Bulletin of the *Transilvania* University of Braşov • Vol 9(58), No. 1 - 2016
Series III: Mathematics, Informatics, Physics, 91-96

AN ANISOTROPIC GEOMETRICAL APPROACH FOR EXTENDED RELATIVISTIC DYNAMICS

Mircea NEAGU[1] **and Alexandru OANĂ**[2]

Abstract

In this paper we present the distinguished (d-) pseudo-Riemannian geometry (in the sense of nonlinear connection, Cartan canonical linear connection, together with its d-torsions) for a Lagrangian inspired by relativistic optics in non-uniform media.

2010 *Mathematics Subject Classification:* 53C60, 53C80, 83C10.
Key words: Minkowski metric, anisotropic optics, nonlinear connection, Cartan linear connection, d-torsions.

1 Introduction

In the geometrical optics [3], a central role is played by the Synge-Beil metric [1], [6] and [7]

$$g_{\alpha\beta}(x,y) = \varphi_{\alpha\beta}(x) + \gamma^2 y_\alpha y_\beta, \tag{1}$$

where $\gamma(x) \geq 0$ is a positive smooth function on the space-time M^4, and $\varphi_{\alpha\beta}(x)$ is a pseudo-Riemannian metric on M^4. One assumes that the four-dimensional manifold M^4 (which is connected and simply connected) is endowed with the local coordinates $(x^\alpha)_{\alpha=\overline{1,4}} = \left(x^1 = t, x^2, x^3, x^4\right)$; for simplicity we use the system of units where the light velocity is $c = 1$. Obviously, the following rule holds good: $y_\alpha = \varphi_{\alpha\mu} y^\mu$. Because the components of $\varphi_{\alpha\beta}(x)$ are dimensionless, the same are γy_α: $[\varphi_{\alpha\beta}(x)] = 1,\ [\gamma y_\alpha] = 1$.

In such a context, let us restrict our geometric-physical study to the Minkowski manifold $\mathcal{M}^4 = \left(\mathbb{R}^4, \eta_{ij}\right)$ which has the local coordinates $(x) := \left(x^i\right)_{i=\overline{1,4}}$. It follows that the corresponding tangent bundle $T\mathbb{R}^4$ has the dimension equal to

[1]Department of Mathematics and Informatics, University Transilvania of Braşov, 50 Iuliu Maniu Bd., 500091 Braşov, Romania, e-mail: mircea.neagu@unitbv.ro
[2]Department of Mathematics and Informatics, University Transilvania of Braşov, 50 Iuliu Maniu Bd., 500091 Braşov, Romania, e-mail: alexandru.oana@unitbv.ro

eight, and its local coordinates are[3]

$$(x,y) := \left(x^i, y^i\right)_{i=\overline{1,4}} = \left(\underbrace{x^1, x^2, x^3, x^4}_{\text{space-time coordinates}}, \underbrace{y^1, y^2, y^3, y^4}_{\text{tangent vector}} \right).$$

Let us introduce a metric on $T\mathbb{R}^4$ inspired by optics in a non-uniform medium (see formula (1)):

$$\mathfrak{g}_{ij}(x,y) = \eta_{ij} + \gamma^2(x) y_i y_j, \tag{2}$$

where $\eta = (\eta_{ij}) = \text{diag } (-1,1,1,1)$ is the Minkowski metric, and $y_i = \eta_{ir} y^r$. Usually, we have $\gamma^2(x) = n^2(x) - 1$, where $n = n(x)$ is the refractive index of the non-uniform medium (see [1], [6] and [7]). Using the metric (2), below we will examine the special case of a possible anisotropic relativistic dynamical model, which is governed by the Lagrangian (in this model one considers that the particle has the mass $m = 1$)

$$\begin{aligned} L(x,y) &= \frac{1}{2}\mathfrak{g}_{ij}(x,y) y^i y^j = \\ &= \frac{1}{2}\left[\eta_{ij} + \gamma^2 y_i y_j\right] y^i y^j = \\ &= \frac{1}{2}\eta_{ij} y^i y^j + \frac{\gamma^2}{2}||y||^4, \end{aligned} \tag{3}$$

where $||y||^2 = -(y^1)^2 + (y^2)^2 + (y^3)^2 + (y^4)^2 = \eta_{ij} y^i y^j$.

Remark 1. *Because the Minkowski metric η_{ij} is invariant with respect to the linear transformations of coordinates induced by the Lorentz group $O(3,1)$, it immediately follows that the Lagrangian (3) has a global geometrical character with respect to these Lorentz transformations.*

Following as a pattern the geometrical ideas from Lagrangian geometry of tangent bundles [5] or jet bundles [2], in what follows we construct the pseudo-Riemann-Lagrange geometrical objects (the canonical nonlinear connection, the Cartan canonical linear connection, together with its d-torsions) produced by the Lagrangian (3).

2 Geometrical objects in the relativistic extended dynamics

The fundamental metrical distinguished tensor induced by the Lagrangian (3) is given by

$$g_{ij}(x,y) = \frac{1}{2}\frac{\partial^2 L}{\partial y^i \partial y^j} = \sigma(x,y)\eta_{ij} + 2\gamma^2(x) y_i y_j,$$

where $\sigma(x,y) = (1/2) + \gamma^2(x)||y||^2 > 0$.

[3]In this paper the Latin letters i, j, k, ... run from 1 to 4. The Einstein convention of summation is adopted all over this work.

Remark 2. *Because the quadratic form $q(\xi) = (y_i y_j)\,\xi^i\xi^j$ is degenerate and has the signature $(3,1,0)$, while the Minkovski metric $\eta(\xi) = \eta_{ij}\xi^i\xi^j$ is non-degenerate and has the signature $(0,3,1)$, we easily deduce that the quadratic form $g(\xi) = g_{ij}(x,y)\xi^i\xi^j$ has the constant signature $(0,3,1)$. It follows that it is invariant under Lorentz linear transformation of coordinates. Consequently, all the subsequent geometrical objects constructed in this paper will have the same form in any chart of coordinates induced by a linear transformation of coordinates produced by the Lorentz group $O(3,1)$.*

The inverse matrix $g^{-1} = (g^{jk})$ has the entries

$$g^{jk}(x,y) = \frac{1}{\sigma(x,y)}\eta^{jk} - \frac{2\gamma^2(x)}{\sigma(x,y)\cdot\tau(x,y)}y^j y^k,$$

where $\eta^{jk} = \eta_{jk}$ and $\tau(x,y) = (1/2) + 3\gamma^2(x)||y||^2 = \sigma(x,y) + 2\gamma^2(x)||y||^2$.

Proposition 1. *For the anisotropic Lagrangian (3), the action*

$$\mathbb{E}(x(t)) = \int_a^b L(x(t),y(t))dt,$$

where $y = dx/dt$, produces on the tangent bundle $T\mathbb{R}^4$ the ***canonical nonlinear connection*** $N = \left(N^i_j\right)$, *whose components are*

$$\begin{aligned} N^i_j \;=\; & \frac{2\gamma}{\sigma}y^i y_j(\gamma_s y^s) + \frac{\gamma||y||^2}{\sigma}\left[\delta^i_j(\gamma_s y^s) + y^i\gamma_j - \gamma^i y_j - \frac{2\gamma^2}{\sigma}y^i y_j(\gamma_s y^s) - \right. \\ & \left. - \frac{6\gamma^2}{\tau}y^i y_j(\gamma_s y^s)\right] + \\ & + \frac{\gamma^3||y||^4}{2\sigma}\left[\frac{1}{\sigma}\gamma^i y_j - \frac{3}{\tau_M}y^i\gamma_j - \frac{3}{\tau}\delta^i_j(\gamma_s y^s) + \frac{6\gamma^2}{\sigma\tau^2}(\tau+3\sigma)y^i y_j(\gamma_s y^s)\right], \end{aligned} \tag{4}$$

where $\gamma_s = \partial\gamma/\partial x^s$ and $\gamma^i = \eta^{ir}\gamma_r$.

Proof. For the energy action functional $\mathbb{E}$, the associated Euler-Lagrange equations can be written in the equivalent form (see [5] and [2])

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x^k(t), y^k(t)\right) = 0, \qquad \forall\; i = \overline{1,4}, \tag{5}$$

where the local components

$$\begin{aligned} G^i \;&\stackrel{def}{=}\; \frac{g^{ir}}{4}\left[\frac{\partial^2 L}{\partial y^r\partial x^s}y^s - \frac{\partial L}{\partial x^r}\right] = \\ &=\; \frac{\gamma}{\sigma}||y||^2 y^i(\gamma_s y^s) - \frac{\gamma}{4\sigma}||y||^4\gamma^i - \frac{3\gamma^3}{2\sigma\tau}||y||^4 y^i(\gamma_s y^s) \end{aligned}$$

represent, from a geometrical point of view, a *semispray* on the tangent vector bundle $T\mathbb{R}^4$. The *canonical nonlinear connection* associated to this semispray has the components (see [5]) $N^i_j \stackrel{def}{=} \dfrac{\partial G^i}{\partial y^j}$.

In conclusion, by direct computations, we find the expression (4). □

Remark 3. *In a uniform medium with the constant refractive index $n(x) = n \in [1, \infty)$, we have $\gamma_s = 0$. Consequently, in this case we obtain $G^i = 0$ and $N^i_j = 0$.*

The nonlinear connection (4) produces the dual *adapted bases* of d-vector fields

$$\left\{\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^r_i \frac{\partial}{\partial y^r} \ ; \ \frac{\partial}{\partial y^i}\right\} \subset \mathcal{X}(T\mathbb{R}^4) \tag{6}$$

and d-covector fields

$$\left\{dx^i \ ; \ \delta y^i = dy^i + N^i_r dx^r\right\} \subset \mathcal{X}^*(T\mathbb{R}^4). \tag{7}$$

The naturalness of the geometrical adapted bases (6) and (7) is coming from the fact that, via a transformation of coordinates, their elements transform as tensors on $\mathbb{R}^4$. Therefore, the description of all subsequent geometrical objects on the tangent space $T\mathbb{R}^4$ (e.g., the Cartan canonical linear connection and its torsion) will be done in local adapted components.

For instance, using the notations $N_{ij} := N^r_i \eta_{rj}$, $N_{i0} := N_{ir} y^r$, $N_{0j} := N_{rj} y^r$, $N_{00} := N_{ij} y^i y^j$, by direct local computations, we obtain the following geometrical result:

Proposition 2. *The Cartan canonical N-linear connection produced by the anisotropic Lagrangian (3) has the adapted local components $C\Gamma(N) = \left(L^i_{jk},\ C^i_{jk}\right)$, where*

$$\begin{aligned} L^i_{jk} \ = \ & -\frac{\gamma}{\sigma}\Big[\gamma\left(\delta^i_j N_{k0} + \delta^i_k N_{j0} - \eta^{ir}\eta_{jk} N_{r0}\right) + ||y||^2\left(\eta_{jk}\gamma^i - \delta^i_j \gamma_k - \delta^i_k \gamma_j\right) + \\ & +\gamma\left\{\left(N_{jk} + N_{kj}\right) y^i + \left(N^i_k - \eta^{ir} N_{rk}\right) y_j + \left(N^i_j - \eta^{ir} N_{rj}\right) y_k\right\} + \\ & +2\left(\gamma^i y_j y_k - y^i y_j \gamma_k - y^i y_k \gamma_j\right)\Big] + \frac{2\gamma^3 y^i}{\sigma\tau}\left[\gamma\left(y_j N_{k0} + y_k N_{j0} - \eta_{jk} N_{00}\right) + \right. \\ & +||y||^2\left(\delta_{jk}\gamma_r y^r - y_j \gamma_k - y_k \gamma_j\right) + 2\left(y_j y_k \gamma_r y^r - y_j \gamma_k ||y||^2 - y_k \gamma_j ||y||^2\right) + \\ & \left. +\gamma\left\{\left(N_{jk} + N_{kj}\right)||y||^2 + \left(N_{k0} - N_{0k}\right) y_j + \left(N_{j0} - N_{0j}\right) y_k\right\}\right], \end{aligned} \tag{8}$$

$$C^i_{jk} = \frac{\gamma^2}{\sigma}\left(y^i \eta_{jk} + \delta^i_j y_k + \delta^i_k y_j\right) - \frac{2\gamma^4}{\sigma\tau}\left(||y||^2 \eta_{jk} + 2 y_j y_k\right) y^i. \tag{9}$$

Proof. The adapted components of the Cartan canonical connection are given by the general formulas (see [5])

$$L^i_{jk} \overset{def}{=} \frac{g^{ir}}{2}\left(\frac{\delta g_{jr}}{\delta x^k} + \frac{\delta g_{kr}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^r}\right),$$

$$C^i_{jk} \overset{def}{=} \frac{g^{ir}}{2}\left(\frac{\partial g_{jr}}{\partial y^k} + \frac{\partial g_{kr}}{\partial y^j} - \frac{\partial g_{jk}}{\partial y^r}\right) = \frac{g^{ir}}{2}\frac{\partial g_{jr}}{\partial y^k}.$$

Using the derivative operators (6), direct calculations lead us to the claimed results. □

Proposition 3. *The Cartan canonical N-linear connection produced by the anisotropic optical Lagrangian (3) is characterized by three effective local torsion d-tensors, namely R^i_{jk}, P^i_{jk}, C^i_{jk}, where*

$$\begin{aligned}
R^i_{jk} &= y^i\left(y_j\gamma_{0k}-y_k\gamma_{0j}\right)\varphi+(\delta^i_j\gamma_{0k}-\delta^i_k\gamma_{0j})\varepsilon+\eta^{is}(\gamma_{sk}y_j-\gamma_{sj}y_k)\omega\\
&\quad+y^i(y_j\gamma_k-\gamma_jy_k)(\gamma_s\gamma^s)[\tfrac{7\varphi}{\gamma}-\tfrac{6}{\tau^2}-18\varphi\varepsilon+\varphi^2\|y\|^2+2\varphi\omega-\varepsilon\phi\|y\|^2]\\
&\quad+(\delta^i_j\gamma_k-\delta^i_k\gamma_j)(\gamma_s\gamma^s)[\tfrac{2\varepsilon}{\gamma}-\tfrac{\|y\|^2}{\sigma\tau}-9\varepsilon^2-\varphi\varepsilon\|y\|^2]\\
&\quad+\gamma^i(y_j\gamma_k-\gamma_jy_k)[\tfrac{\omega}{\gamma}+\tfrac{2\|y\|^2}{\sigma^2}+\tfrac{2\omega}{\sigma\tau}+\tfrac{\varepsilon\gamma\|y\|^2}{2\sigma^3}+\omega^2]\\
&\quad-(\delta^i_jy_k-\delta^i_ky_j)(\gamma_s\gamma^s)^2\varphi(\varphi\|y\|^2+\omega+\varepsilon)-(\delta^i_jy_k-\delta^i_ky_j)(\gamma_s\gamma^s)\omega\varepsilon,\\
P^i_{jk} &= y^iy_jy_k\gamma_0[12\varphi\gamma^2\tau(\tfrac{1}{\sigma}-\sigma^2)-\tfrac{9\gamma^3}{\sigma^2\tau}-\tfrac{4\gamma^3}{\sigma\tau}]+(\delta^i_jy_k+\delta^i_ky_j)\gamma_0\varphi\\
&\quad+y^i(y_j\gamma_k+\gamma_jy_k)[\varphi-\tfrac{2\gamma}{\sigma}+\tfrac{6\gamma^3\|y\|^2}{\sigma\tau}]+y^i\eta_{jk}\gamma_0[\varphi-\tfrac{2\gamma^3\|y\|^2}{\sigma\tau}]\\
&\quad+(\delta^i_j\gamma_k+\delta^i_k\gamma_j)(-\tfrac{3\gamma^3\|y\|^4}{2\sigma\tau})+\gamma^i\eta_{jk}\tfrac{\gamma^3\|y\|^4}{2\sigma^2}+\gamma^iy_jy_k\tfrac{\gamma(4\sigma^2-1)}{2\sigma^3}\\
&\quad+(\delta^i_jN_{k0}+\delta^i_kN_{j0}-\eta^{ii}\eta_{jk}N_{i0})\tfrac{\gamma^2}{\sigma}-y^i(N_{kj}+N_{jk})(\tfrac{2\gamma^4}{\sigma\tau}-\tfrac{\gamma^2}{\sigma})\\
&\quad+[(N^i_j-\eta^{ii}N_{ij})y_k+((N^i_k-\eta^{ii}N_{ik})y_j)]\tfrac{\gamma^2}{\sigma}\\
&\quad-(y^iN_{j0}y_k+y^iy_jN_{k0})\tfrac{4\gamma^4}{\sigma\tau}+(y^i\eta_{jk}N_{00}+y^iy_jN_{0k}+y^iN_{0j}y_k)\tfrac{2\gamma^4}{\sigma\tau},
\end{aligned}$$

where

$$\begin{aligned}
\varphi &= \frac{\gamma(12\sigma^2-6\sigma+1)}{4\sigma^2\tau^2}\\
\varepsilon &= \frac{\gamma\|y\|^2(2\tau+1)}{4\sigma\tau}\\
\phi &= \frac{\gamma^3(12\sigma-3)}{\sigma^3\tau^2}-\frac{12\sigma^2\gamma^2\varphi}{\tau}\\
\omega &= -\frac{\gamma\|y\|^2(2\sigma+1)}{4\sigma^2}.
\end{aligned}$$

Proof. The local components of the torsion tensor are given by the general formulas (see [5])

$$R^i_{jk}=\frac{\delta N^i_j}{\delta x^k}-\frac{\delta N^i_k}{\delta x^j},\quad P^i_{jk}=\frac{\partial N^i_j}{\partial y^k}-L^i_{kj}.$$

Using formula (4) and derivative operators (6), after laborious calculations we get the claimed results. □

Acknowledgements. The present work was developed under the auspices of Grant AR-FRBCF 2565/2015–BRFFR–RA, No. F14RA-006, within the cooperation framework between Romanian Academy and Belarusian Republican Foundation for Fundamental Research.

References

[1] Balan, V., *Synge-Beil and Riemann-Jacobi jet structures with applications to physics*, Int. J. Math. Math. Sci., vol. **2003**, no. 27, 1693-1702.

[2] Balan, V. and Neagu, M., *Jet single-time Lagrange geometry and its applications*, John Wiley & Sons, Inc., Hoboken, New Jersey, 2011.

[3] Landau, L, D. and Lifshitz, E. M., *Physique théorique. **1.** Mécanique. **2.** Théorie des Champs* (in French), Éditions Mir, Moscou, 1982, 1989.

[4] Minnaert, M. G. J., *Light and color in the outdoors*, Springer Verlag, New York, 1993.

[5] Miron, R., Anastasiei, M., *The geometry of Lagrange spaces: theory and applications*, Kluwer Academic Publishers, Dordrecht, 1994.

[6] Miron, R. and Kawaguchi, T., *Relativistic geometrical optics*, Int. J. Theor. Phys., **30** (1991), no. 11, 1521-1543.

[7] Neagu, M., *Riemann-Lagrange geometry for relativistic multi-time optics*, Semin. Mech. - Differ. Dyn. Syst., West Univ. Timişoara, Romania, **87** (2004), 1-16.

[8] Neagu, M., Oana, A. and Red'kov, V. M., *An anisotropic geometrical approach for non-relativistic extended dynamics*, Ricerche Mat., **62** (2013), no. 2, 323–340.

[9] Stam, J. and Languénou, E., *Ray tracing in non-constant media*, In "*Rendering Techniques '96*", Proc. of the 7^{th} Eurographics Workshop on Rendering, Porto, Portugal, June 17-19 (1996); (X. Pueyo and P. Schroeder Eds), Springer-Verlag, 225-234, 1996.

[10] Voicu, N., *On the fundamental equations of electromagnetism in Finslerian spacetimes*, Prog. Electromagn. Res., **113** (2011), 83-102.