_1\right] + N_{00}\tfrac{4\gamma^5\left(4\gamma^3-1\right)}{\sigma^2\tau^2}\right\} \\ & +y^iy_j\left(\gamma_ky_l - y_k\gamma_l\right)\gamma_0\left[2\left(\varphi-6\gamma^2\right)\tfrac{\gamma}{\sigma\tau} + (\omega-2\varepsilon)\mathcal{T}_3 + \tfrac{\gamma^2}{\sigma}\left(\tfrac{4\gamma}{\tau}+\varphi\right)(\omega-3\varepsilon) - \right. \\ & -\mathcal{A} - 16\varphi\gamma^2 - \tfrac{8\varphi}{\sigma\tau^2} - \tfrac{4\gamma^4}{\sigma\tau}\left(\mathcal{A}\left\|y\right\|^2 + \mathcal{B} + 2\mathcal{C}\right) + \mathcal{R}_1\tfrac{\gamma^2(\tau-\sigma)}{\sigma\tau} + \varphi\mathcal{T}_7 \\ & \left. + \left(\mathcal{R}_1\left\|y\right\|^2 + \mathcal{R}_2 + \mathcal{R}_3\right)\tfrac{4\gamma^4}{\sigma\tau}\right] \\ & +y^iy_j\left(y_k\gamma_{0l} - \gamma_{0k}y_l\right)\left[\varphi\tfrac{\sigma\tau+\gamma^2(\sigma+\tau)}{\sigma\tau} + \tfrac{4\gamma^3}{\sigma\tau}(1+\omega) - \tfrac{2\gamma^2\sigma(2\tau+1)}{\sigma^2\tau^2}\right] \\ & +y^iy_j\left(N_{k0}N_{0l} - N_{0k}N_{l0}\right)\tfrac{8\gamma^6\left(\gamma^2-\tau\right)}{\sigma^2\tau^2} \\ & +y^iy_j\left(N_{0k}y_l - y_kN_{0l}\right)\gamma_0\left[\left(\varphi\left\|y\right\|^2+\omega+\varepsilon\right)\tfrac{4\gamma^6(2\tau+1)}{\sigma^2\tau^2} - \varphi\tfrac{6\gamma^4}{\sigma\tau}\right. \\ & \left. +\tfrac{4\gamma^5\left(\sigma\tau-\left\|y\right\|^2+2\gamma^2-\tau\right)}{\sigma^2\tau^2}\right] \\ & +y^iy_j\left(N_{0k}\gamma_l - \gamma_kN_{0l}\right)\left[8\varphi\gamma^3 - \tfrac{4\gamma^2}{\sigma\tau^2} - \tfrac{8\gamma^3}{\sigma\tau} + \varepsilon\tfrac{6\gamma^4}{\sigma\tau} - \varepsilon\left\|y\right\|^2\tfrac{4\gamma^6(2\tau+1)}{\sigma^2\tau^2} + \tfrac{2\gamma^5}{\sigma^2\tau^2}\right] \\ & +y^iy_j\left(N_{k0}\gamma_l - \gamma_kN_{l0}\right)\left\{\tfrac{16\gamma^3}{\sigma\tau} - 16\varphi\gamma^3 + \tfrac{8\gamma^2}{\sigma\tau^2} - \varepsilon\tfrac{12\gamma^4}{\sigma\tau} + \varepsilon\tfrac{8\gamma^6\left\|y\right\|^2(2\tau+1)}{\sigma^2\tau^2}\gamma_0\right. \\ & \left. -\tfrac{\gamma^3(1-\sigma)}{\sigma^2\tau} - \tfrac{\gamma^3\left[(4\sigma-1)\tau-2\gamma^2+16\gamma^5\left\|y\right\|^2\right]}{\sigma^2\tau^2}\right\} - y^iy_j\left(N_{k0}y_l - y_kN_{l0}\right)N_{00}\tfrac{16\gamma^8}{\sigma^2\tau^2} \\ & +y^iy_j\left(N_{k0}y_l - y_kN_{l0}\right)\gamma_0\left\{\left(\varphi\left\|y\right\|^2+\omega+\varepsilon\right)\tfrac{8\gamma^6(2\tau+1)}{\sigma^2\tau^2} - \varphi\tfrac{12\gamma^4}{\sigma\tau}\right. \\ & \left. +\tfrac{\gamma^5\left[\left(4+27\gamma^2\right)\left\|y\right\|^2+8\sigma\right]}{\sigma^2\tau^2} - \tfrac{4\gamma^5}{\sigma^2\tau}\right\} \\ & +y^iy_j\left(N_{rk}\gamma^ry_l - y_kN_{rl}\gamma^r\right)\omega\tfrac{2\gamma^4}{\sigma\tau} - y^iy_j\left(N_{kr}\gamma^ry_l - y_kN_{lr}\gamma^r\right)\omega\tfrac{4\gamma^4}{\sigma\tau} \\ & +y^iy_j\left(N_{rk}y_l - y_kN_{rl}\right)\left[\gamma^r\tfrac{\gamma^3\left(4\gamma^2-2\tau-1\right)}{\sigma^2\tau} + \eta^{rr}N_{r0}\tfrac{4\gamma^6}{\sigma^2\tau}\right] \\ & +y^iy_j\left(N_{kr}y_l - y_kN_{lr}\right)\gamma^r\tfrac{2\gamma^3\left(2\gamma^2-\tau\right)}{\sigma^2\tau} + y^iy_j\left(N_{rk}N_{lr} - N_{kr}N_{rl}\right)\eta^{rr}\tfrac{\gamma^4\left(2\gamma^2-\tau\right)}{\sigma^2\tau} \\ & +y^iy_j\left[\left(N_{rk}+N_{kr}\right)N^r_l - \left(N_{rl}+N_{lr}\right)N^r_k\right]\tfrac{\gamma^4\left(2\gamma^2-\tau\right)}{\sigma^2\tau} \end{aligned}$$

$$+y^i y_j\left(N^r_k y_l - y_k N^r_l\right)\left(\gamma^r\frac{\gamma^3}{\sigma^2\tau} + N_{r0}\frac{4\gamma^6}{\sigma^2\tau}\right)$$
$$+y^i y_j\left(N_{kl} - N_{lk}\right)\left(\frac{4\gamma^6 N_{00}}{\sigma^2\tau} - \gamma_0\varepsilon\frac{6\gamma^4}{\sigma\tau}\right)$$
$$+y^i\gamma_j\left(y_k\gamma_l - \gamma_k y_l\right)\Big[\mathcal{C}\frac{2\gamma^4\left(1-\|y\|^2\right)}{\sigma\tau} - \frac{18\gamma^2\|y\|^2}{\sigma\tau} - 24\varphi\gamma^2\|y\|^2 + \frac{12\gamma\|y\|^2}{\sigma\tau^2}$$
$$+\varepsilon\frac{\gamma^2(\varepsilon-\omega)}{\sigma} + \mathcal{R}_3\frac{\gamma^2}{\tau} + \frac{2(1-\sigma)}{\sigma^2} + \frac{\gamma^2\|y\|^2}{\sigma^2\tau^2} + \mathcal{T}_7(2\omega-\varepsilon) - \mathcal{T}_3\varepsilon\|y\|^2 - \frac{2\gamma(\varepsilon-\omega)}{\sigma\tau}\Big]$$
$$+y^i\gamma_j\left(N_{0k}y_l - y_k N_{0l}\right)\left[\omega\frac{2\gamma^4}{\sigma\tau} + \frac{\gamma^3\left(2\gamma^2-\sigma\right)}{\sigma^2\tau^2}\right]$$
$$+\left(y^i\gamma_{jl}y_k - \gamma_{jk}y_l\right)\left[\frac{2\gamma^4\omega\left(1-\|y\|^2\right)}{\sigma\tau} + \frac{\gamma}{\sigma\tau} + \omega\frac{\gamma^2}{\tau}\right]$$
$$-y^i\gamma_j\left(N_{k0}y_l - y_k N_{l0}\right)\left[\omega\frac{4\gamma^4}{\sigma\tau} + \frac{\gamma^3\left(7\sigma-3-2\gamma^2+16\gamma^5\|y\|^2\right)}{\sigma^2\tau^2}\right]$$
$$+y^i N_{j0}\left(y_k\gamma_l - \gamma_k y_l\right)\Big[\frac{16\gamma^3}{\sigma\tau} - 16\varphi\gamma^3 + \frac{8\gamma^2}{\sigma\tau^2} - \varepsilon\frac{12\gamma^4}{\sigma\tau} + \varepsilon\frac{8\gamma^6\|y\|^2(2\tau+1)}{\sigma^2\tau^2}\gamma_0$$
$$+\omega\frac{2\gamma^4}{\sigma\tau} + \frac{\gamma^3(\sigma+\tau)}{\sigma\tau^2}\Big]$$
$$+y^i N_{j0}\left(N_{k0}y_l - y_k N_{l0}\right)\frac{8\gamma^6\left(\tau-\gamma^2\right)}{\sigma^2\tau} - y^i N_{j0}\left(N_{0k}y_l - y_k N_{0l}\right)\frac{2\gamma^6\left(4\gamma^2-7\sigma+2\right)}{\sigma^2\tau^2}$$
$$+y^i N_{0j}\left(y_k N_{0l} - N_{0k}y_l\right)\frac{4\gamma^8\left(||y||^2-1\right)}{\sigma^2\tau^2} + y^i N_{0j}\left(y_k N_{l0} - N_{k0}y_l\right)\frac{2\gamma^6\left(2\sigma-2\gamma^2+\tau\right)}{\sigma^2\tau}$$
$$+y^i N_{0j}\left(y_k\gamma_l - \gamma_k y_l\right)\Big[8\varphi\gamma^3 + \frac{\gamma^2(\gamma-4)}{\sigma\tau^2} - \frac{8\gamma^3}{\sigma\tau} + \frac{2(3\varepsilon+\omega)\gamma^4}{\sigma\tau}$$
$$-\varepsilon\|y\|^2\frac{4\gamma^6(2\tau+1)}{\sigma^2\tau^2} + \frac{2\gamma^5}{\sigma^2\tau^2}\Big]$$
$$+y^i\left(N_{jk}y_l - y_k N_{jl}\right)\gamma_0\Big[\left(\varphi\|y\|^2 + \omega + \varepsilon\right)\mathcal{T}_5 + \varepsilon\frac{4\gamma^4}{\sigma\tau}$$
$$+\varphi\frac{\gamma^2\left(2\gamma^2-\tau\right)}{\sigma\tau} + \frac{2\gamma^3\tau\left(2\gamma^2-\tau\right)}{\sigma^2\tau^2}\Big]$$
$$+y^i\left(\eta_{jk}\gamma_l - \gamma_k\eta_{jl}\right)\gamma_0\Big[\mathcal{B} + \frac{6\gamma^2||y||^2}{\sigma\tau} - 8\varphi\gamma^2||y||^2 + \frac{4\gamma||y||^2}{\sigma\tau^2}$$
$$-\frac{2\gamma^4||y||^2}{\sigma\tau}\left(\mathcal{A}\,||y||^2 + \mathcal{B} + \mathcal{C}\right) - \varepsilon\frac{2\gamma^3||y||^2}{\sigma\tau} + \varepsilon\left(\omega + \varphi\,||y||^2\right)\frac{\gamma^2}{\sigma} - \varepsilon\,||y||^2\,\mathcal{T}_4$$
$$+\varepsilon\mathcal{T}_7 - \varepsilon\mathcal{T}_6 + \mathcal{R}_2\frac{\gamma^2}{\tau} + \frac{(2\sigma-1)(2\sigma-2\sigma\tau-1)}{2\sigma^2\tau^2}\Big]$$
$$+y^i\left(\eta_{jk}\gamma_{0l} - \gamma_{0k}\eta_{jl}\right)\left[\frac{\gamma^2}{\tau}\left(\omega + 2\varepsilon + \varphi\,||y||^2\right) + \frac{2\gamma^3||y||^2}{\sigma\tau} + \varepsilon\frac{\gamma^2\left(2\gamma^2-\tau\right)}{\sigma\tau}\right]$$
$$+y^i\left(\eta_{jk}\gamma_l - \gamma_k\eta_{jl}\right)N_{00}\Big\{8\varphi\gamma^3 - \frac{4\gamma^2(1+2\gamma\tau)}{\sigma\tau^2} + \varepsilon\frac{\gamma^4(6\sigma\tau-4\sigma+4)}{\sigma\tau} - 2\varepsilon\mathcal{T}_6$$
$$+\frac{\gamma^3[1-\sigma-\tau(4\sigma-2)]}{\sigma^2\tau^2}\Big\}$$
$$-y^i\left(\eta_{jk}y_l - y_k\eta_{jl}\right)\gamma_0 N_{00}\left[\left(\omega + 2\varepsilon + \varphi\,||y||^2\right)\frac{4\gamma^6(1+2\tau)}{\sigma^2\tau^2} - \frac{6\varphi\gamma^4}{\sigma\tau} + \frac{12\gamma^5\tau}{\sigma^2\tau^2}\right]$$
$$-y^i\left(\eta_{jk}y_l - y_k\eta_{jl}\right)(\gamma_0)^2\Big[\varphi\mathcal{T}_6 + \left(\omega + \varepsilon + \varphi\,||y||^2\right)\mathcal{T}_4 - \frac{\gamma^2}{\sigma}\left(\omega + \varphi\,||y||^2\right)$$
$$-\varepsilon\mathcal{T}_3 + \frac{2\varphi\gamma^3||y||^2}{\sigma\tau} + \mathcal{R}_4\frac{\gamma^2}{\tau} + \frac{12\gamma^4||y||^2(2\sigma-1)}{\sigma^2\tau^2}\Big]$$
$$-y^i\left(\eta_{jk}y_l - y_k\eta_{jl}\right)\gamma_0\left(\omega\mathcal{T}_6 + \omega\varepsilon\frac{\gamma^2}{\tau} + \frac{\gamma^2\|y\|^2}{\sigma^2\tau}\right)$$
$$+y^i\left(\eta_{jk}y_l - y_k\eta_{jl}\right)\left(N_{0r}\gamma^r\right)\left(\frac{2\gamma^4\omega}{\sigma\tau} + \frac{2\gamma^5\|y\|^2}{\sigma^2\tau}\right)$$
$$+y^i\left(\eta_{jk}y_l - y_k\eta_{jl}\right)\left(N_{r0}\gamma^r\right)\left[\omega\frac{2\gamma^4}{\sigma\tau} + \frac{2\gamma(3-4\sigma)-2\sigma+1}{2\sigma^2\tau}\right]$$

$$
\begin{aligned}
&+y^i\left(\eta_{jk}y_l-y_k\eta_{jl}\right)\left[\eta^{rr}\left(N_{r0}\right)^2\tfrac{4\gamma^6}{\sigma^2\tau}-\eta^{rr}N_{r0}N_{0r}\tfrac{2\gamma^6}{\sigma^2\tau}\right]\\
&+y^i\left(\eta_{jk}N_{0l}-N_{0k}\eta_{jl}\right)\gamma_0\left(\tfrac{2\gamma^4\varepsilon}{\sigma\tau}+\tfrac{4\gamma^7\|y\|^2}{\sigma^2\tau^2}\right)\\
&+y^i\left(\eta_{jk}N_{0l}-N_{0k}\eta_{jl}\right)N_{00}\tfrac{4\gamma^8\left(\|y\|^2-1\right)}{\sigma^2\tau^2}\\
&+y^i\left(\eta_{jk}N_{l0}-N_{k0}\eta_{jl}\right)\gamma_0\left(\tfrac{2\gamma^4\varepsilon}{\sigma\tau}+\tfrac{8\sigma\gamma^7\|y\|^2}{\sigma^2\tau^2}\right)\\
&+y^i\left(\eta_{jk}N_{l0}-N_{k0}\eta_{jl}\right)N_{00}\tfrac{4\gamma^6\left(\sigma+2\tau-\gamma^2\right)}{\sigma^2\tau^2}\\
&+y^i\left(\delta_{jk}y_l-\delta_{jl}y_k\right)\left(\gamma_0\right)^2\tfrac{4\gamma^3\varepsilon}{\sigma\tau}+y^i\left(\delta_{jk}\gamma_l-\delta_{jl}\gamma_k\right)\gamma_0\tfrac{\gamma\varepsilon}{\sigma\tau}\\
&+y^i\left[\eta_{jk}\left(N_{lr}+N_{rl}\right)-\eta_{jl}\left(N_{rk}+N_{kr}\right)\right]\left[\eta^{rr}N_{r0}\tfrac{\gamma^4\left(2\gamma^2-\tau\right)}{\sigma^2\tau}-\gamma^r\tfrac{\gamma^3||y||^2\left(2\gamma^2-\tau\right)}{\sigma^2\tau}\right]\\
&-y^i\left(\delta_{jk}N_{0l}-\delta_{jl}N_{0k}\right)\gamma_0\tfrac{2\varepsilon\gamma^4}{\sigma\tau}+y^i\left(\delta_{jk}N_{l0}-\delta_{jl}N_{k0}\right)\gamma_0\tfrac{4\gamma^4}{\sigma\tau}\\
&+y^i\left[\left(N_{jk}+N_{kj}\right)N_{l0}-\left(N_{jl}+N_{lj}\right)N_{k0}\right]\tfrac{\gamma^4\left(2\gamma^2-\tau\right)\left(2\gamma^2+\sigma-1\right)}{\sigma^2\tau^2}\\
&+y^i\left[\left(N_{jk}+N_{kj}\right)N_{0l}-\left(N_{jl}+N_{lj}\right)N_{0k}\right]\tfrac{\gamma^4\left(2\gamma^2-\tau\right)\left(2\gamma^2-5\sigma+2\right)}{\sigma^2\tau^2}\\
&+N^i_j\left(y_k\gamma_l-\gamma_ky_l\right)\left[\left(\varepsilon-\omega\right)\tfrac{\gamma^2}{\sigma}-\tfrac{2\gamma^4\varepsilon||y||^2}{\sigma^2}+\tfrac{2\gamma^3||y||^2}{\sigma^2}\right]\\
&-\left(N^i_ky_jy_l-N^i_ly_jy_k\right)\gamma_0\left[\left(\varphi\,||y||^2+\omega+\varepsilon\right)\tfrac{2\gamma^4}{\sigma^2}-\varphi\tfrac{\gamma^2}{\sigma}\right]\\
&+\left(N^i_ky_j\gamma_l-N^i_ly_j\gamma_k\right)\left(\tfrac{\gamma^3||y||^2(2\tau+1)}{\sigma^2\tau}-\tfrac{2\gamma^4\varepsilon||y||^2}{\sigma^2}\right)\\
&+\left(N^i_ky_jN_{l0}-N^i_ly_jN_{k0}\right)\tfrac{\gamma^4(\sigma-1)}{\sigma^2\tau}+\left(N^i_ky_jN_{0l}-N^i_ly_jN_{0k}\right)\tfrac{\gamma^4}{\sigma\tau}\\
&+\left(N^i_k\gamma_jy_l-N^i_l\gamma_jy_k\right)\left(\omega\tfrac{\gamma^2}{\sigma}+\tfrac{\gamma^3||y||^2}{\sigma^2\tau}\right)+\left(N^i_k\gamma_j\gamma_l-N^i_l\gamma_j\gamma_k\right)\left(\varepsilon\tfrac{\gamma^2}{\sigma}+\tfrac{2\gamma^3}{\sigma\tau}\right)\\
&+\left(N^i_kN_{j0}y_k-N^i_lN_{j0}y_l\right)\left[\tfrac{\gamma^2}{\sigma^2}+\tfrac{\gamma^4(\sigma-1)}{\sigma^2\tau}\right]+\left(N^i_kN_{j0}y_k-N^i_lN_{j0}y_l\right)\tfrac{\gamma^4}{\sigma\tau}\\
&+\left[N^i_k\left(N_{jl}+N_{lj}\right)-N^i_l\left(N_{jk}+N_{kj}\right)\right]\tfrac{\gamma^4||y||^2\left(2\gamma^2-\tau\right)}{\sigma^2\tau}\\
&+N^i_ry_j\left(N^r_ky_l-y_kN^r_l\right)\tfrac{\gamma^4}{\sigma^2}+N^i_ry_j\left(N_{rk}y_l-y_kN_{rl}\right)\tfrac{\gamma^4}{\sigma^2}\\
&+\left(N^i_k\eta_{jl}-N^i_l\eta_{jk}\right)\left\{\left[\varepsilon\tfrac{\gamma^2}{\sigma}-\tfrac{\gamma^3||y||^2(5\sigma-2)}{\sigma^2\tau}\right]\gamma_0+\tfrac{\gamma^4}{\sigma\tau}N_{00}\right\}\\
&+N^i_r\left(y_k\eta_{jl}-\eta_{jk}y_l\right)\left[\eta^{rr}N_{r0}\tfrac{\gamma^4}{\sigma^2}-\gamma^r\tfrac{\gamma^3||y||^2}{\sigma^2}\right]+N^i_0y_j\left(N_{0k}y_l-y_kN_{0l}\right)\tfrac{2\gamma^6}{\sigma^2\tau}\\
&+N^i_0y_j\left(y_k\gamma_l-\gamma_ky_l\right)\tfrac{\gamma^3}{\sigma^2\tau}+N^i_0y_j\left(y_kN_{l0}-N_{k0}y_l\right)\tfrac{4\gamma^6}{\sigma^2\tau}\\
&+N^i_0\left[y_k\left(N_{jl}+N_{lj}\right)-\left(N_{jk}+N_{kj}\right)y_l\right]\tfrac{\gamma^4\left(2\gamma^2-\tau\right)}{\sigma^2\tau}\\
&+N^i_0\left(\eta_{jk}y_l-y_k\eta_{jl}\right)\left(\tfrac{2\gamma^6}{\sigma^2\tau}N_{00}+\tfrac{2\gamma^5\|y\|^2}{\sigma^2\tau}\gamma_0\right)\\
&+\gamma^iy_j\left(y_k\gamma_l-\gamma_ky_l\right)\left[\tfrac{4\gamma^2||y||^2}{\sigma^2}-\tfrac{2\gamma^4C}{\sigma^2}-\omega\tfrac{\gamma}{\sigma\tau}+\tfrac{2\gamma}{\sigma}\left(\omega-2\varepsilon\right)-\right.\\
&\left.-2\varepsilon\left(\omega-\varepsilon\right)\tfrac{\gamma^2}{\sigma}+\varepsilon\,||y||^2\,\mathcal{T}_{10}+\mathcal{R}_3\tfrac{\gamma^2}{\sigma}+\tfrac{\gamma^2||y||^2\left(1+2\tau+2\tau^2\right)}{\sigma^2\tau^2}\right]\\
&+\gamma^iy_j\left(N_{kl}-N_{lk}\right)\tfrac{2\gamma^3||y||^2}{\sigma^2}-\gamma^iy_j\left(y_kN_{0l}-N_{0k}y_l\right)\left(\omega\tfrac{2\gamma^4}{\sigma\tau}+\tfrac{\gamma^3}{\sigma^2\tau}\right)\\
&-\gamma^iy_j\left(y_kN_{l0}-N_{k0}y_l\right)\left(\omega\tfrac{4\gamma^4}{\sigma\tau}+\tfrac{2\gamma^3(1-\tau)}{\sigma^2\tau}\right)\\
&-\gamma^i\left(\delta_{jk}y_l-y_k\delta_{jl}\right)\gamma_0\varepsilon\tfrac{2\gamma}{\sigma}+\gamma^i\left(\eta_{jk}y_l-\eta_{jl}y_k\right)N_{00}\tfrac{2\gamma^3(\omega\sigma-\gamma\tau)}{\sigma^2\tau}\\
&+\gamma^i\left(\eta_{jk}y_l-\eta_{jl}y_k\right)\gamma_0\left\{\left(\varphi\,||y||^2+\omega+\varepsilon\right)\mathcal{T}_{10}-\omega\left(\omega+||y||^2\right)\tfrac{\gamma^2}{\sigma}\right.\\
&\left.-\varepsilon\left(\varepsilon-\omega\right)\tfrac{\gamma^2}{\sigma}+\omega^2\tfrac{2\gamma^3||y||^2}{\sigma\tau}-\tfrac{\gamma^2}{\sigma^2\tau}\right\}+\gamma^i\left(\eta_{jk}N_{l0}-\eta_{jl}N_{k0}\right)\tfrac{\gamma^2||y||^2(1-2\gamma)}{\sigma^2}
\end{aligned}
$$

$$
\begin{aligned}
&+\gamma^i\left(\eta_{jk}\gamma_l-\eta_{jl}\gamma_k\right)\left[\tfrac{(\sigma-1)||y||^2}{\sigma^2}+\tfrac{\gamma^2||y||^4}{\sigma^2}-\varepsilon\,||y||^2\,\mathcal{T}_{10}\right]\\
&-\gamma^i\left[\left(N_{jk}+N_{kj}\right)y_l-\left(N_{jl}+N_{lj}\right)y_k\right]\left[\omega\frac{\gamma^2\left(2\gamma^2-\tau\right)}{\sigma\tau}-\frac{2\gamma^3||y||^2\left(3\gamma^2-\tau\right)}{\sigma^2\tau}\right]\\
&+\eta^{ii}N_{i0}\left(\eta_{jk}\gamma_l-\eta_{jl}\gamma_k\right)\left[\tfrac{\gamma^2||y||^2(\gamma-2)}{\sigma^2}+\tfrac{\gamma}{\sigma^2}+\tfrac{\varepsilon\gamma^2(3\sigma-1)}{\sigma^2}\right]\\
&-\eta^{ii}N_{i0}\left(\eta_{jk}y_l-\eta_{jl}y_k\right)\gamma_0\left\{\tfrac{4\gamma^6}{\sigma^2\tau}N_{00}-\left[\left(\varphi\,||y||^2+\omega+\varepsilon\right)\tfrac{\gamma^4}{\sigma^2}+\varphi\tfrac{\gamma^2}{\sigma}\right]\right\}\\
&+\eta^{ii}N_{i0}\left(\eta_{jk}N_{l0}-\eta_{jl}N_{k0}\right)\tfrac{\gamma^2}{\sigma^2}+\eta^{ii}N_{i0}y_j\left(N_{k0}y_l-y_kN_{l0}\right)\tfrac{4\gamma^6}{\sigma^2\tau}\\
&+\eta^{ii}N_{i0}y_j\left(\gamma_ky_l-y_k\gamma_l\right)\tfrac{\gamma^3(1+\tau)}{\sigma^2\tau}-\eta^{ii}N_{i0}y_j\left(N_{kl}-N_{lk}\right)\tfrac{2\gamma^4}{\sigma^2}\\
&+\eta^{ii}N_{i0}y_j\left(y_kN_{0l}-N_{0k}y_l\right)\tfrac{2\gamma^6\left(1+||y||^2\right)}{\sigma^2\tau}+\eta^{ii}N_{i0}N_{0j}\left(y_kN_{l0}-N_{k0}y_l\right)\tfrac{2\gamma^6||y||^2}{\sigma^2\tau}\\
&+\eta^{ii}N_{i0}\left(N_{jl}y_k-N_{jk}y_l\right)\tfrac{\gamma^2\left(\tau||y||^2-2\gamma^4\right)}{\sigma^2\tau}+\eta^{ii}N_{i0}\left(N_{lj}y_k-N_{kj}y_l\right)\tfrac{\gamma^2\left(\tau||y||^2+2\gamma^2\tau-2\gamma^4\right)}{\sigma^2\tau}\\
&+\eta^{ii}N_{ir}y_j\left(N_{rk}y_l-y_kN_{rl}\right)\eta^{rr}\tfrac{\gamma^4}{\sigma^2}-\eta^{ii}N_{ir}y_j\left(N^r_ky_l-y_kN^r_l\right)\eta^{rr}\tfrac{\gamma^4}{\sigma^2}\\
&+\eta^{ii}N_{ir}\left(\eta_{jk}y_l-\eta_{jl}y_k\right)\left(\eta^{rr}N_{r0}\tfrac{\gamma^4}{\sigma^2}-\gamma^r\tfrac{\gamma^r||y||^2}{\sigma^2}\right)\\
&+\eta^{ii}N_{ij}\left(y_k\gamma_l-\gamma_ky_l\right)\left[\omega\tfrac{\gamma^2}{\sigma}+\varepsilon\tfrac{2\gamma^4||y||^2}{\sigma^2}-\varepsilon\tfrac{\gamma^2}{\sigma}-\tfrac{2\gamma^3||y||^2}{\sigma^2}\right]\\
&-\eta^{ii}\left(N_{ik}y_jy_l-N_{il}y_jy_k\right)\gamma_0\left[\varphi\tfrac{\gamma^2}{\sigma}-\tfrac{2\gamma^3}{\sigma\tau}-\left(\varphi\,||y||^2+\omega+\varepsilon\right)\tfrac{2\gamma^4}{\sigma^2}\right]\\
&+\eta^{ii}\left(N_{ik}y_j\gamma_l-N_{il}y_j\gamma_k\right)\left[\varepsilon\tfrac{2\gamma^2||y||^2}{\sigma^2}-\tfrac{\gamma^3||y||^2\left(4\gamma^3+1\right)}{\sigma^2\tau}-\varepsilon\tfrac{\gamma^2}{\sigma}-\tfrac{2\gamma^3||y||^2}{\sigma^2\tau}\right]\\
&-\eta^{ii}\left(N_{ik}y_jN_{0l}-N_{il}y_jN_{0k}\right)\tfrac{\gamma^4}{\sigma\tau}+\eta^{ii}\left(N_{ik}y_jN_{l0}-N_{il}y_jN_{k0}\right)\tfrac{\gamma^4}{\sigma\tau}\\
&+\eta^{ii}\left(N_{ik}N_{j0}y_l-N_{il}N_{j0}y_k\right)\tfrac{\gamma^4}{\sigma\tau}-\eta^{ii}\left(N_{ik}N_{0j}y_l-N_{il}N_{0j}y_k\right)\tfrac{\gamma^4}{\sigma\tau}\\
&-\eta^{ii}\left[N_{ik}\left(N_{jl}+N_{lj}\right)-N_{il}\left(N_{jk}-N_{kj}\right)\right]\tfrac{\gamma^4||y||^2\left(2\gamma^2-\tau\right)}{\sigma^2\tau}\\
&-\eta^{ii}\left(N_{ik}\eta_{jl}-N_{il}\eta_{jk}\right)\left\{\gamma_0\left[\varepsilon\tfrac{2\gamma^2}{\sigma}-\tfrac{\gamma^3||y||^2(5\sigma-2)}{\sigma^2}\right]+N_{00}\tfrac{\gamma^4}{\sigma\tau}\right\}\\
&-\eta^{ii}\left(N_{ik}\gamma_jy_l-N_{il}\gamma_jy_k\right)\omega\tfrac{\gamma^2}{\sigma}+\eta^{ii}N_{ir}\left(\eta_{jk}y_l-\eta_{jl}y_k\right)\gamma^r\omega\tfrac{\gamma^2}{\sigma}\\
&-\left(\eta^{ir}\gamma_{rl}y_jy_k-\eta^{ir}\gamma_{rk}y_jy_l\right)\left(\tfrac{2\gamma}{\sigma}-\tfrac{\omega\gamma^2}{\sigma}\right)\\
&+\left(\eta^{ii}\eta_{ij}y_k\gamma_l-\eta^{ii}\eta_{ij}\gamma_ky_l\right)\gamma_0\tfrac{\gamma^2}{\sigma}\left[\mathcal{B}-2\varepsilon\left(\varphi\,||y||^2+\varepsilon\right)\right]\\
&+\left(\eta^{ii}\eta_{ij}y_k\gamma_{0l}-\eta^{ii}\eta_{ij}\gamma_{0k}y_l\right)\varepsilon\tfrac{\gamma^2}{\sigma}\\
&-\left(\eta^{ir}\eta_{jk}\gamma_{rl}-\eta^{ir}\eta_{jl}\gamma_{rk}\right)\tfrac{\gamma||y||^2}{\sigma}-\left(\eta^{ii}\eta_{ik}y_jy_l-\eta^{ii}\eta_{il}y_jy_k\right)\left(\gamma^r\gamma_r\right)\varepsilon\omega\tfrac{\gamma^2}{\sigma}\\
&-\left(\eta^{ii}\eta_{ik}y_jy_l-\eta^{ii}\eta_{il}y_jy_k\right)\left(\gamma_0\right)^2\varphi\tfrac{\gamma^2}{\sigma}\left(\varphi\,||y||^2+\omega\right)\\
&+\left(\eta^{ii}\eta_{ik}y_j\gamma_l-\eta^{ii}\eta_{il}y_j\gamma_k\right)\gamma_0\tfrac{\gamma^2}{\sigma}\left[\mathcal{B}-\varepsilon\left(\varphi\,||y||^2+2\varepsilon-\omega\right)\right]\\
&+\left(\eta^{ii}\eta_{ik}y_j\gamma_{0l}-\eta^{ii}\eta_{il}y_j\gamma_{0k}\right)\varepsilon\tfrac{\gamma^2}{\sigma}+\left(\eta^{ii}\eta_{ik}\gamma_jy_l-\eta^{ii}\eta_{il}\gamma_j\gamma_k\right)\varepsilon\omega\tfrac{\gamma^2}{\sigma}\\
&-\delta^i_j\left(y_k\gamma_l-\gamma_ky_l\right)\gamma_0\tfrac{\gamma^2}{\sigma}\Big[\mathcal{A}\,||y||^2+2\mathcal{B}+\mathcal{C}+\varphi\left(\omega-4\varepsilon\right)||y||^2-\alpha\varepsilon\,||y||^2\\
&-\omega\varepsilon++\omega^2-4\varepsilon^2+\left(\varphi\,||y||^2+\omega+\varepsilon\right)\mathcal{T}_9+\mathcal{R}_2+\mathcal{R}_3-\mathcal{R}_1\,||y||^2\Big]\\
&-\delta^i_j\left(y_k\gamma_{0l}-\gamma_{0k}y_l\right)\tfrac{\varepsilon\gamma^2}{\sigma}-\delta^i_j\left(N_{k0}y_l-y_kN_{l0}\right)\gamma_0\tfrac{\gamma^2}{\sigma}\left[\left(\varphi\,||y||^2+\omega+\varepsilon\right)\tfrac{2\gamma^2}{\sigma}-\varphi\right]\\
&-\delta^i_j\left(N_{k0}\gamma_l-\gamma_kN_{l0}\right)\tfrac{\gamma^2\left[\varepsilon(1-\sigma)+||y||^2(1-\gamma)\right]}{\sigma^2}+\delta^i_j\left(N_{kr}y_l-y_kN_{l0}\right)\gamma^r\tfrac{\omega\gamma^2}{\sigma}\\
&+\left(\delta^i_ky_jy_l-\delta^i_ly_jy_k\right)\left(\gamma_0\right)^2\left[\tfrac{\gamma^2}{\sigma}\left(\alpha+2\varphi\right)\left(\varphi\,||y||^2+\omega+\varepsilon\right)+\left(\varphi\,||y||^2+\varepsilon\right)\varphi\right.
\end{aligned}
$$

$$+\varepsilon\frac{4\gamma}{\tau}+\mathcal{R}_4\frac{\gamma^2}{\sigma}-\frac{4\gamma^4||y||^2}{\sigma^2\tau}\Big]$$
$$+\left(\delta^i_k y_j y_l-\delta^i_l y_j y_k\right)(\gamma^r\gamma_r)\left[\left(\varphi\,||y||^2+\varepsilon\right)\omega\frac{\gamma^2}{\sigma}+\frac{2\gamma^2||y||^2}{\sigma^2}\right]$$
$$+\left(\delta^i_k y_j y_l-\delta^i_l y_j y_k\right)\left(\frac{4\gamma^5}{\sigma^2\tau}\gamma_0 N_{00}-\frac{2\gamma^3}{\sigma^2}\gamma^r N_{r0}\right)$$
$$+\left(\delta^i_k y_j \gamma_l-\delta^i_l y_j \gamma_k\right)\gamma_0\left\{\frac{\gamma^2}{\sigma}\left[\mathcal{R}_2-\left(\mathcal{A}\,||y||^2+2\mathcal{B}+\mathcal{C}\right)\right]\right.$$
$$\left.-\frac{\gamma^2}{\sigma}\varepsilon\left(4\varphi\,||y||^2+\omega+4\varepsilon+\alpha+1\right)-\frac{\gamma^2||y||^2}{\sigma^2\tau}\right\}+\left(\delta^i_k y_j \gamma_l-\delta^i_l y_j \gamma_k\right)N_{00}\frac{\gamma^3}{\sigma^2\tau}$$
$$+\left(\delta^i_k y_j N_{l0}-\delta^i_l y_j N_{k0}\right)\left(\varepsilon\frac{4\gamma^4}{\sigma\tau}+N_{00}\frac{4\gamma^6}{\sigma^2\tau}-\gamma_0\frac{4\gamma^5||y||^2}{\sigma^2\tau}\right)$$
$$-\left(\delta^i_k y_j N_{0l}-\delta^i_l y_j N_{0k}\right)\left[\gamma_0\left(\varepsilon\frac{2\gamma^4}{\sigma\tau}-\frac{2\gamma^5||y||^2}{\sigma^2\tau}\right)+\frac{2\gamma^6}{\sigma^2\tau}N_{00}\right]$$
$$+\left(\delta^i_k y_j N^r_l-\delta^i_l y_j N^r_k\right)\left(\gamma_r\frac{\gamma^3||y||^2}{\sigma^2}-N_{r0}\frac{\gamma^4}{\sigma^2}\right)$$
$$+\left(\delta^i_k y_j N_{rl}-\delta^i_l y_j N_{rk}\right)\left(\eta^{rr}N_{r0}\frac{\gamma^4}{\sigma^2}-\gamma^r\frac{\gamma^3||y||^2}{\sigma^2}\right)$$
$$-\left(\delta^i_k y_j \gamma_{0l}-\delta^i_l y_j \gamma_{0k}\right)\frac{\gamma^2}{\sigma}\left(\varphi\,||y||^2+\omega+\varepsilon\right)$$
$$-\left(\delta^i_k N_{0j} y_l-\delta^i_l N_{0j} y_k\right)\left[\gamma_0\left(\varepsilon\frac{2\gamma^4}{\sigma\tau}-\frac{2\gamma^5||y||^2}{\sigma^2\tau}\right)+N_{00}\frac{2\gamma^6}{\sigma^2\tau}\right]$$
$$-\left(\delta^i_k N_{j0}\gamma_l-\delta^i_l N_{j0}\gamma_k\right)\left(\frac{\gamma}{\sigma^2}-\frac{\gamma^2||y||^2}{\sigma^2}-\varepsilon\frac{\gamma}{\sigma}+\varepsilon\frac{2\gamma^4||y||^2}{\sigma^2}\right)$$
$$-\left(\delta^i_k N_{j0} y_l-\delta^i_l N_{j0} y_k\right)\gamma_0\left[\frac{2\gamma^4}{\sigma^2}\left(\varphi\,||y||^2+\omega+\varepsilon\right)-\varphi\frac{\gamma^2}{\sigma}+\varepsilon\frac{4\gamma^4}{\sigma\tau}+\frac{4\gamma^5||y||^2}{\sigma^2\tau}\right]$$
$$+\left(\delta^i_k N_{j0} y_l-\delta^i_l N_{j0} y_k\right)N_{00}\frac{4\gamma^6}{\sigma^2\tau}-\left(\delta^i_k N_{j0}N_{l0}-\delta^i_l N_{j0}N_{k0}\right)\frac{\gamma^4}{\sigma^2}$$
$$-\left(\delta^i_k N_{jl}-\delta^i_l N_{jk}\right)\left\{\gamma_0\left[\frac{\gamma^3||y||^2\left(2\gamma^2-\tau\right)}{\sigma^2\tau}-\varepsilon\frac{2\gamma^4}{\sigma\tau}\right]+N_{00}\frac{\gamma^2||y||^2}{\sigma^2}\right\}$$
$$-\left(\delta^i_k N_{lj}-\delta^i_l N_{kj}\right)\left\{\gamma_0\left[\frac{\gamma^3||y||^2\left(2\gamma^2-\tau\right)}{\sigma^2\tau}-\varepsilon\frac{2\gamma^4}{\sigma\tau}\right]+N_{00}\frac{\gamma^2||y||^2}{\sigma^2}\right\}$$
$$+\left(\delta^i_k N^r_j y_l-\delta^i_l N^r_j y_k\right)\left[\gamma_r\left(\omega\frac{\gamma^2}{\sigma}+\frac{\gamma^3||y||^2}{\sigma^2}\right)-N_{r0}\frac{\gamma^4}{\sigma^2}\right]$$
$$-\left(\delta^i_k N_{rj} y_l-\delta^i_l N_{rj} y_k\right)N_{r0}\frac{\gamma^4}{\sigma^2}$$
$$-\left(\delta^i_k \gamma_j y_l-\delta^i_l \gamma_j y_k\right)\gamma_0\left\{\left[\mathcal{T}_8\left(\varphi\,||y||^2+\omega+\varepsilon\right)+\omega\left(\gamma+\varphi\,||y||^2\right)\right]\left(-\frac{\gamma^2}{\sigma}\right)\right.$$
$$\left.+\varepsilon\left(\varepsilon\frac{\gamma^2}{\sigma}-\frac{\gamma}{\sigma\tau}\right)+\frac{\gamma^2||y||^2}{\sigma^2\tau}\right\}+\left(\delta^i_k \gamma_j y_l-\delta^i_l \gamma_j y_k\right)N_{00}\frac{\gamma^3}{\sigma^2\tau}$$
$$+\left(\delta^i_k \gamma_j \gamma_l-\delta^i_l \gamma_j \gamma_k\right)\left[\frac{(1-\sigma)||y||^2}{\sigma^2}-\varepsilon\,||y||^2\,\mathcal{T}_8-\frac{\gamma^2||y||^4}{\sigma^2}\right]$$
$$-\left(\delta^i_k \gamma_j N_{rl}-\delta^i_l \gamma_j N_{rk}\right)\gamma^r\frac{\gamma^3||y||^2}{\sigma^2}+\left(\delta^i_k \gamma_j N_{l0}-\delta^i_l \gamma_j N_{k0}\right)\frac{\gamma^3||y||^2}{\sigma^2}$$
$$+\left(\delta^i_k \gamma_{jl}-\delta^i_l \gamma_{jk}\right)\frac{\gamma||y||^2}{\sigma}$$
$$-\left(\delta^i_k \eta_{jl}-\delta^i_l \eta_{jk}\right)\left[\gamma_0 N_{00}\frac{2\gamma^4\left(\sigma\varepsilon-2\gamma||y||^2\right)}{\sigma^2\tau}+(\gamma_0)^2\,\frac{2\gamma^3||y||^2\left(\sigma\varepsilon-\gamma||y||^2\right)}{\sigma^2\tau}\right]$$
$$-\left(\delta^i_k \eta_{jl}-\delta^i_l \eta_{jk}\right)\left[(N_{00})^2\,\frac{2\gamma^6}{\sigma^2\tau}-\frac{\gamma^2||y||^2(1+\gamma)}{\sigma^2}-\eta^{rr}\,(N_{r0})^2\,\frac{\gamma^4}{\sigma^2}-\frac{\gamma^2||y||^4}{\sigma^2}\,(\gamma^r\gamma_r)\right],$$

$$P^i_{\ jkl}=y^i y_j y_k y_l\left\{\gamma_0\left[\frac{3\gamma^2M(\tau-\sigma)}{\sigma\tau}+\frac{4M\gamma^4}{\sigma\tau}-2\varphi\frac{\gamma^2}{\sigma}+\frac{8\gamma^5(2\tau+1)}{\sigma^2\tau^2}+\alpha\frac{8\gamma^4}{\sigma\tau}-\frac{4\varphi\gamma^4}{\sigma\tau}\right.\right.$$
$$\left.\left.-\frac{4\gamma^5\left(4\sigma^2-1\right)}{2\gamma^4\tau}-\frac{4\gamma^5}{\sigma^2\tau}+\frac{12\gamma^5(1-2\sigma)}{\sigma^2\tau^2}\right]+\frac{\gamma^3(\tau+2\sigma-8\gamma\varphi)}{\sigma^2\tau^2}\right\}$$
$$+y^i y_j\left(y_k\gamma_l-\gamma_k y_l\right)\left\{\left(\varphi-\frac{\gamma}{2\sigma^3}\right)\left(\frac{2\gamma^4}{\sigma\tau}-\frac{\gamma^2}{\sigma}\right)+\left(\varphi+\frac{\gamma}{2\sigma^3}\right)\left(-\frac{\gamma^2}{\sigma}\right)\right.$$

$$
\begin{aligned}
&+\left(\varepsilon+\beta+\varphi\|y\|^2\right)-\left(\omega-\frac{\gamma\|y\|^2}{2\sigma^3}\right)\frac{2\gamma^4}{\sigma\tau}+\left(\varphi-\frac{\gamma}{\sigma\tau}\right)\left[1+\frac{\gamma^2(1-\sigma)}{\sigma\tau}\right]\\
&-\frac{2\gamma^3}{\sigma^2}-\frac{4\gamma^5(2-3\sigma)}{\sigma^2\tau^2}+\frac{8\varepsilon\gamma^4}{\sigma\tau}-\frac{\gamma^3(2\sigma-1)}{\sigma^2\tau}-\frac{27\gamma^7\|y\|^4}{\sigma^2\tau^2}+\frac{\gamma^5\|y\|^2(42\sigma+9)}{2\sigma^2\tau^2}\Bigg\}\\
&+y^i\gamma_j y_k y_l\left[\left(\varphi-\frac{\gamma}{2\sigma^3}\right)\left(\frac{2\gamma^4}{\sigma\tau}-\frac{\gamma^2}{\sigma}\right)+\left(\varphi+\frac{\gamma}{2\sigma^3}\right)\left(-\frac{\gamma^2}{\sigma}\right)\right.\\
&-\frac{2\gamma^3\left(2\gamma^2+\sigma+2\tau\right)}{\sigma^2\tau}+\frac{\gamma^5\|y\|^2(9-8\sigma)}{2\sigma^2\tau}-\omega\frac{4\gamma^4}{\sigma\tau}\\
&\left.+\left(\varepsilon+\beta+\varphi\|y\|^2\right)-\left(\omega-\frac{\gamma\|y\|^2}{2\sigma^3}\right)\frac{2\gamma^4}{\sigma\tau}+\frac{\gamma^3\left(4\sigma^2-1\right)}{2\sigma^3\tau}\right]\\
&+y^iy_j\left(N_{0k}y_l-y_kN_{0l}\right)\frac{2\gamma^6\left(14\tau+3-4\gamma^2\right)}{\sigma^2\tau^2}-y^iy_j\left(N_{k0}y_l-y_kN_{l0}\right)\frac{\gamma^6\left(+21\tau+21+8\gamma^2\right)}{\sigma^2\tau^2}\\
&+y^i\left(N_{kj}y_l-N_{lj}y_k\right)\frac{2\gamma^4\left(2\sigma+3\gamma^2\right)}{\sigma^2\tau}+y^iy_j\left(N_{kl}-N_{lk}\right)\frac{\gamma^4\left[\tau(5+2\sigma)-2\left(3+2\gamma^2\right)\right]}{\sigma^2\tau^2}\\
&+y^i\left(N_{jk}y_l-N_{jl}y_k\right)\frac{2\gamma^4\left(\tau-9\gamma^2\right)}{\sigma\tau^2}+y^iN_{0j}\left[y_ky_l\frac{2\gamma^6(17\sigma-3)}{\sigma^2\tau^2}+\delta_{kl}\frac{2\gamma^4}{\sigma\tau}\right]\\
&-y^iN_{j0}\left\{y_ky_l\frac{4\gamma^6[11\sigma-2(1+\sigma\tau)]}{\sigma^2\tau^2}+\eta_{kl}\frac{\gamma^4}{\sigma\tau}+\delta_{kl}\frac{4\gamma^4}{\sigma\tau}\right\}\\
&+y^iy_j\eta_{kl}\gamma_0\left\{\left[8(1-\sigma)+\frac{\sigma-\tau}{\sigma\tau}\right]\frac{\gamma^3(16\sigma-7)}{4\sigma^2\tau^2}-\frac{4\gamma^7||y||^4}{2\sigma^3\tau}+2\varphi\frac{\gamma^2\left(2\gamma^2-\tau\right)}{\sigma\tau}\right.\\
&\left.+\left(2\omega-\varepsilon+\varphi||y||^2\right)\frac{2\gamma^4}{\sigma\tau}-\frac{4\gamma^5}{\sigma^2\tau}-\frac{2\gamma^5||y||^2}{\sigma\tau^2}\right\}+y^iy_j\eta_{kl}N_{00}\frac{\gamma^6(35\sigma-11)}{\sigma^2\tau^2}\\
&+y^iy_j\delta_{kl}\gamma_0\frac{4\gamma^3}{\sigma\tau}+y^i\left(\eta_{jk}y_l-\eta_{jl}y_k\right)\left[-\frac{2\gamma^2}{\sigma\tau}\left(\alpha||y||^2+2\varphi||y||^2+\omega+\varepsilon\right)\right.\\
&\left.+4\varepsilon\frac{\gamma^4}{\sigma\tau}-8\varphi\gamma^2-\alpha\frac{\gamma^2}{\sigma}+\frac{\gamma^3(8\sigma-1)}{\sigma^2\tau}+\frac{2\gamma^2\left(2-\gamma^3\right)}{\sigma\tau^2}-\frac{2\gamma^5(5\sigma-1)}{\sigma^2\tau^2}\right]\\
&+y^i\left(\delta_{jk}y_l-\delta_{jl}y_k\right)\gamma_0\frac{4\gamma^3}{\sigma\tau}+y^i\left(\eta_{jk}y_l-\eta_{jl}y_k\right)N_{00}\frac{4\gamma^6(5\sigma-1)}{\sigma^2\tau^2}\\
&+y^i\left(\eta_{jk}y_l-\eta_{jl}y_k\right)\frac{\varphi\gamma^2}{\tau}-y^i\left(\eta_{jk}N_{l0}-\eta_{jl}N_{k0}\right)\frac{\gamma^4\left[\tau+2\gamma^2\left(1+2||y||^2\right)\right]}{\sigma\tau^2}\\
&-y^i\left(\eta_{jk}N_{0l}-\eta_{jl}N_{0k}\right)\frac{\gamma^4\left(2\gamma^2-\sigma\right)}{\sigma\tau^2}+y^i\left(\delta_{jk}y_l-\delta_{jl}y_k\right)\frac{\gamma}{\sigma\tau}+y^i\gamma_j\delta_{kl}\frac{\gamma}{\sigma\tau}\\
&+y^i\left(\eta_{jk}\gamma_l-\eta_{jl}\gamma_k\right)\left[(\omega+\varepsilon)\frac{\gamma^2\left(2\gamma^2-\tau\right)}{\sigma\tau}+\frac{\gamma^3||y||^2(14\tau+25)}{2\sigma\tau^2}+\frac{8\varphi\gamma^3\varepsilon^2+2\gamma}{\tau}\right]\\
&+y^i\gamma_j\eta_{kl}\left[(\omega+\varepsilon)\frac{\gamma^2\left(2\gamma^2-\tau\right)}{\sigma\tau}+\frac{\gamma^5||y||^4}{2\sigma^2\tau}\right]\\
&-y^i\left[\left(N_{jk}-N_{kj}\right)y_l-\left(N_{jl}-N_{lj}\right)y_k\right]\frac{\gamma^4}{\sigma\tau}+\delta^i_jy_ky_l\left[\frac{2\varphi\gamma^2}{\sigma}+\frac{\gamma^2||y||^2}{\sigma}\left(\varphi-\frac{2\gamma}{\sigma\tau}\right)\right]\\
&+\delta^i_jy_ky_l\gamma_0\left[\frac{M\gamma^2||y||^2}{\sigma}+\frac{\gamma^3\left(4\sigma^2-1\right)}{2\sigma^3}-\frac{4\gamma^5||y||^2}{\sigma^2\tau}-\frac{\gamma^2(\alpha+\varphi)}{\sigma}\right]\\
&+\delta^i_j\eta_{kl}\gamma_0\left\{\frac{\gamma^3}{4\sigma^2\tau}\left[8(1-\sigma)+\frac{\sigma-\tau}{\sigma\tau}+\frac{\gamma^5||y||^4}{2\sigma^3}-\frac{\gamma^2\left(\omega+\varphi||y||^2\right)}{\sigma}\right]\right\}\\
&+\delta^i_j\eta_{kl}N_{00}\frac{2\gamma^6\left(\tau+\sigma-2\tau||y||^2\right)}{\sigma^2\tau^2}-\delta^i_j\left(\gamma_ky_l-y_k\gamma_l\right)\frac{\gamma^2\left[2\varepsilon(1-\tau)+\varphi||y||^2\right]}{\sigma}\\
&+\delta^i_j\left(y_kN_{0l}-N_{0k}y_l\right)\frac{2\gamma^6\left(1-||y||^2\right)}{\sigma^2\tau}-\delta^i_j\left(N_{kl}-N_{lk}\right)\frac{\gamma^2}{\sigma}+\left(\delta^i_ky_jy_l-\delta^i_ly_jy_k\right)\frac{\varphi\gamma^2}{\sigma}\\
&+\left(\delta^i_ky_jy_l-\delta^i_ly_jy_k\right)\gamma_0\left[\frac{2\gamma^3}{\sigma\tau}-\frac{2\gamma^4}{\sigma^2}\left(\varphi||y||^2-\frac{2\gamma^3||y||^4}{4\sigma^2\tau}\right)\right]-\left(\delta^i_ky_jy_l-\delta^i_ly_jy_k\right)N_{00}\frac{4\gamma^6}{\sigma^2\tau}\\
&-\left(\delta^i_ky_j\gamma_l-\delta^i_ly_j\gamma_k\right)\frac{\gamma^2}{\sigma}\left(4\varepsilon+\varphi||y||^2\right)+\left(\delta^i_k\gamma_jy_l-\delta^i_l\gamma_jy_k\right)\varepsilon\left(\frac{2}{||y||^2}-\frac{\gamma^2}{\sigma}\right)\\
&+\left(\delta^i_k\eta_{jl}-\delta^i_l\eta_{jk}\right)\left\{\gamma_0\frac{\gamma^3\left[2\gamma^2+(2\sigma-\tau)||y||^2\right]}{\sigma^2\tau}+N_{00}\frac{\gamma^4}{\sigma\tau}\right\}\\
&+\left(\delta^i_kN_{jl}-\delta^i_lN_{jk}\right)\frac{\gamma^2(4\sigma-1)\left(2\gamma^2-\tau\right)}{2\sigma^2\tau}+\left(\delta^i_kN_{lj}-\delta^i_lN_{kj}\right)\frac{\gamma^2(8\sigma-3)\left(2\sigma\gamma^2-\tau\right)}{2\sigma^2\tau}
\end{aligned}
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\begin{aligned}
&+\left(\delta^i_k N_{j0} y_l - \delta^i_l N_{j0} y_k\right) \tfrac{2\gamma^4}{\sigma^2} + \left(\delta^i_k N_{0j} y_l - \delta^i_l N_{0j} y_k\right) \tfrac{\gamma^4}{\sigma\tau} \\
&+\left(\delta^i_k y_j N_{l0} - \delta^i_l y_j N_{k0}\right) \tfrac{\gamma^4(\sigma+3)}{\sigma^2\tau} + \left(\delta^i_k y_j N_{0l} - \delta^i_l y_j N_{0k}\right) \tfrac{\gamma^4}{\sigma\tau} \\
&+\gamma^i y_j \eta_{kl} \left[\tfrac{\gamma^5 ||y||^4}{2\sigma^3} - (\omega-\varepsilon)\tfrac{\gamma^2}{\sigma}\right] + \gamma^i \left(\eta_{jk} y_l - \eta_{jl} y_k\right)\left[\varepsilon \tfrac{\gamma^2}{\sigma} + \tfrac{2\gamma^3 ||y||^2}{\sigma\tau} + \omega \tfrac{\gamma^2(2-5\sigma)}{\sigma\tau}\right] \\
&+\gamma^i y_j y_k y_l \left[\tfrac{\gamma^3\left(4\sigma^2-1\right)}{2\sigma^3} + \tfrac{4\gamma^3}{\sigma^2} + \tfrac{\gamma^2}{\sigma}\left(2\varphi + \tfrac{\gamma}{\sigma^3}\right) - \tfrac{4\gamma^5}{\sigma^2\tau} - \tfrac{8\gamma^5 ||y||^2}{\sigma^2\tau} - \omega \tfrac{4\gamma^4}{\sigma\tau}\right] \\
&+\gamma^i \left(y_j \delta_{kl} - \delta_{jl} y_k\right) \tfrac{2\gamma}{\sigma} + N^i_0 y_j y_k y_l \tfrac{4\gamma^6}{\sigma^2\tau} + N^i_0 \left(\eta_{jk} y_l - \eta_{jl} y_k\right) \tfrac{4\gamma^6}{\sigma\tau} \\
&-N^i_j \eta_{kl} \tfrac{\gamma^2}{\sigma} + N^i_j y_k y_l \tfrac{2\gamma^4}{\sigma^2} + \left(N^i_k \eta_{jl} - N^i_l \eta_{jk}\right) \tfrac{\gamma^2(1-4\sigma)}{2\sigma\tau} + \left(N^i_k y_j y_l - N^i_l y_j y_k\right) \tfrac{\gamma^2\left(\tau\gamma^2+2\sigma\right)}{\sigma^2\tau} \\
&+\eta^{ii} N_{i0} \left(\eta_{jl} y_k - \eta_{jk} y_l\right) \tfrac{2\gamma^4}{\sigma^2} + \eta^{ii} N_{i0} y_j y_k y_l \tfrac{8\gamma^6}{\sigma^2} - \eta^{ii} N_{i0} \left(\eta_{jk} y_l - \eta_{jl} y_k\right) \tfrac{2\gamma^4}{\sigma^2} \\
&-\eta^{ii} N^i_0 y_j \eta_{kl} \tfrac{\gamma^4}{\sigma^2} + \eta^{ii} N_{ij} \eta_{kl} \tfrac{\gamma^2}{\sigma} - \eta^{ii} N_{ij} y_k y_l \tfrac{2\gamma^4}{\sigma^2} \\
&+\eta^{ii} \left(N_{ik} y_j y_l - N_{il} y_j y_k\right) \tfrac{\gamma^4(1-5\sigma)}{\sigma^2\tau} + \eta^{ii} \left(N_{ik} \eta_{jl} - N_{il} \eta_{jk}\right) \tfrac{\gamma^2(4\sigma-1)}{\sigma\tau} \\
&+\eta^{ii} \eta_{ij} y_k y_l \gamma_0 \tfrac{\gamma^2}{\sigma} + \eta^{ii} \eta_{ij} \left(y_k \gamma_l - \gamma_k y_l\right) \varepsilon \tfrac{\gamma^2}{\sigma} \\
&-\eta^{ii} \left(\eta_{il} \eta_{jk} - \eta_{ik} \eta_{jl}\right) \gamma_0 \left(\omega + \varphi \|y\|^2\right) \tfrac{\gamma^2}{\sigma} + \eta^{ii} \left(\eta_{ik} y_j \gamma_l - \eta_{il} y_j \gamma_k\right) \omega \tfrac{\gamma^2}{\sigma} \\
&+\eta^{ii} \left(\eta_{ik} y_j y_l - \eta_{il} y_j y_k\right) \gamma_0 \varphi \tfrac{\gamma^2}{\sigma} + \eta^{ii} \left(\eta_{ik} \gamma_j y_l - \eta_{il} \gamma_j y_k\right) \omega \tfrac{\gamma^2}{\sigma},
\end{aligned}
$$

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\begin{aligned}
S^i_{jkl} = &\left(\delta^i_k N^r_j N^r_l - \delta^i_l N^r_j N^r_k\right) \tfrac{\gamma^4}{\sigma^2} + \left(\delta^i_k N^r_j \gamma_l - \delta^i_l N^r_j \gamma_k\right) \tfrac{\gamma^3 \|y\|^2}{\sigma^2} \\
&+\left(\delta^i_k N^r_j y_l - \delta^i_l N^r_j y_k\right) N_{00} \tfrac{4\gamma^6}{\sigma^2\tau} - \left(\delta^i_k N^r_j y_l - \delta^i_l N^r_j y_k\right) \gamma_0 \tfrac{4\gamma^5 \|y\|^2}{\sigma^2} \\
&-\left(\delta^i_k N_{0j} y_l - \delta^i_l N_{0j} y_k\right) N_{00} \tfrac{2\gamma^6}{\sigma^2\tau} + \left(\delta^i_k N_{0j} y_l - \delta^i_l N_{0j} y_k\right) \gamma_0 \tfrac{2\gamma^5 \|y\|^2}{\sigma^2} \\
&+\left(\delta^i_k N^r_j - \delta^i_l N^r_j + \delta^i_k N^r_l - \delta^i_l N^r_k\right) N_{00} \tfrac{\gamma^2}{\sigma} \left(\tfrac{2\gamma^4}{\sigma\tau} - \tfrac{\gamma^2}{\sigma}\right) \\
&+\left(\delta^i_k y_j N^r_l - \delta^i_l y_j N^r_k\right) N_{00} \tfrac{4\gamma^6}{\sigma^2\tau} + \left(\delta^i_k y_j N_{0l} - \delta^i_l y_j N_{0k}\right) \gamma_0 \tfrac{2\gamma^5 \|y\|^2}{\sigma^2} \\
&+\left(\delta^i_k y_j N_{0l} - \delta^i_l y_j N_{0k}\right) N_{00} \tfrac{2\gamma^6}{\sigma^2\tau} - \left(\delta^i_k y_j N^r_l - \delta^i_l y_j N^r_k\right) \gamma_0 \tfrac{4\gamma^5 \|y\|^2}{\sigma^2} \\
&-\left(\delta^i_k N^r_j - \delta^i_l N^r_j + \delta^i_k N^r_l - \delta^i_l N^r_k\right) \gamma_0 \tfrac{\gamma^2}{\sigma} \left(\tfrac{2\gamma^4}{\sigma\tau} - \tfrac{\gamma^2}{\sigma}\right) \\
&+\left(\delta^i_k \gamma_j N^r_l - \delta^i_l \gamma_j N^r_k\right) \tfrac{(2-N_{00})\gamma^3 \|y\|^2}{\sigma^2} - \left(\delta^i_k \gamma_j \gamma_l - \delta^i_l \gamma_j \gamma_k\right) \tfrac{\gamma^2 \|y\|^4}{\sigma^2} \\
&+\left(\delta^i_k \gamma_j y_l - \delta^i_l \gamma_j y_k\right) \left(N_{00} - \gamma_0\right) \tfrac{\gamma^3}{\sigma^2\tau} + \left(\delta^i_k y_j \gamma_l - \delta^i_l y_j \gamma_k\right) \left(N_{00} - \gamma_0\right) \tfrac{\gamma^3}{\sigma^2\tau} \\
&+\left(\delta^i_k y_j y_l - \delta^i_l y_j y_k\right) \left(\tfrac{2\sigma-1}{2\sigma^2} \left(\gamma_r \gamma^r\right) - \tfrac{4\gamma^4 \|y\|^2}{\sigma^2\tau} \left(\gamma_0\right)^2 - \tfrac{2\gamma^3}{\sigma^2} \gamma^r N_{r0} + \tfrac{4\gamma^5}{\sigma^2\tau} \gamma_0 N_{00}\right) \\
&+\left(\delta^i_k y_j N^r_l - \delta^i_l y_j N^r_k\right) \left(\tfrac{\gamma^3 \|y\|^2}{\sigma^2} \gamma_r - \tfrac{\gamma^4}{\sigma^2} N_{r0}\right) \\
&+\left(\delta^i_k N^r_j y_l - \delta^i_l N^r_j y_k\right) \left(\tfrac{\gamma^3 \|y\|^2}{\sigma^2} \gamma_r + \tfrac{\gamma^4}{\sigma^2} \left(\eta^{rr} - 1\right) N_{r0}\right) \\
&-\left(\delta^i_k N_{rj} y_l - \delta^i_l N_{rj} y_k + \delta^i_k y_j N_{rl} - \delta^i_l y_j N_{rk}\right) \tfrac{\gamma^3 \|y\|^2}{\sigma^2} \gamma^r \\
&+\left(\delta^i_k \eta_{jl} - \delta^i_l \eta_{jk}\right) \left(\tfrac{\gamma^4}{\sigma^2} \eta^{rr} \left(N_{r0}\right)^2 - \tfrac{2\gamma^3 \|y\|^2}{\sigma^2} \gamma^r N_{r0} + \tfrac{\gamma^2 \|y\|^4}{\sigma^2} \left(\gamma_r \gamma^r\right)\right) \\
&+\left(\delta^i_k \eta_{jl} - \delta^i_l \eta_{jk}\right) \left(-\tfrac{2\gamma^6}{\sigma^2\tau} \left(N_{00}\right)^2 - \tfrac{2\gamma^4 \|y\|^4}{\sigma^2\tau} \left(\gamma_0\right)^2 + \tfrac{4\gamma^5}{\sigma^2\tau} \gamma_0 N_{00}\right) \\
&+y^i \left[\left(N_{jl} + N_{lj}\right) N_{k0} - \left(N_{jk} + N_{kj}\right) N_{l0}\right] \tfrac{4\gamma^6 \|y\|^2}{\sigma^2\tau^2} \\
&+y^i \left[\left(N_{jk} + N_{kj}\right) y_l - \left(N_{jl} + N_{lj}\right) y_k\right] \tfrac{6\gamma^5 \|y\|^2 \left(2\gamma^2-\tau\right)}{\sigma^2\tau^2} \\
&+y^i \left[\left(N_{jk} + N_{kj}\right) y_l - \left(N_{jl} + N_{lj}\right) y_k\right] \tfrac{\gamma^3 \gamma_0 \left(2\gamma^2-\tau\right)}{\sigma^2\tau^2} \\
&+y^i \left[\left(N_{jk} + N_{kj}\right) \gamma_l - \left(N_{jl} + N_{lj}\right) \gamma_k\right] \tfrac{\gamma^3 \|y\|^2 \left(2\gamma^2-\tau\right)}{\sigma^2\tau^2}
\end{aligned}
$$

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\begin{aligned}
&+y^i y^j \left[\left(N_{rk}+N_{kr}\right)N_l^r-\left(N_{rl}+N_{lr}\right)N_k^r\right]\frac{\gamma^4\left(2\gamma^2-\tau\right)}{\sigma^2\tau}\\
&+y^i y^j\left(N_{rk}N_{lr}-N_{kr}N_{rl}\right)\eta^{rr}\frac{\gamma^4\left(2\gamma^2-\tau\right)}{\sigma^2\tau}\\
&+y^i\left[N_j^r\left(N_{rk}+N_{kr}\right)y_l-N_j^r y_k\left(N_{rl}+N_{lr}\right)\right]\frac{\gamma^4\left(2\gamma^2-\tau\right)}{\sigma^2\tau}\\
&+y^i N_{rj}\left(y_k N_{rl}-N_{rk}y_l\right)\eta^{rr}\frac{2\gamma^4\left(2\gamma^2-\tau\right)}{\sigma^2\tau}\\
&+y^i N_{i0}\left(\eta_{jl}N_{k0}-\eta_{jk}N_{l0}\right)\frac{\gamma^4}{\sigma^2}\\
&+\eta^{ii}N_{i0}y_j\left(N_{lk}-N_{kl}\right)\frac{2\gamma^4}{\sigma^2}+\eta^{ii}N_{i0}y_j\left(\gamma_k y_l-y_k\gamma_l\right)\frac{\gamma^2(2\tau+1+\gamma)}{\sigma^2}\\
&+\eta^{ii}N_{i0}y_j\left(N_{k0}y_l-y_kN_{l0}\right)\frac{8\gamma^6}{\sigma^2\tau}+\eta^{ii}N_{i0}y_j\left(y_kN_{0l}-N_{0k}y_l\right)\frac{4\gamma^6}{\sigma^2\tau}\\
&+\eta^{ii}N_{i0}\left[\left(N_{jk}+N_{kj}\right)N_{0l}-\left(N_{jl}+N_{lj}\right)N_{0k}\right]\frac{\gamma^4\left(2\gamma^2-\tau\right)}{\sigma^2\tau}\\
&+\eta^{ii}N_{i0}\left[\left(N_{jk}y_l-N_{jl}y_k\right)\frac{\gamma^2\left(-6\sigma^2+15\sigma-3\right)}{\sigma^2\tau}+\left(N_{kj}y_l-N_{lj}y_k\right)\frac{\gamma^4\left(\sigma+\tau\|y\|^2\right)}{\sigma^2\tau}\right]\\
&-\eta^{ii}N_{i0}\left(\eta_{jk}\gamma_l-\eta_{jl}\gamma_k\right)\frac{\gamma^3\|y\|^2}{\sigma^2}+\eta^{ii}N_{i0}\left(\eta_{jk}y_l-\eta_{jl}y_k\right)\left(\gamma_0\frac{4\gamma^5\|y\|^2}{\sigma^2}-\frac{4\gamma^6}{\sigma^2\tau}N_{00}\right)\\
&+\eta^{ii}\left(N_{ik}y_l-N_{il}y_k\right)y_j\gamma_0\frac{2\gamma^3}{\sigma\tau}+\eta^{ii}\left(N_{ik}N_{l0}-N_{il}N_{k0}\right)y_j\frac{\gamma^4(1-\sigma)}{\sigma^2\tau}\\
&-\eta^{ii}\left(N_{ik}N_{0l}-N_{il}N_{0k}\right)y_j\frac{\gamma^4}{\sigma\tau}-\eta^{ii}\left(N_{ik}y_l-N_{il}y_k\right)\left[N_{j0}\frac{\gamma^4(1-\sigma)}{\sigma^2\tau}+N_{0j}\frac{\gamma^4}{\sigma\tau}\right]\\
&-\eta^{ii}\left(N_{ik}y_l-N_{il}y_k\right)\gamma_j\frac{\gamma^4\|y\|^2}{\sigma^2}-\eta^{ii}\left(N_{ik}\gamma_l-N_{il}\gamma_k\right)y_j\frac{\gamma^4\|y\|^2}{\sigma^2}\\
&+\eta^{ii}\left[\left(N_{jk}+N_{kj}\right)N_{il}-\left(N_{jl}+N_{lj}\right)N_{ik}\right]\frac{\gamma^4\|y\|^2\left(2\gamma^2-\tau\right)}{\sigma^2}\\
&+\eta^{ii}\left(\eta_{jk}N_{il}-\eta_{jl}N_{ik}\right)\left(N_{00}\frac{\gamma^4}{\sigma\tau}-\gamma_0\frac{\gamma^3\|y\|^2}{\sigma^2}\right)-\eta^{ii}N_{ir}y_j\left(N_k^r y_l-N_l^r y_k\right)\frac{\gamma^4}{\sigma^2}\\
&-\eta^{ii}N_{ir}\left(\eta_{jk}y_l-\eta_{jl}y_k\right)\left(\gamma_r\frac{\gamma^3\|y\|^2}{\sigma^2}-\eta^{rr}\frac{\gamma^4}{\sigma^2}\right)+\eta^{ii}N_i^r y_j\left(N_{rk}y_l-N_{rl}y_k\right)\frac{\gamma^4}{\sigma^2}\\
&+y^i\left(\eta_{jk}N_{l0}\eta^{ll}N_{ll}-\eta_{jl}N_{k0}\eta^{kk}N_{kk}\right)\frac{2\gamma^4\left(2\gamma^2-\tau\right)}{\sigma^2\tau}\\
&+y^i\left(\eta_{jk}N_{l0}-\eta_{jl}N_{k0}\right)\left(N_{00}\frac{2\gamma^3\left(2\gamma^3\sigma-4\gamma^3-(2\sigma-1)\tau\right)}{\sigma^2\tau^2}+\gamma_0\frac{4\gamma^7\|y\|^2\left(1+2\|y\|^2\right)}{\sigma^2\tau^2}\right)\\
&-y^i\left(\eta_{jk}N_{0l}-\eta_{jl}N_{0k}\right)\left(N_{00}\frac{4\gamma^8\left(1+2\|y\|^2\right)}{\sigma^2\tau^2}+\gamma_0\frac{4\gamma^7\left(1+2\|y\|^2\right)}{\sigma^2\tau^2}\right)\\
&+y^i\left(\eta_{jk}y_l-\eta_{jl}y_k\right)\left(N_{0r}\gamma^r\frac{2\gamma^5\|y\|^2}{\sigma^2\tau}+\gamma_0\frac{8\gamma^6\|y\|^4}{\sigma^2\tau^2}-N_{00}\frac{8\gamma^7\|y\|^4}{\sigma^2\tau^2}\right)\\
&+y^i\left(\eta_{jk}y_l-\eta_{jl}y_k\right)\left(N_{00}\gamma_0\frac{24\gamma^7\|y\|^2}{\sigma^2\tau^2}-\left(\gamma_0\right)^2\frac{2\gamma^4\|y\|^2(\sigma+\tau)}{\sigma^2\tau^2}+\left(\gamma_r\gamma^r\right)\frac{1-2\sigma}{2\sigma^2\tau}\right)\\
&+y^i\left(\eta_{jk}y_l-\eta_{jl}y_k\right)\left(\gamma^r N_{r0}\frac{\gamma^3(4\sigma-3)}{\sigma^2\tau}+\eta^{rr}\left(N_{r0}\right)^2\frac{4\gamma^6}{\sigma^2\tau}-\eta^{rr}N_{r0}N_{0r}\frac{2\gamma^6}{\sigma^2\tau}\right)\\
&+y^i\left(\eta_{jk}\gamma_l-\eta_{jl}\gamma_k\right)\left(N_{00}\frac{\gamma^3(\sigma-\tau+\sigma\tau)}{\sigma^2\tau}-\gamma_0\frac{1-2\sigma}{\sigma\tau}\right)\\
&-y^i\left[\eta_{jk}\left(N_{rl}+N_{lr}\right)-\eta_{jl}\left(N_{rk}+N_{kr}\right)\right]\gamma^r\frac{\gamma^4\left(2\gamma^2-\tau\right)}{\sigma^2\tau}\\
&+y^i N_{j0}\left(y_kN_{l0}-N_{k0}y_l\right)\frac{\gamma^5\left[4(\gamma+1)\left(\gamma^2-\sigma\right)+7-20\sigma\right]}{\sigma^2\tau}\\
&+y^iN_{j0}\left(\gamma_k y_l-y_k\gamma_l\right)\frac{\gamma^3(1-\sigma)}{\sigma^2\tau^2}-y^iN_{0j}\left(\gamma_ky_l-y_k\gamma_l\right)\frac{\gamma^3}{\sigma\tau^2}\\
&+y^iN_{0j}\left(N_{0k}y_l-y_kN_{0l}\right)\frac{2\gamma^6\left(2\gamma^2-\tau-2\sigma+1\right)}{\sigma^2\tau^2}+y^iN_{0j}\left(y_kN_{l0}-N_{k0}y_l\right)\frac{2\gamma^6\left(\tau+2\sigma-2\gamma^2\right)}{\sigma^2\tau^2}\\
&+y^iN_{j0}\left(y_kN_{0l}-N_{0k}y_l\right)\frac{2\gamma^6\left(\sigma-1+4\gamma^2\right)}{\sigma^2\tau^2}+\gamma^i\left(\eta_{jk}\gamma_l-\eta_{jl}\gamma_k\right)\frac{\gamma^2\|y\|^4}{\sigma^2}\\
&-\gamma^i\left(\eta_{jk}N_{l0}-\eta_{jl}N_{k0}\right)\frac{\gamma^3\|y\|^2}{\sigma^2}-\gamma^i\left(\eta_{jk}y_l-\eta_{jl}y_k\right)\left(N_{00}\frac{\gamma^3}{\sigma^2\tau}-\gamma_0\frac{\gamma^2\|y\|^2}{\sigma^2\tau^2}\right)\\
&+\gamma^iy_j\left(N_{ko}y_l-y_kN_{l0}\right)\frac{2\gamma^3(2-3\sigma)}{\sigma^2\tau}+\gamma^iy_j\left(N_{0k}y_l-y_kN_{0l}\right)\frac{\gamma^3}{\sigma^2\tau}
\end{aligned}
$$

$$
\begin{aligned}
&+\gamma^i y_j \left(y_k\gamma_l-\gamma_k y_l\right)\tfrac{(2\sigma-1)(6\sigma+1)}{2\sigma^2\tau}+\gamma^i y_j\left(N_{kl}-N_{lk}\right)\tfrac{2\gamma^3\|y\|^2}{\sigma^2}\\
&-\gamma^i\left(N_{jk}y_l-N_{jl}y_k\right)\left(\tfrac{\gamma^4\left(\tau-2\gamma^2\right)}{\sigma^2\tau}+\tfrac{\gamma^3\|y\|^2\left(\tau-4\gamma^2\right)}{\sigma^2\tau}\right)\\
&-\gamma^i\left(N_{kj}y_l-N_{lj}y_k\right)\left(\tfrac{\gamma^4\left(\tau-2\gamma^2\right)}{\sigma^2\tau}+\tfrac{\gamma^3\|y\|^2\left(3\tau-4\gamma^2\right)}{\sigma^2\tau}\right)\\
&+\left(N^i_l y_j N_{k0}-N^i_k y_j N_{l0}\right)\tfrac{\gamma^4(1-\sigma)}{\sigma^2\tau}-\left(N^i_l y_j N_{0k}-N^i_k y_j N_{0l}\right)\tfrac{\gamma^4}{\sigma\tau}\\
&+\left(N^i_l N_{j0}y_k-N^i_k N_{j0}y_l\right)N_{0j}\tfrac{\gamma^4}{\sigma^2}-\left(N^i_l N_{0j}y_k-N^i_k N_{0j}y_l\right)\tfrac{\gamma^4}{\sigma\tau}\\
&+\left(N^i_l y_j y_k-N^i_k y_j y_l\right)\tfrac{2\gamma^3}{\sigma\tau}+\left(N^i_l y_j\gamma_k-N^i_k y_j\gamma_l\right)\tfrac{6\gamma^5\|y\|^4}{\sigma^2\tau}\\
&-\left(N^i_l\gamma_j y_k-N^i_k\gamma_j y_l\right)\tfrac{\gamma^3\|y\|^2}{\sigma^2\tau}+N^i_j\left(y_k\gamma_l-\gamma_k y_l\right)\tfrac{2\gamma^3\|y\|^2}{\sigma^2}\\
&+N^i_r y_j\left(N^r_k y_l-N^r_l y_k\right)\tfrac{\gamma^4}{\sigma^2}-N^i_r\left(\eta_{jk}y_l-\eta_{jk}y_l\right)\eta^{rr}N_{r0}\tfrac{\gamma^4}{\sigma^2}\\
&-\left(N^i_l\eta_{jk}-N^i_k\eta_{jl}\right)N_{00}\tfrac{\gamma^4}{\sigma\tau}+\left(N^i_l\eta_{jk}-N^i_k\eta_{jl}\right)\gamma_0\tfrac{\gamma^3\|y\|^2}{\sigma^2\tau}\\
&+N^i_r\left(\eta_{jk}y_l-\eta_{jl}y_k\right)\gamma^r\tfrac{\gamma^3\|y\|^2}{\sigma^2\tau}-N^i_r y_j\left(N_{rk}y_l-N_{rl}y_k\right)\eta^{rr}\tfrac{\gamma^4}{\sigma^2}\\
&-\left[N^i_l\left(N_{jk}+N_{kj}\right)-N^i_k\left(N_{jl}+N_{lj}\right)\right]\tfrac{\gamma^4\left(2\gamma^2-\tau\right)}{\sigma\tau}\\
&-N^i_0\left[\left(N_{jk}+N_{kj}\right)y_l-\left(N_{jl}+N_{lj}\right)y_k\right]\tfrac{\gamma^4\left(2\gamma^2-\tau\right)}{\sigma\tau}\\
&-N^i_0 y_j\left(\gamma_k y_l-y_k\gamma_l\right)\tfrac{\gamma^3}{\sigma^2\tau}-N^i_0 y_j\left(N_{k0}y_l-y_k N_{l0}\right)\tfrac{4\gamma^6}{\sigma^2\tau}\\
&+N^i_0 y_j\left(N_{0k}y_l-y_k N_{0l}\right)\tfrac{2\gamma^6}{\sigma^2\tau}+N^i_0\left[\left(\eta_{jk}y_l-\eta_{jl}y_k\right)N_{00}\tfrac{2\gamma^6}{\sigma^2\tau}-\gamma_0\tfrac{2\gamma^5\|y\|^2}{\sigma^2\tau}\right]\\
&+y^i y_j\left(N_{kl}-N_{lk}\right)\left(-\gamma_0\tfrac{4\gamma^5\|y\|^2}{\sigma^2\tau}+N_{00}\tfrac{4\gamma^6}{\sigma^2\tau}\right)\\
&+y^i\left(N_{jk}y_l-N_{jl}y_k\right)\left(\gamma_0\tfrac{2\gamma^5\|y\|^2}{\sigma^2\tau}-N_{00}\tfrac{2\gamma^6}{\sigma^2\tau}\right)\\
&+y^i\left(N_{kj}y_l-N_{lj}y_k\right)\left(-\gamma_0\tfrac{2\gamma^5\|y\|^2}{\sigma^2\tau}+N_{00}\tfrac{2\gamma^6}{\sigma^2\tau}\right)\\
&+y^i y_j\left(N^r_k y_l-y_k N^r_l\right)\left(-\gamma_r\tfrac{\gamma^3}{\sigma^2\tau}-N_{r0}\tfrac{4\gamma^6}{\sigma^2\tau}+N_{0r}\tfrac{2\gamma^6}{\sigma^2\tau}\right)\\
&+y^i y_j\left(N_{rk}y_l-y_k N_{rl}\right)\left[\gamma^r\tfrac{2\gamma^5\left(2-3\|y\|^2\right)}{\sigma^2\tau}+\eta^{rr}N_{r0}\tfrac{4\gamma^6}{\sigma^2\tau}-N^r_0\tfrac{2\gamma^6}{\sigma^2\tau}\right]\\
&+y^i y_j\left(N_{kr}y_l-y_k N_{lr}\right)\gamma^r\tfrac{2\gamma^3\left(2\gamma^2-\tau\right)}{\sigma^2\tau}\\
&+y^i y_j\left(N_{k0}y_l-y_k N_{l0}\right)\left[\gamma_0\tfrac{4\gamma^4\left[\gamma\left(\tau-\sigma-2\gamma^2\right)-4\tau\|y\|^2\right]}{\sigma^2\tau^2}+N_{00}\tfrac{32\gamma^8}{\sigma^2\tau^2}\right]\\
&+y^i y_j\left(y_k N_{0l}-N_{0k}y_l\right)\left[\gamma_0\tfrac{4\gamma^4\left[2\sigma-1-\tau\left(\gamma+\|y\|^2\right)\right]}{\sigma^2\tau^2}-\tfrac{\gamma^3\left(\gamma+\sigma-4\gamma^2\right)}{\sigma^2\tau^2}\right]\\
&+y^i y_j\left(N_{k0}\gamma_l-\gamma_k N_{l0}\right)\tfrac{\gamma^3\left(2\gamma^2-3\tau\right)}{\sigma^2\tau^2}+y^i y_j\left(N_{0k}\gamma_l-\gamma_k N_{0l}\right)\tfrac{\gamma^3\left(\sigma+\tau-4\gamma^2\right)}{\sigma^2\tau^2}\\
&+y^i y_j\left(N_{k0}N_{0l}-N_{0k}N_{l0}\right)\tfrac{4\gamma^6\left(2\gamma^2-\sigma\right)}{\sigma^2\tau^2}\\
&+y^i y_j\left(\gamma_k y_l-y_k\gamma_l\right)\left[N_{00}\tfrac{4\gamma^5(\tau+2)}{\sigma^2\tau^2}+\gamma_0\tfrac{2\sigma\gamma^2(\tau+1)}{\sigma^2\tau^2}\right]\\
&+y^i\gamma_j\left(y_k\gamma_l-\gamma_k y_l\right)\tfrac{4\gamma\varepsilon}{\sigma\tau}+y^i\gamma_j\left(y_k N_{l0}-N_{k0}y_l\right)\tfrac{2\gamma^5\left[1-6\|y\|^2(\gamma-\sigma)\right]}{\sigma^2\tau^2}\\
&+y^i\gamma_j\left(N_{0k}y_l-y_k N_{0l}\right)\tfrac{\gamma^3\left(\sigma-2\gamma^2\right)}{\sigma^2\tau^2},
\end{aligned}
$$

where

$$\mathcal{A}=\tfrac{7\varphi}{\gamma}-\tfrac{6}{\tau^2}-16\varphi\varepsilon,\qquad \mathcal{R}_1=\tfrac{7\varphi}{\gamma}-\tfrac{6}{\tau^2}-18\varphi\varepsilon+\varphi^2\,||y||^2+2\varphi\omega-\varepsilon\phi\,||y||^2\,,$$

$$\mathcal{B}=\tfrac{2\varepsilon}{\gamma}-\tfrac{||y||^2}{\sigma\tau}-8\varepsilon^2,\qquad \mathcal{R}_2=\tfrac{2\varepsilon}{\gamma}-\tfrac{\|y\|^2}{\sigma\tau}-9\varepsilon^2-\varphi\varepsilon\,\|y\|^2\,,$$

$$\mathcal{C}=\frac{\omega}{\gamma}+\frac{2||y||^2}{\sigma^2}+\frac{2\omega}{\sigma\gamma},\qquad \mathcal{R}_3=\frac{\omega}{\gamma}+\frac{2\|y\|^2}{\sigma^2}+\frac{2\omega}{\sigma\tau}+\frac{\varepsilon\gamma\|y\|^2}{2\sigma^3}+\omega^2,$$
$$\mathcal{R}_4=\varphi(\varphi\|y\|^2+\omega+\varepsilon)$$

and

$$\mathcal{T}_1=2\varphi\frac{\gamma^2\left(2\gamma^2-\sigma\right)}{\sigma\tau}+\frac{2\varphi-\gamma^2}{\sigma}+\frac{\left(2-4\gamma^4\right)\left(\varepsilon+\varphi||y||^2\right)}{\sigma\tau},$$
$$\mathcal{T}_2=\left(\varphi-\frac{\gamma}{2\sigma^3}\right)\frac{\gamma^2\left(2\gamma^2-\sigma\right)}{\sigma\tau}-\frac{\gamma^2}{\sigma}\left(\varphi+\frac{\gamma}{2\sigma^3}\right)+\varepsilon+\varphi\,||y||^2+\beta-$$
$$-\frac{2\gamma^4}{\sigma\tau}\left(\omega-\frac{\gamma||y||^2}{2\sigma^3}\right),$$
$$\mathcal{T}_3=-\frac{2\gamma^4}{\sigma\tau}\left[\omega+2\varepsilon+(3\varphi+\alpha)\,||y||^2\right]+2\varphi\frac{\gamma^2\left(2\gamma^2-\sigma\right)}{\sigma\tau}+\frac{4\gamma^4}{\sigma\tau}\left(\omega+\varphi\,||y||^2\right),$$
$$\mathcal{T}_4=\frac{4\gamma^4}{\sigma\tau}-\alpha\frac{\gamma^2}{\sigma}-8\varphi\gamma^2+\frac{4\gamma^2}{\sigma\tau^2}-2\frac{\gamma^4}{\sigma\tau}\left[(\alpha+2\varphi)\,||y||^2+\omega+\varepsilon\right],$$
$$\mathcal{T}_5=\frac{2\gamma^4}{\sigma^2}-\frac{4\gamma^6}{\sigma^2\tau}-\frac{12\gamma^6}{\sigma\tau^2},$$
$$\mathcal{T}_6=4\varepsilon\frac{\gamma^4}{\sigma\tau}-\frac{2\gamma^4||y||^2}{\sigma\tau}\left(\varepsilon+\varphi\,||y||^2\right)-\frac{\gamma^2}{\sigma}\left(3\varepsilon+\varphi\,||y||^2\right)+\frac{2\gamma^3||y||^2}{\sigma\tau},$$
$$\mathcal{T}_7=(\omega+\varepsilon)\,\frac{\gamma^2\left(2\gamma^2-\sigma\right)}{\sigma\tau},$$
$$\mathcal{T}_8=-\frac{\gamma^2}{\sigma}\left(\varepsilon+\varphi\,||y||^2\right)-\frac{2\gamma^3||y||^4}{\sigma^2},$$
$$\mathcal{T}_9=\frac{4\gamma^3}{\sigma^2}+\frac{\gamma^2}{\sigma}\left(2\varphi+\frac{\gamma}{\sigma^3}\right),$$
$$\mathcal{T}_{10}=-\frac{\gamma^2}{\sigma}\varphi\,||y||^2+\varepsilon+\frac{2\gamma}{\sigma^2}\left(\gamma^2\,||y||^4-\sigma\right).$$

Remark 4. *In our opinion, the very complicated form of the d-curvatures, produced by the Lagrangian (3), suggests that the Miron-Anastasiei geometrical theory applied for our anisotropic optical Lagrangian does not offer interesting geometrical aspects for the physical properties of the anisotropic medium. Consequently, the physical interpretations of the geometrical results of this paper still remain an open problem. It follows that another geometrical approach for the anisotropic optical Lagrangian (3) is required.*

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