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ON GENERALIZED TANAKA-WEBSTER CONNECTION IN SASAKIAN MANIFOLDS

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Abstract

The object of this paper is to characterize Sasakian manifolds admitting generalized Tanaka-Webster connection. We study certain curvature conditions of Sasakian manifolds with respect to the generalized Tanaka-Webster connection. Finally, we give an example of a 5-dimensional Sasakian manifold admitting the generalized Tanaka-Webster connection to illustrate the results.

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1 Introduction

Tanaka-Webster connection has been introduced by Tanno [21] as a generalization of the well-known connection defined at the end of the 1970's by Tanaka in [22] and independently by Webster in [25]. This connection coincides with the Tanaka-Webster connection if the associated CR-structure is integrable. The Tanaka-Webster connection is defined as the cannonical affine connection on a non-degenerate, pseudo-Harmitian CR-manifold. For a real hypersurface in a Kähler manifold with almost contact structure (ϕ, ξ, η, g), Cho ([5],[6]) adapted Tanno's g-Tanaka-Webster connection for a non-zero real number k. Using the g-Tanaka-Webster connection, some geometers have studied some characterizations of real hypersurfaces in complex space forms [24]. In 1960 Sasakian manifolds introduced by Sasaki [16], can be described as an odd-dimensional counterpart of Kähler manifolds. The notion of local symmetry of a Riemannian manifold began with the work of Cartan [4]. The notion of locally symmetry of a Riemannian manifold has been weakened by many authors in several directions. As a weaker

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version of local symmetry, in 1977 Takahashi [20] introduced the notion of local ϕ-symmetry on a Sasakian manifold. In this paper we are interested in studying g-Tanaka-Webster connection and finding certain curvature properties of this connection on Sasakian manifolds.
A Sasakian manifold is said to be η-Einstein if

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

where S is the Ricci tensor of type $(0,2)$ and a, b are smooth functions. It is known that [2] in an η-Einstein Sasakian manifold the associated scalars are constant.

A Ricci soliton is a generalization of an Einstein metric. In a Riemannian manifold (M, g), g is called a Ricci soliton if [11, 26]

$$(\pounds_V g + 2S + 2\lambda g)(X,Y) = 0,$$

where $\pounds$ is the Lie derivative, S is the Ricci tensor, V is a complete vector field on M and λ is a constant. The Ricci soliton is said to be shrinking, steady and expanding according to whether λ is negative, zero and positive respectively. For more details we refer to the reader [13, 7].

This article is arranged as follows: Section 2 is a review of all the necessary background in Sasakian manifolds. In section 3, we obtain the expressions of the curvature tensor and Ricci tensor with respect to the g-Tanaka-Webster connection and then prove some results. Section 4 covers ϕ-sectional curvature admitting g-Tanaka-Webster connection connection. Section 5, is devoted to study locally ϕ-symmetric Sasakian manifolds and we prove that local ϕ- symmetry with respect to the Levi-Civita connection and g-Tanaka-Webster connection are equivalent. Next in section 6, we prove that if a Sasakian manifold admits Ricci soliton with respect to the g-Tanaka-Webster connection then the manifold is an η-Einstein manifold. Also we prove that the converse is true if the Ricci soliton is steady. Finally, we construct an example of a 5-dimensional Sasakian manifold admitting g-Tanaka-Webster connection to verify some results.

2 Sasakian manifolds

A $(2n+1)$-dimensional Riemannian manifold with the structure (ϕ, ξ, η, g) is called an almost contact manifold if the following relations hold [1, 2]

$$\phi^2 = -I + \eta \otimes \xi, \eta(\xi) = 1, \ \phi\xi = 0, \ \eta \circ \phi = 0, \tag{1}$$

$$g(X,Y) = g(\phi X, \phi Y) + \eta(X)\eta(Y), \tag{2}$$

$$g(X, \phi Y) = -g(\phi X, Y), \ g(X, \xi) = \eta(X), \tag{3}$$

for all vectors field X, Y. An almost contact metric manifold is called contact metric manifold if

$$d\eta(X,Y) = \Phi(X,Y) = g(X, \phi Y).$$

A normal contact metric manifold is known as Sasakian manifold. An almost contact metric manifold is Sasakian if and only if [3, 17, 18]

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X \tag{4}$$

for all vectors field X, Y, where ∇ is the Levi-Civita connection of the Riemannian metric. From the above equation it follows that

$$\nabla_X \xi = -\phi X, \tag{5}$$

$$(\nabla_X \eta)Y = g(X, \phi Y). \tag{6}$$

Moreover, the curvature tensor R, the Ricci tensor S and the Ricci operator Q satisfy [19, 9, 15]

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{7}$$

$$S(X,\xi) = 2n\eta(X), \tag{8}$$

$$Q\xi = 2n\xi. \tag{9}$$

$$S(\phi X, \phi Y) = S(X,Y) - 2n\eta(X)\eta(Y). \tag{10}$$

$$\eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{11}$$

Sasakian manifolds have been studied by several authors such as Boyer [3], Tachibana [19], Tanno [23], Godliński et al [10], De et al [8], Mihai et al [12, 14] and many others.

3 Expression of the Curvature tensor and Ricci tensor with respect to the g-Tanaka-Webster connection

The g-Tanaka-Webster connection $\bar{\nabla}$ is given by [21],

$$\bar{\nabla}_X Y = \nabla_X Y - \eta(Y)\nabla_X \xi + (\nabla_X \eta)(Y)\xi - \eta(X)\phi Y, \tag{12}$$

for any X, Y vector fields tangent to M and ∇ is the Levi-Civita connection. With the help of (5) and (6) the above equation takes the form,

$$\bar{\nabla}_X Y = \nabla_X Y + g(X, \phi Y)\xi + \eta(Y)\phi X - \eta(X)\phi Y. \tag{13}$$

Putting $Y = \xi$ in (13) and using (1) we have

$$\bar{\nabla}_X \xi = \nabla_X \xi + \phi X, \tag{14}$$

Using (5) in (14) we get

$$\bar{\nabla}_X \xi = 0. \tag{15}$$

Also using (4) and (12) we have,

$$(\bar{\nabla}_X \phi)Y = 0. \tag{16}$$

Let R and $\bar{R}$ denote the curvature tensor ∇ and $\bar{\nabla}$ respectively. Then

$$\bar{R}(X,Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z \tag{17}$$

Using (13) in (17) yields

$$\begin{aligned} \bar{R}(X,Y)Z &= R(X,Y)Z + [g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]\xi - g(Y,\phi Z)\phi X \\ &+ g(X,\phi Z)\phi Y - 2g(Y,\phi X)\phi Z - \eta(Y)\eta(Z)X \\ &+ \eta(X)\eta(Z)Y. \end{aligned} \tag{18}$$

Using (7) and putting $Z = \xi$ in (18) we get

$$\bar{R}(X,Y)\xi = 0.$$

Now taking inner product with W in (18) yields

$$\begin{aligned} g(\bar{R}(X,Y)Z,W) &= g(R(X,Y)Z,W) + [g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]\eta(W) \\ &- g(Y,\phi Z)g(\phi X,W) + g(X,\phi Z)g(\phi Y,W) - 2g(Y,\phi X)g(\phi Z,W) \\ &- \eta(Y)\eta(Z)g(X,W) + \eta(X)\eta(Z)g(Y,W). \end{aligned}$$

Let $\{e_1, e_2, e_3, ..., e_{2n+1}\}$ be a local orthonormal basis of the tangent space at a point of the manifold M. Then by putting $X = W = e_i$ in the above equation and taking summation over i, $1 \leqslant i \leqslant (2n+1)$, we obtain

$$\bar{S}(Y,Z) = S(Y,Z) - 2g(Y,Z) - 2(n-1)\eta(Y)\eta(Z), \tag{19}$$

where $\bar{S}$ and S are the Ricci tensor of M with respect to $\bar{\nabla}$ and ∇ respectively.

Let $\bar{r}$ and r denote the scalar curvature of M with respect to $\bar{\nabla}$ and ∇ respectively. Let $\{e_1, e_2, e_3, ..., e_{2n+1}\}$ be a local orthonormal basis of the tangent space at each point of the manifold M. Then by putting $Y = Z = e_i$ and taking summation over i, $1 \leqslant i \leqslant (2n+1)$, we have

$$\bar{r} = r - 4n. \tag{20}$$

From the above discussions we can state the following:

Proposition 3.1. *For a Sasakian manifold M admitting g-Tanaka-Webster connection $\bar{\nabla}$*
(i) The curvature tensor $\bar{R}$ of $\bar{\nabla}$ is given by (18),
(ii) The Ricci tensor $\bar{S}$ of $\bar{\nabla}$ is given by (19),
(iii) The scalar curvature $\bar{r}$ of $\bar{\nabla}$ is given by (20),
(iv) $\bar{R}(X,Y)Z = -\bar{R}(Y,X)Z$,
(v) $\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = 0$,
(vi) The Ricci tensor $\bar{S}$ is symmetric,

Also from (15) and (16) we state the following:

Proposition 3.2. *In a Sasakian manifold the structure tensor is parallel with respect to the g-Tanaka-Webster connection.*

Remark 3.1. *It is known [2] that, an almost contact metric structure* $(\phi, \xi.\eta, g)$ *is cosympletic if and only if* ϕ *is parallel.*

Hence from (16) it follows that, a Sasakian manifold is cosympletic manifold with respect to the g-Tanaka-Webster connection.

Now suppose that the Sasakian manifold is Ricci flat with respect to the g-Tanaka-Webster connection . Then from (19) we get

$$S(Y, Z) = 2g(Y, Z) + 2(n-1)\eta(Y)\eta(Z).$$

Conversely, if the manifold is an η-Einstein manifold of the form $S(X, Y) = 2g(X, Y) + 2(n-1)\eta(X)\eta(Y)$, then from (19) it follows that $\bar{S}(X, Y) = 0$.
This leads to the following:

Theorem 3.1. *The manifold* M^{2n+1} *is Ricci flat with respect to the g-Tanaka-Webster connection if and only if* M^{2n+1} *is an* η*-Einstein manifold.*

4 ϕ-sectional curvature of Sasakian manifolds admitting g-Tanaka-Webster connection

A plane section in M is called a ϕ-section if there exists a unit vector X in M orthogonal to ξ such that $\{X, \phi X\}$ is an orthonomal basis of the plane section. Then the sectional curvature $K(X, \phi X) = g(R(X, \phi X)\phi X, X)$ is called ϕ-sectional curvature [1].
Taking inner product with W and then putting $Y = Z = \phi X$ and $W = X$ in (18) we get

$$g(\bar{R}(X, \phi X)\phi X, X) = g(R(X, \phi X)\phi X, X) - [g(X, X)]^2.$$

Thus we can state the following:

Theorem 4.1. *If the* ϕ*-sectional curvature of a Sasakian manifold is a constant* c *with respect to the Levi-Civita connection, then the* ϕ*-sectional curvature of the manifold with respect to the g-Tanaka-Webster connection is* $(c - 1)$.

5 Local ϕ symmetry

The notion of local ϕ-symmetry for Sasakian manifolds was introduced by Takahashi [20]. A Sasakian manifold is said to be locally ϕ-symmetric if

$$\phi^2(\nabla_W R)(X, Y)Z = 0,$$

for all vector fields X, Y, Z, W orthogonal to ξ. Now we have,

$$\begin{aligned}(\bar{\nabla}_W \bar{R})(X,Y)Z &= \bar{\nabla}_W \bar{R}(X,Y)Z - \bar{R}(\bar{\nabla}_W X, Y)Z - \bar{R}(X, \bar{\nabla}_W Y)Z \\ &\quad - \bar{R}(X,Y)\bar{\nabla}_W Z. \end{aligned} \tag{21}$$

Using (12), (15) and (17) in (21) yields

$$\begin{aligned}(\bar{\nabla}_W \bar{R})(X,Y)Z &= (\nabla_W R)(X,Y)Z + g(W, \phi R(X,Y)Z)\xi + \eta((R(X,Y)Z)\phi W \\ &\quad - g(Y,\phi Z)\nabla_W \phi X + g(Y,\phi Z)g(W,X)\xi - g(Y,\phi Z)\eta(X)\eta(W)\xi \\ &\quad + g(X,\phi Z)\nabla_W \phi Y - g(X,\phi Z)g(W,Y)\xi + g(X,\phi Z)\eta(Y)\eta(W)\xi \\ &\quad - 2g(Y,\phi X)\nabla_W \phi Z - 2g(Y,\phi X)g(W,Z)\xi + 2g(Y,\phi X)\eta(Z)\eta(W)\xi \\ &\quad + \eta(\nabla_W X)g(Y,Z)\xi - g(\nabla_W X, \phi Z)\phi Y + g(Y,\phi Z)\phi\nabla_W X \\ &\quad - 2g(Y,\phi\nabla_W X)\phi Z - \eta(\nabla_W Y)g(X,Z)\xi - g(X,\phi Z)\phi\nabla_W Y \\ &\quad + g(\nabla_W Y,\phi Z)\phi X - 2g(\nabla_W Y,\phi X)\phi Z - g(X,\phi\nabla_W Z)\phi Y \\ &\quad + g(Y,\phi\nabla_W Z)\phi X - 2g(Y,\phi X)\phi\nabla_W Z. \end{aligned} \tag{22}$$

Operating ϕ^2 on both sides of (22) and using (1) we get,

$$\begin{aligned}\phi^2((\bar{\nabla}_W \bar{R})(X,Y)Z) &= \phi^2((\nabla_W R)(X,Y)Z) + \eta(R(X,Y)Z)\phi^2(\phi W) \\ &\quad - g(Y,\phi Z)\phi^2(\nabla_W \phi X) + g(X,\phi Z)\phi^2(\nabla_W \phi Y) \\ &\quad - 2g(Y,\phi X)\phi^2(\nabla_W \phi Z) - g(\nabla_W X,\phi Z)\phi^2(\phi Y) \\ &\quad + g(Y,\phi Z)\phi^2(\phi\nabla_W X) - 2g(Y,\phi\nabla_W X)\phi^2(\phi Z) \\ &\quad - g(X,\phi Z)\phi^2(\phi\nabla_W Y) + g(\nabla_W Y,\phi Z)\phi^2(\phi X) \\ &\quad - 2g(\nabla_W Y,\phi X)\phi^2(\phi Z) - g(X,\phi\nabla_W Z)\phi^2(\phi Y) \\ &\quad + g(Y,\phi\nabla_W Z)\phi^2(\phi X) - 2g(Y,\phi X)\phi^2(\phi\nabla_W Z). \end{aligned} \tag{23}$$

Using (4), (11) in (23) and considering X, Y, Z, W orthogonal to ξ the above equation yields,

$$\phi^2((\bar{\nabla}_W \bar{R})(X,Y)Z) = \phi^2((\nabla_W R)(X,Y)Z).$$

This leads to the following:

Theorem 5.1. *In a Sasakian manifold local ϕ-symmetry with respect to the Levi-Civita connection and g-Tanaka-Webster connection are equivalent.*

6 Ricci solitons

Suppose that a Sasakian manifold admits Ricci solitons with respect to the connection $\bar{\nabla}$. Then

$$(\pounds_V g + 2\bar{S} + 2\lambda g)(X,Y) = 0,$$

which implies that

$$g(\bar{\nabla}_X \xi, Y) + g(X, \bar{\nabla}_Y \xi) + 2\bar{S}(X,Y) + 2\lambda g(X,Y) = 0. \tag{24}$$

Using (14) in (24) yields

$$\bar{S}(X,Y) + \lambda g(X,Y) = 0. \tag{25}$$

Again using (18) in (25) we have

$$S(X,Y) - 2g(Y,Z) - 2(n-1)\eta(X)\eta(Y) = 0. \tag{26}$$

Putting $X = Y = e_i$ in (26), where $\{e_1, e_2, e_3, ..., e_{2n+1}\}$ is a local orthonormal basis of the tangent space at a point of the manifold M and taking summation over i, $1 \leqslant i \leqslant (2n+1)$ we have

$$\lambda = \frac{(6n - r)}{(2n+1)}. \tag{27}$$

From (26) and (27) we have the following:

Theorem 6.1. *If a Sasakian manifold admits Ricci soliton with respect to the g-Tanaka-Webster connection, then the manifold is an η-Einstein manifold and the Ricci soliton is shrinking, steady or expanding according to $r > 6n$, $r = 6n$ or $r < 6n$.*

The converse of the above theorem is not true, in general. Let M be an η-Einstein Sasakian manifold. Then

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{28}$$

where a and b are certain scalars. Now using (17) we have,

$$\begin{aligned} & g(\bar{\nabla}_X \xi, Y) + g(X, \bar{\nabla}_Y \xi) + 2\bar{S}(X,Y) + 2\lambda g(X,Y) \\ = \; & S(X,Y) - 2g(X,Y) - 2(n-1)\eta(X)\eta(Y) + \lambda g(X,Y). \end{aligned} \tag{29}$$

Using (28) in the above equation yields

$$\begin{aligned} & g(\bar{\nabla}_X \xi, Y) + g(X, \bar{\nabla}_Y \xi) + 2\bar{S}(X,Y) + 2\lambda g(X,Y) \\ = \; & (a + \lambda - 2)g(X,Y) + (b - 2n + 2)\eta(X)\eta(Y). \end{aligned} \tag{30}$$

Now if the η-Einstein manifolds admits a Ricci soliton with respect to g-Tanaka-Webster connection then from above equation it follows that,

$$a + \lambda - 2 = 0 \tag{31}$$

and

$$b - 2n + 2 = 0. \tag{32}$$

Also from (28) it follows that

$$a + b = 2n. \tag{33}$$

Using (31), (32) and (33) we have, $\lambda = 0$. This leads to the following:

Theorem 6.2. *If an η-Einstein Sasakian manifold, admits a Ricci soliton with respect to the g-Tanaka-Webster connection, then the Ricci soliton is steady.*

7 Example of a 5-dimensional Sasakian manifold admitting g-Tanaka-Webster connection

Consider the 5-dimensional manifold $M = \{(x, y, z, u, v) \in \mathbb{R}^5\}$, where (x, y, z, u, v) are the standard coordinates in $\mathbb{R}^5$.
We choose the vector fields
$e_1 = 2(y\frac{\partial}{\partial z} - \frac{\partial}{\partial x})$, $e_2 = \frac{\partial}{\partial y}$, $e_3 = -2\frac{\partial}{\partial z}$, $e_4 = 2(v\frac{\partial}{\partial z} - \frac{\partial}{\partial u})$, $e_5 = -2\frac{\partial}{\partial v}$,
which are linearly independent at each point of M.
Let g be the Riemannian metric defined by $g(e_i, e_j) = 0$, $i \neq j$, $i, j = 1, 2, 3, 4, 5$ and
$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = g(e_4, e_4) = g(e_5, e_5) = 1$.
Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$, for any $Z \in \chi(M)$, where $\chi(M)$ is the set of all differentiable vector fields on M.
Let ϕ be the $(1, 1)$-tensor field defined by
$\phi e_1 = e_2$, $\phi e_2 = -e_1$, $\phi e_3 = 0$, $\phi e_4 = e_5$, $\phi e_5 = -e_4$.
Using the linearity of ϕ and g, we have
$\eta(e_3) = 1$, $\phi^2 Z = -Z + \eta(Z)e_5$ and $g(\phi Z, \phi U) = g(Z, U) - \eta(Z)\eta(U)$, for any $U, Z \in \chi(M)$. Thus, for $e_3 = \xi$, $M(\phi, \xi, \eta, g)$ defines an almost contact metric manifold.
Also we have
$[e_1, e_2] = 2e_3, [e_4, e_5] = 2e_3$ and $[e_i, e_j] = 0$ for others i, j.
The Levi-Civita connection ∇ of the metric tensor g is given by Koszul's formula which is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) \\ &\quad - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned} \tag{34}$$

Taking $e_3 = \xi$ and using Koszul's formula we get the following

$\nabla_{e_1} e_1 = 0, \nabla_{e_1} e_2 = e_3, \nabla_{e_1} e_3 = -e_2, \nabla_{e_1} e_4 = 0, \nabla_{e_1} e_5 = 0,$
$\nabla_{e_2} e_1 = -e_3, \nabla_{e_2} e_2 = 0, \nabla_{e_2} e_3 = e_1, \nabla_{e_2} e_4 = 0, \nabla_{e_2} e_5 = 0,$
$\nabla_{e_3} e_1 = -e_2, \nabla_{e_3} e_2 = e_1, \nabla_{e_3} e_3 = 0, \nabla_{e_3} e_4 = 0, \nabla_{e_3} e_5 = e_4,$
$\nabla_{e_4} e_1 = 0, \nabla_{e_4} e_2 = 0, \nabla_{e_4} e_3 = -e_5, \nabla_{e_4} e_4 = 0, \nabla_{e_4} e_5 = e_3,$
$\nabla_{e_5} e_1 = \nabla_{e_5} e_2 = \nabla_{e_5} e_3 = \nabla_{e_5} e_4 = \nabla_{e_5} e_5 = 0.$

From the above results we see that (ϕ, ξ, η, g) structure satisfies the formula

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X,$$

where $\eta(e_3) = 1$. Hence $M(\phi, \xi, \eta, g)$ is a 5-dimensional Sasakian manifold.
Using the above relation in (13), we obtain

$\bar{\nabla}_{e_1} e_1 = \bar{\nabla}_{e_1} e_2 = \bar{\nabla}_{e_1} e_3 = \bar{\nabla}_{e_1} e_4 = \bar{\nabla}_{e_1} e_5 = 0,$
$\bar{\nabla}_{e_2} e_1 = \bar{\nabla}_{e_2} e_2 = \bar{\nabla}_{e_2} e_3 = \bar{\nabla}_{e_2} e_4 = \bar{\nabla}_{e_2} e_5 = 0,$

$\bar{\nabla}_{e_3}e_1 = \bar{\nabla}_{e_3}e_2 = \bar{\nabla}_{e_3}e_3 = \bar{\nabla}_{e_3}e_4 = \bar{\nabla}_{e_3}e_5 = 0,$
$\bar{\nabla}_{e_4}e_1 = \bar{\nabla}_{e_4}e_2 = \bar{\nabla}_{e_4}e_3 = \bar{\nabla}_{e_4}e_4 = \bar{\nabla}_{e_4}e_5 = 0,$
$\bar{\nabla}_{e_5}e_1 = \bar{\nabla}_{e_5}e_2 = \bar{\nabla}_{e_5}e_3 = \bar{\nabla}_{e_5}e_4 = \bar{\nabla}_{e_5}e_5 = 0.$

By the above results, we can easily obtain that the non-vanishing components of the curvature tensor with respect to the Levi-Civita connection are as follows:

$R(e_1, e_2)e_1 = 3e_2, R(e_1, e_3)e_1 = -e_3, R(e_2, e_4)e_1 = -e_5, R(e_2, e_5)e_1 = e_4,$
$R(e_4, e_5)e_1 = 2e_2, R(e_1, e_2)e_2 = -e_1, R(e_1, e_4)e_2 = e_5, R(e_2, e_3)e_2 = -e_3,$
$R(e_4, e_5)e_2 = -2e_1, R(e_1, e_3)e_3 = e_1, R(e_2, e_3)e_1 = -e_3, R(e_3, e_4)e_3 = -e_4,$
$R(e_4, e_5)e_4 = 2e_5, R(e_1, e_2)e_5 = -2e_4, R(e_1, e_4)e_5 = e_2, R(e_2, e_4)e_5 = e_1,$
$R(e_4, e_5)e_5 = -2e_4, R(e_1, e_4)e_5 = -e_2.$
Now the components of the curvature tensor with respect to the g-Tanaka-webster connection are as follows:

$\bar{R}(e_1, e_2)e_2 = \bar{R}(e_1, e_3)e_3 = \bar{R}(e_1, e_4)e_4 = 0,$
$\bar{R}(e_1, e_2)e_1 = \bar{R}(e_1, e_3)e_1 = \bar{R}(e_2, e_3)e_2 = 0,$
$\bar{R}(e_2, e_3)e_3 = \bar{R}(e_2, e_4)e_4 = \bar{R}(e_2, e_5)e_5 = 0,$
$\bar{R}(e_3, e_4)e_4 = \bar{R}(e_2, e_5)e_2 = \bar{R}(e_1, e_5)e_1 = 0,$
$\bar{R}(e_3, e_5)e_3 = \bar{R}(e_1, e_4)e_1 = \bar{R}(e_2, e_4)e_2 = 0,$
$\bar{R}(e_1, e_5)e_5 = \bar{R}(e_3, e_5)e_5 = \bar{R}(e_4, e_5)e_5 = 0.$

With the help of the above results we get the Ricci tensor are as follows:

$$S(e_1, e_1) = -2, S(e_2, e_2) = 3, S(e_3, e_3) = S(e_4, e_4) = 4, S(e_5, e_5) = -1, \quad (35)$$

and

$$\bar{S}(e_1, e_1) = \bar{S}(e_2, e_2) = \bar{S}(e_3, e_3) = \bar{S}(e_4, e_4) = \bar{S}(e_5, e_5) = 0. \quad (36)$$

Therefore $r = \sum_{i=1}^{5} S(e_i, e_i) = 8$ and $\bar{r} = \sum_{i=1}^{5} \bar{S}(e_i, e_i) = 0$.
Now from the expressions of the curvature tensor and Ricci tensor we can easily verify Proposition 3.1 and Theorem 3.1.

References

[1] Blair, D. E., *Contact manifolds in Riemannian geometry* , Lecture notes in math. **509**, Springer-Verlag., (1976).

[2] Blair, D. E., *Riemannian geometry of contact and Symplectic Manifolds,* Progress in Math. **203**, Birkhäuser, Boston, (2002).

[3] Boyer, Charles P. and Galicki, K. *Sasakian geometry* , Oxford Mathematical Monographs, Oxford University Press, Oxford.,(2008), MR 2382957, Zbl 1155.53002.

[4] Cartan, E., *Sur une classe remarquable d'espaces de Riemannian,* Bull. Soc. Math. France, **54** (1926), 214-264.

[5] Cho, J. T., *CR-structures on real hypersurfaces of a complex space form,* Publ. Math. Debrecen **54** (1999), 473-487.

[6] Cho, J. T., *Pseudo-Einstein CR-structures on real hypersurfaces in a complex space form,* Hokkaido Math. J. **37** (2008), 1-17.

[7] Chow, B. and Knopf, D., *The Ricci flow: An introduction,* Math. Surv. and Monogr. **110** (2004).

[8] De, U. C., Gazi, A. K. and Özgur, C., *On almost pseudo symmetries of Sasakian manifolds,* Math. Pannon. **19** (2008), 81-88.

[9] Futaki, A.,Ono, H. and Wang, G., *Transverse Kähler geometry of Sasakian manifolds and toric Sasaki Einstein manifolds,* J. Diff. Geom. **83** (2009), 585-636.

[10] Godliński, M., Kopczyński, W. and Nurowski, P., *Locally Sasakian manifolds,* Class. Quantum Grav. **17** (2000).

[11] Hamilton, R. S., *The Ricci flow on surfaces,* Contemp. Math. **71** (1988).

[12] Hasegawa, I. and Mihai, I., *Contact CR-warped product submanifolds in Sasakian manifolds,* Geom. Dedicata. **102** (2003), 143-150.

[13] Ivey, T., *Ricci solitons on compact* 3*-manifolds,* Different. Geom. Appl. **3** (1993), 301-307.

[14] Mihai, I., *Certain submanifolds of a Sasakian manifolds.* Geometry and topology of submanifolds, VIII (Brussels, 1995/Nordfjordeid,1995), 265-268, World Sci. Publ. River Edge, NJ, 1996.

[15] Martelli, D., Sparks, J. and Yau, S. T., *Sasaki Einstein manifolds and volume minimisation,* Commun. Math. Phys., **280** (2007), 611-673.

[16] Sasaki, S., *On differentiable manifolds with certain structures which are closely related to almost contact structure,* Tohoku Math J., **12** (1960), 456-476.

[17] Sasaki, S., *Lectures note on Almost contact manifolds, Part I*, Tohoku University, 1965.

[18] Sasaki, S., *Lectures note on Almost contact manifolds, Part II*, Tohoku University, 1967.

[19] Tachibana, S., *On harmonic tensors in compact Sasakian spaces,* Tohoku Math J.(2), **17** (1965), 271-284.

[20] Takahasi, T., *Sasakian ϕ-symmetryc spaces,* Tohoku Math J. **29** (1977), 91-113.

[21] Tanno, S., *The automorphism groups of almost contact Riemannian manifold,* Tohoku Math. J. **21** (1969), 21-38.

[22] Tanaka, N., *On non-degenerate real hypersurface, graded Lie algebra and Cartan connections,* Japan . J. Math. **2** (1976), 131-190.

[23] Tanno, S., *Sasakian manifolds with constant ϕ-holomorphic sectional curvature,* Tohoku Math J. **23** (1969), 21-38.

[24] Takagi, R., *Real hypersurfaces in complex projective space with constant principal curvatures,* J. Math. Soc. Japan **27** (1975), 45-53.

[25] Webster, S. M., *Pseudohermitian structures on a real hypersurface,* J. Diff. Geom. **13** (1978), 25-41.

[26] Yildiz, A., De, U. C. and Turan, M., *On* 3*-dimensional f-Kenmotsu manifolds and Ricci soliton,* Ukrainian J. Math. **65** (2013), 684-693.