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UNIFICATION OF MAXIMALITY AND MINIMALITY OF OPEN AND μ-OPEN SETS

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Abstract

In this paper, we introduce and study the notions of maximal and minimal generalized open sets of a GTS (X, μ) with respect to open sets of a topological space (X, τ) along with the notions of maximal and minimal open sets of a topological space (X, τ) with respect to generalized open sets of a GTS (X, μ). We observe that contrary to maximal and minimal μ-open sets of a GTS (X, μ), the unified notions of maximality and minimality of generalized open sets behave differently.

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Key words: maximal open set, minimal open set, maximal μ-open set, minimal μ-open set.

1 Introduction

The concept of generalized topological spaces was introduced by Császár [2]. A subcollection μ of the power set $exp(X)$ of a nonempty set X is a generalized topology due to Császár [2] if the empty set $\emptyset \in \mu$ and the union of arbitrary numbers of members of μ is a member of μ. A nonempty set X equipped with a generalized topology μ is called a generalized topological space [2] and it is denoted by (X, μ). Generally, we write a 'GT' for a 'generalized topology' μ on a nonempty set X and a 'GTS' for a 'generalized topological space' (X, μ). A member of μ is called a μ-open set of (X, μ). The complement of a μ-open set is called a μ-closed set.

Nakaoka and Oda [6] introduced and studied the concept of minimal open sets (Definition 1) in a topological space. Dualizing the concept of minimal open sets, Nakaoka and Oda [5] introduced and studied the notion of maximal open sets

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(Definition 2) in a topological space. Generalizing the concept of maximal open sets, Roy and Sen [7] introduced and studied the notion of maximal μ-open sets (Definition 3) (and also minimal μ-closed sets) in a GTS. Since certain classes of sets like semi-open [3], preopen [4] sets on a topological space $(X, \mathscr{T})$ form generalized topologies on X, the notion of a GTS may be unified in terms of those sets including open sets of a topological spaces. That's why, we find a considerable number of articles on the unification of existing topological notions via a GT. In this paper, we unified the notions of maximality and minimality of μ-open sets of a GTS (X, μ) on a topological space (X, τ), and we introduce and study the notions of maximal (τ, μ)-open sets (Definition 5) and minimal (τ, μ)-open sets (Definition 7). Contrary to maximal and minimal μ-open sets of a GTS, maximal (τ, μ)-open and minimal (τ, μ)-open sets behave differently e.g., Theorem 1 and Theorem 6 under operations of union and intersection respectively. Hence it follows the relevance of the study of properties of maximal (τ, μ)-open and minimal (τ, μ)-open sets.

For $A \subset X$, we write 'A is τ-open in X' (resp. 'μ-open in X') to mean '$A \in \tau$' (resp.'$A \in \mu$') without referring the 'topology τ on X' (resp. 'GT μ on X') to be comprehended from the context. The meanings of terms 'τ-closed', 'μ-closed' used in the sequel are apparent. By a 'proper open set' (resp. 'proper μ-open set') of a 'topological space (X, τ)' (resp. 'GTS (X, μ)'), we mean an 'open set' (resp. 'μ-open set') $G \neq \emptyset, X$. For a subset A of a topological space (X, τ), $c_\tau(A)$ denotes the closure of A with respect to the topological space (X, τ). Throughout the paper, R denotes the set of real numbers.

2 Unification of maximality of μ-open sets

Firstly, we recall some definitions to use in the sequel.

Definition 1 (Nakaoka and Oda [6]). *A proper open set U of (X, τ) is said to be a minimal open set if any open set which is contained in U is U or $\emptyset$.*

Definition 2 (Nakaoka and Oda [5]). *A proper open set U of (X, τ) is said to be a maximal open set if any open set which contains U is X or U.*

Definition 3 (Roy and Sen [7]). *A proper μ-open set U of a GTS (X, μ) is said to be a maximal μ-open set if any μ-open set which contains U is X or U.*

Definition 4 (S. Al Ghour et al. [1]). *A proper μ-open set U of X is said to be a minimal μ-open set if the only nonempty μ-open set which is contained in U is U itself.*

We introduce the notions following henceforth.

Definition 5. *A proper μ-open set A of a GTS (X, μ) is said to be maximal (τ, μ)-open on X if B is a τ-open set of (X, τ) containing A, then either $B = A$ or $B = X$. A is said to be absolutely maximal (τ, μ)-open on X if A satisfies the following condition: if B is a τ-open set of (X, τ) containing A, then $B = X$.*

It follows that a maximal (τ, μ)-open set on X is an absolutely maximal (τ, μ)-open set on X iff it is not τ-open in X.

We note that a maximal (τ, μ)-open set on X switches to a maximal open set in (X, τ), if we choose $\mu = \tau$. Also the existence of a notion like absolutely maximal (τ, μ)-open sets on X in a topological space (X, τ) as well as in a GTS (X, μ) is absurd.

Example 1. *For $a, b \in R$ with $a < b$, we define*

$$\begin{aligned} \tau &= \{\emptyset, R, \{a\}, (-\infty, a), (-\infty, a]\}, \\ \mu &= \{\emptyset, (-\infty, b), (-\infty, b]\}. \end{aligned}$$

Then τ is a topology in R and μ is a GT in R. $(-\infty, b)$ is maximal (τ, μ)-open on R but it is not maximal μ-open in (R, μ).

Example 2. *For $a \in R$, we define*

$$\begin{aligned} \tau &= \{\emptyset, R, \{a\}, (-\infty, a), (-\infty, a]\}, \\ \mu &= \{\emptyset, (-\infty, a)\}. \end{aligned}$$

Then τ is a topology on R and μ is a GT on R. $(-\infty, a)$ is maximal μ-open in (R, μ) but it is not maximal (τ, μ)-open on R.

So it follows that notions of maximal μ-open sets in X and maximal (τ, μ)-open sets on X are independent.

Theorem 1. *If A is maximal (τ, μ)-open on X and B is μ-open in X with $A \cup B \neq X$, then either $A \cup B$ is absolutely maximal (τ, μ)-open on X or A is both τ-open and μ-open with $B \subset A$.*

Proof. We note that A is a proper μ-open set. Suppose there exists a τ-open set $U \neq X, A \cup B$ such that $A \cup B \subset U$. Then we get $A \subset A \cup B \subset U$. Since A is maximal (τ, μ)-open on X and $U \neq X$, we have $A = U \Rightarrow A \cup B = U$, a contradiction to our assumption $U \neq A \cup B$. So we have $U = X$ or $U = A \cup B$ which imply that $A \cup B$ is maximal (τ, μ)-open on X. If $U = A \cup B \neq X$, then $A \cup B$ is both μ-open and τ-open, and so by maximal (τ, μ)-openness of A on X, we get $A = A \cup B$ which implies A is both τ-open and μ-open, and $B \subset A$. □

The assumption of $A \cup B \neq X$ in Theorem 1 is totally reasonable since $X \in \mu$ is not ensured in a GTS (X, μ), and we make certain that $X \in \mu$, if opted $A \cup B = X$.

If A is maximal open (resp. maximal μ-open) in a topological space (X, τ) (resp. GTS (X, μ)), then there does not exist a maximal open set (resp. maximal μ-open set) distinct from A and containing A. But we see from Theorem 1 that if A is maximal (τ, μ)-open on X, then there may exist another maximal (τ, μ)-open set distinct from A and containing A. It is an odd property of maximal (τ, μ)-open sets on X in comparison to maximal μ-open sets which also prompted us to investigate the notion of maximal (τ, μ)-open sets on X and some similar notions due to their such behaviour.

Corollary 1. *If A is absolutely maximal (τ,μ)-open on X and B is μ-open with $A\cup B\neq X$, then $A\cup B$ is absolutely maximal (τ,μ)-open on X.*

Proof. It follows easily. □

Corollary 2. *If $A\notin\tau$ is maximal (τ,μ)-open on X and B is μ-open with $A\cup B\neq X$, then $A\cup B$ is absolutely maximal (τ,μ)-open on X.*

Proof. Similar to the proof of Theorem 1. □

Theorem 2. *If A, B are distinct maximal (τ,μ)-open sets on X, then $A\cup B$ is absolutely maximal (τ,μ)-open on X if $A\cup B\neq X$.*

Proof. Proceeding like the proof of Theorem 1, we see that if there exists a τ-open set U such that $A\cup B\subset U$, then $U=X$ or $U=A\cup B$. If $U=A\cup B\neq X$, then $A\cup B$ is proper τ-open and so by maximal (τ,μ)-openness of A on X, we get $A=A\cup B$ which implies A is τ-open and $B\subset A$. Since B is maximal (τ,μ)-open on X and $A\neq X$, we get $A=B$ which is not possible by hypothesis. Similarly, considering $B\subset A\cup B\subset U$, we may have $U=X$ or $A=B$. Hence $A\cup B$ is absolutely maximal (τ,μ)-open on X if $A\cup B\neq X$. □

Corollary 3. *If A, B are distinct absolutely maximal (τ,μ)-open sets on X, then $A\cup B$ is absolutely maximal (τ,μ)-open on X if $A\cup B\neq X$.*

Theorem 3. *If a τ-open set A is maximal (τ,μ)-open on X, then either A is the only such set in X or X is the union of two such sets.*

Proof. Let another τ-open set B be also maximal (τ,μ)-open on X. Then $A\cup B$ is τ-open. As $A\subset A\cup B$ and A is maximal (τ,μ)-open on X, we have $A=A\cup B\Rightarrow B\subset A$ or $A\cup B=X$. Similarly, for B, we get $A\subset B$ or $A\cup B=X$. $B\subset A$ and $A\cup B=X$ imply that $A=X$ which is not possible. Similarly, $A\subset B$ together with $A\cup B=X$ is not possible. The only possible cases are $B\subset A, A\subset B\Rightarrow A=B$ and $A\cup B=X$. □

Definition 6. *A proper τ-open set A of a topological space (X,τ) is said to be maximal (μ,τ)-open on X if B is a μ-open set of X containing A, then either $B=A$ or $B=X$. A is said to be absolutely maximal (μ,τ)-open on X if A satisfies the following condition: if B is a μ-open set of X containing A, then $B=X$.*

If there exists no proper μ-open set containing A, then A is said to be absolutely maximal (μ,τ)-open on X.

It follows that a maximal (μ,τ)-open set on X is an absolutely maximal (μ,τ)-open set on X iff it is not μ-open in X.

We note that a maximal (μ,τ)-open set on X switches to a maximal open set in (X,τ), if we choose $\tau=\mu$.

Example 3. *For $a, b \in R$ with $a < b$, we define*

$$\begin{aligned} \tau &= \{\emptyset, R, \{b\}, (-\infty, b), (-\infty, b]\}, \\ \mu &= \{\emptyset, (-\infty, a), (-\infty, a]\}. \end{aligned}$$

Then τ is a topology in R and μ is a GT in R. $(-\infty, b)$ is maximal (μ, τ)-open on R but it is not maximal open in (R, τ).

Example 4. *For $a, b \in R$ with $a < b$, we define*

$$\begin{aligned} \tau &= \{\emptyset, R, \{a\}, (-\infty, a), (-\infty, a]\}, \\ \mu &= \{\emptyset, (-\infty, b)\}. \end{aligned}$$

Then τ is a topology in R and μ is a GT in R. $(-\infty, a]$ is maximal open in (R, τ) but it is not maximal (μ, τ)-open on R.

So it follows that notions of maximal open sets on X and maximal (μ, τ)-open sets on X are independent.

Lemma 1. *If a subset A of X is both τ-open in (X, τ) and maximal (τ, μ)-open on X, then A is maximal open in (X, τ).*

Proof. Let U be a τ-open set such that $A \subset U$. Since A is maximal (τ, μ)-open on X, we have $U = A$ or $U = X$. Since A is τ-open, it follows from the definition that A is maximal open in (X, τ). □

Lemma 2. *If a subset A of X is both μ-open in (X, μ) and maximal (μ, τ)-open on X, then A is maximal μ-open in (X, μ).*

Proof. Similar to the proof of Lemma 1. □

Theorem 4. *A subset of X is both maximal (τ, μ)-open and maximal (μ, τ)-open on X iff it is both maximal open in (X, τ) and maximal μ-open in (X, μ).*

Proof. Firstly, suppose that A is both maximal open in (X, τ) and maximal μ-open in (X, μ). So A is proper τ-open as well as μ-open. As A is maximal open (resp. maximal μ-open) in X, we have $A = U$ or $U = X$ for any τ-open (resp. μ-open) set U containing A. Considering A as μ-open, we do not get a τ-open set U such that $A \subset U$ and $U \neq A, X$. So A is maximal (τ, μ)-open on X. Similarly, A is maximal (μ, τ)-open on X.

Conversely, if A is both maximal (τ, μ)-open and maximal (μ, τ)-open on X, then A is both τ-open and μ-open. The result follows by Lemma 1 and Lemma 2. □

Theorem 5. *If A is τ-open in X and maximal (τ, μ)-open on X, then either $c_\tau(A) = X$ or $c_\tau(A) = A$.*

Proof. Let $x \in X - A$ and G be τ-open in X. Since $A \cup G$ is τ-open, we get $A \cup G = A$ or $A \cup G = X$. But $A \cup G = A$ is impossible. It follows that for any $x \in X - A$ and any τ-open nbd G of x, we have $A \cup G = X$. The two cases arise. Case I: There is $x \in X - A$ and a τ-open nbd G of x such that $A \cap G = \emptyset$. Then $c_\tau(A) = A$.

Case II: For any $x \in X - A$ and any τ-open nbd G of x we have $A \cap G \neq \emptyset$. Then $c_\tau(A) = X$. □

3 Unification of minimality of μ-open sets

We introduce the concept of minimal (τ, μ)-open sets on X by dualizing the concept of maximal (τ, μ)-open sets on X.

Definition 7. *A proper μ-open set A of (X, μ) is said to be minimal (τ, μ)-open on X if B is a τ-open set of (X, τ) contained in A, then either $B = A$ or $B = \emptyset$. A is said to be absolutely minimal (τ, μ)-open on X if A satisfies the following condition: if B is a τ-open set of (X, τ) contained in A, then $B = \emptyset$.*

It follows that a minimal (τ, μ)-open set on X is an absolutely minimal (τ, μ)-open set on X iff it is not τ-open.

We note that a minimal (τ, μ)-open set on X switches to a minimal open set in (X, τ), if we choose $\mu = \tau$. Also the existence of a notion like absolutely minimal (τ, μ)-open sets on X in a topological space (X, τ) as well as in a GTS (X, μ) is absurd.

In Example 1, $(-\infty, b)$ is minimal μ-open in (R, μ) but it is not minimal (τ, μ)-open on R. In Example 3, $(-\infty, a]$ is minimal (τ, μ)-open on R but it is not minimal μ-open in (R, μ). So the notions of minimal μ-open sets in X and minimal (τ, μ)-open sets on X are independent.

By dualizing some earlier results, we have the results through Theorem 6 to Theorem 9. The proofs of these results are omitted as the proofs are similar to the proofs of corresponding results already established.

Theorem 6. *If A is minimal (τ, μ)-open on X and B is μ-open with $A \cap B \neq \emptyset$ and $A \cap B \in \mu$, then either $A \cap B$ is absolutely minimal (τ, μ)-open on X or A is both τ-open and μ-open with $A \subset B$.*

Corollary 4. *If A is absolutely minimal (τ, μ)-open on X and B is μ-open with $A \cap B \neq \emptyset$ and $A \cap B \in \mu$, then $A \cap B$ is absolutely minimal (τ, μ)-open on X.*

Corollary 5. *If $A \notin \tau$ is minimal (τ, μ)-open on X and B is μ-open with $A \cap B \neq \emptyset$ and $A \cap B \in \mu$, then $A \cap B$ is absolutely minimal (τ, μ)-open on X.*

Theorem 7. *If A, B are distinct minimal (τ, μ)-open sets on X with $A \cap B \in \mu$, then $A \cap B$ is absolutely minimal (τ, μ)-open on X if $A \cap B \neq \emptyset$.*

Corollary 6. *If A, B are distinct absolutely minimal (τ, μ)-open sets on X with $A \cap B \in \mu$, then $A \cap B$ is absolutely minimal (τ, μ)-open on X if $A \cap B \neq \emptyset$.*

Theorem 8. *If A, B are τ-open sets in X as well as minimal (τ, μ)-open sets on X with $A \cap B \in \mu$, then either $A = B$ or $A \cap B = \emptyset$.*

Definition 8. *A proper τ-open set A of a topological space (X, τ) is said to be minimal (μ, τ)-open on X if B is a μ-open set of X contained in A, then either $B = A$ or $B = \emptyset$. A is said to be absolutely minimal (μ, τ)-open on X if A satisfies the following condition: if B is a μ-open set of X contained in A, then $B = \emptyset$.*

It follows that a minimal (μ, τ)-open set on X is an absolutely minimal (μ, τ)-open set on X iff it is not μ-open.

We note that a maximal (μ, τ)-open set on X switches to a maximal open set in (X, τ), if we choose $\tau = \mu$.

In Example 3, $(-\infty, b)$ is minimal open in (R, τ) but it is not minimal (μ, τ)-open on R. In Example 1, $(-\infty, a]$ is minimal (μ, τ)-open on R but it is not minimal open in (R, τ). So it follows that the notions of minimal open sets in X and minimal (μ, τ)-open sets on X are independent.

Lemma 3. *If a subset A of X is both τ-open in (X, τ) and minimal (τ, μ)-open on X, then A is minimal open in (X, τ).*

Lemma 4. *If a subset A of X is both μ-open in (X, μ) and minimal (μ, τ)-open on X, then A is minimal μ-open in (X, μ).*

Theorem 9. *A subset of X is both minimal (τ, μ)-open and minimal (μ, τ)-open on X iff it is both minimal open in (X, τ) and minimal μ-open in (X, μ).*

Suppose that a topological space (X, τ) has only one proper open set A and a GTS (X, μ) has only one proper μ-open set B with $A \cap B = \emptyset$. Then B is absolutely maximal (τ, μ)-open as well as absolutely minimal (τ, μ)-open on X and A is absolutely maximal (μ, τ)-open as well as minimal (μ, τ)-open on X. Again suppose that a topological space (X, τ) has only one proper open set A and a GTS (X, μ) has only one proper μ-open set B with $A \subsetneqq B$. Then B is absolutely maximal (τ, μ)-open and A is absolutely minimal (μ, τ)-open on X.

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