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TYPES OF INTEGER HARMONIC NUMBERS (II)

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Abstract

In the first part of this paper we obtained several bi-unitary harmonic numbers which are higher than 10^9, using the Mersenne prime numbers. In this paper we investigate bi-unitary harmonic numbers of some particular forms: $2^k \cdot n$, pqt^2, p^2q^2t, with different primes p, q, t and a squarefree integer n.

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1 Introduction

The *harmonic numbers* introduced by O. Ore in [8] were named in this way by C. Pomerance in [11]. They are defined as positive integers for which the harmonic mean of their divisors is an integer. O. Ore linked the perfect numbers with the harmonic numbers, showing that every perfect number is harmonic. A list of the harmonic numbers less than $2 \cdot 10^9$ is given by G. L. Cohen in [1], finding a total of 130 of harmonic numbers, and G. L. Cohen and R. M. Sorli in [2] have continued to this list up to 10^{10}.

The notion of harmonic numbers is extended to unitary harmonic numbers by K. Nageswara Rao in [7] and then to bi-unitary harmonic numbers by J. Sándor in [12].

Our paper is inspired by [12], where J. Sándor presented a table containing all the 211 bi-unitary harmonic numbers up to 10^9. We extend the J. Sándors's study, looking for other bi-unitary harmonic numbers, greater than 10^9. In the first part of our paper, [9], we start with some Mersenne primes and we found new bi-unitary harmonic numbers, different from those on the Sándor's list.

In this paper, after a brief revision of basic notions and results about bi-unitary numbers in Section 2, we study bi-unitary harmonic numbers of certain forms.

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Section 3 is dedicated to bi-unitary harmonic numbers of the form $2^k p_1 p_2 ... p_r$, with $p_1, p_2, ..., p_r$ different primes and $k \leq 10$, $r \leq 7$. We investigate how many bi-unitary harmonic numbers exist for a fixed k. For example, we obtain that any bi-unitary harmonic number of the form $2^k p_1 p_2 ... p_r$ with $k \in \{3, 4, 7, 8, 9\}$ does not exist, since there is only one bi-unitary harmonic number, $2^5 \cdot 3 \cdot 7$, for $k = 5$ and there are 17 bi-unitary harmonic numbers of the form $2^{10} p_1 p_2 ... p_r$ with $r \leq 7$. We also obtain new bi-unitary harmonic numbers greater than 10^9.

In the last section we study bi-unitary harmonic numbers with another particular factorization into prime numbers. We prove that the only even number that is also a perfect number and bi-unitary harmonic number is 6. We also obtain that there are only two bi-unitary harmonic numbers of the form pqt^2: 60 and 90, since $5^2 \cdot 7^2 \cdot 13$ is the only bi-unitary harmonic number with prime factorization $p^2 q^2 t$.

2 Preliminaries

We briefly recall the notion of bi-unitary harmonic numbers. Let n be a positive integer and $1 = d_1 < d_2 < ... < d_s = n$ all its natural divisors.

We denote by $\sigma(n)$ and $\tau(n)$ the sum of divisors of n and the number of divisors of n, respectively. The harmonic mean of divisors $H(n)$ can be written as

$$H(n) = \frac{n\tau(n)}{\sigma(n)}, \tag{1}$$

Therefore, we remark that $H(n)$ is an integer if and only if $\sigma(n)|n\tau(n)$. These numbers were studied by O. Ore in [8].

A number n satisfying the condition $\sigma(n)|n\tau(n)$ is called, [11], *harmonic number*. It is proved, [8], that every perfect number is harmonic.

A divisor d of a positive integer n is called, [7], *unitary divisor* of n if $\left(d, \frac{n}{d}\right) = 1$. Let us denote by $\sigma^*(n)$, $\tau^*(n)$ the sum and the number of unitary divisors of n, respectively.

A positive integer n is called, [7], *unitary harmonic number* when $\sigma^*(n)|n\tau^*(n)$. This definition shows that a unitary perfect number n, which means it satisfies $\sigma^*(n) = 2n$, is also a unitary harmonic number.

The notion of unitary divisor was extended to bi-unitary divisors. We recall that a divisor d of n is called *bi-unitary divisor* if the largest unitary common divisor of d and $\frac{n}{d}$ is 1. We denote by $\sigma^{**}(n)$ the sum of bi-unitary divisors of n.

In [13], Ch. Wall introduces the concept of bi-unitary perfect numbers, in the following way. A number n is called *bi-unitary perfect* number if $\sigma^{**}(n) = 2n$. It is proved that the only bi-unitary perfect numbers are 6, 60 and 90.

We remark that the function $\sigma^{**}(n)$ is multiplicative and we have

$$\sigma^{**}(p^a) = \begin{cases} \sigma(p^a) = \frac{p^{a+1}-1}{p-1}, & \text{for } a \text{ odd} \\ \sigma(p^a) - p^{\frac{a}{2}} = \frac{p^{a+1}-1}{p-1} - p^{\frac{a}{2}}, & \text{for } a \text{ even} \end{cases} \tag{2}$$

We denote by $\tau^{**}(n)$ the number of bi-unitary divisors of n and it is easy to see that if $n = p_1^{a_1} p_2^{a_2} ... p_r^{a_r} > 1$, is the prime factorization of n, then

$$\tau^{**}(n) = \prod_{a_i = even} a_i \prod_{a_i = odd} (a_i + 1) \, . \tag{3}$$

Definition 1. *A natural number n is called bi-unitary harmonic number if, [12]:*

$$\sigma^{**}(n) | n\tau^{**}(n).$$

Remark 1. *In [14] there are all unitary harmonic numbers with at most 4 primes in their factorization, since in [12] there are all bi-unitary harmonic numbers smaller than 10^9. From these, we can remark that there are unitary harmonic numbers which are not bi-unitary harmonic numbers, for example $2^3 \cdot 3^3 \cdot 5 \cdot 7$, and there are bi-unitary harmonic numbers which are not unitary harmonic, for example $2^3 \cdot 3^3 \cdot 5^4 \cdot 7$.*

Every bi-unitary perfect number is also a bi-unitary harmonic number, [12]. In the same paper it is also proved that if n has the prime decomposition $n = p_1^{a_1} p_2^{a_2} ... p_r^{a_r} > 1$ with all exponents $\{a_i\}_{i=\overline{1,r}}$ odd numbers, then n is a bi-unitary harmonic number if and only if n is harmonic. It is also proved that bi-unitary harmonic numbers are not of the following forms: pq^4, p^3q^2 and p^3q^4, and the only number $5 \cdot 3^2$ is bi-unitary harmonic number in form pq^2, where p, q are primes.

3 Bi-unitary harmonic numbers $2^k n$, with n an odd squarefree number and $k \le 10$

In this section we search the even bi-unitary harmonic numbers n which have the primes factorisation

$$n = 2^k p_1 p_2 ... p_r, \tag{4}$$

with $1 \le k \le 10$ and $1 \le r \le 7$.

Proposition 1. *Number 6 is the only bi-unitary harmonic number of the form (4) with $k = 1$.*

Proof. Let $n = 2p_1p_2...p_r$ be a bi-unitary harmonic number with odd primes $p_1 < p_2 < ... < p_r$. From (2) and (3) we compute

$$\sigma^{**}(n) = (1 + 2)(1 + p_1)(1 + p_2)...(1 + p_r), \quad \tau^{**}(n) = 2^{r+1},$$

and from Definition 1 we have

$$3(1 + p_1)(1 + p_2)...(1 + p_r) | 2^{r+2} p_1 p_2 ... p_r.$$

It follows that $p_1 = 3$ and the above relation becomes

$$12(1 + p_2)...(1 + p_r) | 12 \cdot 2^r p_2 ... p_r. \tag{5}$$

For $r = 1$ we find $n = 6$.

For $r \geq 2$ we write (5) as it follows

$$\frac{1+p_2}{2}\frac{1+p_3}{2}...\frac{1+p_r}{2}|2p_2...p_r.$$

Let d be a prime divisor of the integer $\frac{1+p_2}{2}$. Then $d \leq \frac{1+p_2}{2} < p_2$ which means that $d = 2$ and that $\frac{1+p_2}{2}$ could not have another prime divisor. Then $\frac{1+p_2}{2}$ has to be 2, hence $p_2 = 3$ which is not true from condition $p_1 < p_2$. We obtained that 6 is the only bi-unitary harmonic number $n = 2p_1p_2...p_r$. $\square$

Proposition 2. *Numbers* $2^2 \cdot 3 \cdot 5$*,* $2^2 \cdot 3 \cdot 5 \cdot 7$ *and* $2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 13$ *are the only bi-unitary harmonic numbers of the form* (4) *with* $k = 2$*.*

Proof. Let $n = 2^2p_1p_2...p_r$ be a bi-unitary harmonic number. From (2) and (3) we compute

$$\sigma^{**}(n) = (1+2^2)(1+p_1)(1+p_2)...(1+p_r), \quad \tau^{**}(n) = 2^{r+1},$$

and from Definition 1 we have

$$5(1+p_1)(1+p_2)...(1+p_r)|2^{r+3}p_1p_2...p_r.$$

It follows that $p_1 = 5$ and the above relation becomes

$$30(1+p_2)...(1+p_r)|10 \cdot 2^{r+2}p_2...p_r.$$

It follows that $p_2 = 3$ and

$$12(1+p_3)...(1+p_r)|12 \cdot 2^r p_3...p_r \tag{6}$$

For $r = 2$ we obtain $n = 2^2 \cdot 3 \cdot 5$.

For $r \geq 3$, let us suppose $p_3 < p_4 < ... < p_r$ and we write (6) as it follows:

$$\frac{1+p_3}{2}\frac{1+p_4}{2}...\frac{1+p_r}{2}|2^2p_3...p_r. \tag{7}$$

If d is a prime divisor of $\frac{1+p_3}{2}$, so smaller than p_3, it could be only $d = 2$. We obtain $\frac{1+p_3}{2} = 2m$ and let d' be a prime divisor of m. Since we also have $d'|m|2p_3...p_r$ and $d' < p_3$, hence $d' = 2$ and we don't have other values for it. It results $p_3 = 3$ (if $m = 1$) or $p_3 = 7$. But $p_3 > 5$, then $p_3 = 7$.

For $r = 3$ we have a solution $n = 2^2 \cdot 3 \cdot 5 \cdot 7$.

For $r \geq 4$, relation (7) becomes

$$4\frac{1+p_4}{2}...\frac{1+p_r}{2}|2^2 \cdot 7p_4...p_r. \tag{8}$$

We obtain in this case $\frac{1+p_4}{2} = 7$, so $p_4 = 13$.

For $r = 4$, the solution is $n = 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 13$.

For $r \geq 5$, relation (7) is

$$7\frac{1+p_5}{2}...\frac{1+p_r}{2}|7 \cdot 13p_5...p_r.$$

By a similar reasoning with the above one, we obtain $\frac{1+p_5}{2} = 13$, so $p_5 = 25$ which is not prime. It follows that bi-unitary harmonic numbers $n = 2^2p_1p_2...p_r$ with $r \geq 5$ do not exist. $\square$

Proposition 3. *Bi-unitary harmonic numbers of the form* (4) *with* $k \in \{3, 4, 7, 8, 9\}$ *do not exist* .

Proof. For $k = 3$, let $n = 2^3p_1p_2...p_r$ a bi-unitary harmonic number. We compute

$$\sigma^{**}(n) = (1 + 2 + 2^2 + 2^3)(1 + p_1)(1 + p_2)...(1 + p_r), \quad \tau^{**}(n) = 2^{r+2},$$

and from Definition 1 we have

$$15(1 + p_1)(1 + p_2)...(1 + p_r)|2^{r+5}p_1p_2...p_r.$$

It follows that $p_1 = 3$, $p_2 = 5$ and the above relation becomes

$$4 \cdot 6(1 + p_3)...(1 + p_r)|2^{r+2}p_3...p_r.$$

It follows that $3|2^{r+2}p_3...p_r$ with primes $p_3, ..., p_r$ greater than 3, which is impossible. So, any bi-unitary harmonic number of the form $n = 2^3p_1p_2...p_r$ does not exist.

In the following we use this: for any odd prime p we have

$$2^kp_1p_2...p_r \neq p^a \cdot b, \quad \forall a \geq 2, \tag{9}$$

where b is a positive integer.

Now, let $k = 4$ and $n = 2^4p_1p_2...p_r$ be a bi-unitary harmonic number. We compute

$$\sigma^{**}(n) = (1 + 2 + 2^3 + 2^4)(1 + p_1)(1 + p_2)...(1 + p_r), \quad \tau^{**}(n) = 2^{r+2},$$

and from Definition 1 we have

$$27(1 + p_1)(1 + p_2)...(1 + p_r)|2^{r+5}p_1p_2...p_r.$$

It follows that

$$3^3|2^{r+2}p_3...p_r,$$

which is false from relation (9). Hence, any bi-unitary harmonic number of the form $n = 2^4p_1p_2...p_r$ doesn't exist.

For $k = 7$ and $n = 2^7p_1p_2...p_r$ a bi-unitary harmonic number. We compute

$$\sigma^{**}(n) = (1+2+2^2+2^3+2^4+2^5+2^6+2^7)(1+p_1)(1+p_2)...(1+p_r), \quad \tau^{**}(n) = 2^{r+3},$$

and from Definition 1 we have

$$255(1+p_1)(1+p_2)...(1+p_r)|2^{r+10}p_1p_2...p_r.$$

It follows that $p_1 = 3$, $p_2 = 5$, $p_3 = 17$, then

$$4 \cdot 6 \cdot 18(1+p_4)...(1+p_r)|2^{r+10}p_4p_5...p_r,$$

which gives us

$$3^3|2^{r+2}p_4...p_r,$$

which is false from relation (9). Hence, any bi-unitary harmonic number of the form $n = 2^7p_1p_2...p_r$ does not exist.

For $k = 8$, let $n = 2^8p_1p_2...p_r$ a bi-unitary harmonic number. We compute

$$\sigma^{**}(n) = (1+2+2^2+2^3)(1+2^5)(1+p_1)(1+p_2)...(1+p_r), \quad \tau^{**}(n) = 2^{r+3},$$

and from Definition 1 we have

$$15 \cdot 33(1+p_1)(1+p_2)...(1+p_r)|2^{r+3}p_1p_2...p_r.$$

It follows that $3^2|2^{r+3}p_1p_2p_3...p_r$, which is false from relation (9). Hence, any bi-unitary harmonic number of the form $n = 2^8p_1p_2...p_r$ does not exist.

For $k = 9$, let $n = 2^9p_1p_2...p_r$ be a bi-unitary harmonic number. We compute

$$\sigma^{**}(n) = (2^{10}-1)(1+p_1)(1+p_2)...(1+p_r), \quad \tau^{**}(n) = 10 \cdot 2^r,$$

and from Definition 1 we have

$$3 \cdot 11 \cdot 31(1+p_1)(1+p_2)...(1+p_r)|2^{r+11} \cdot 5p_1p_2...p_r.$$

It follows that $p_1 = 3$, $p_2 = 11$, $p_3 = 31$ and the above relation becomes

$$4 \cdot 12 \cdot 32(1+p_4)...(1+p_r)|2^{r+11} \cdot 5p_4...p_r.$$

It follows that $3|2^{r+1}p_4...p_r$ with primes $p_4, ..., p_r$ greater than 3, which is imposible. So, any bi-unitary harmonic number of the form $n = 2^9p_1p_2...p_r$ does not exist.

□

Proposition 4. *There is only one bi-unitary harmonic number,* $2^5 \cdot 3 \cdot 7$*, of the form* (4) *with* $k = 5$*.*

Proof. Let $n = 2^5p_1p_2...p_r$ be a bi-unitary harmonic number and we compute

$$\sigma^{**}(n) = (1+2+2^2+2^3+2^4+2^5)(1+p_1)(1+p_2)...(1+p_r), \quad \tau^{**}(n) = 2^r \cdot 6.$$

From Definition 1 we have

$$63(1+p_1)(1+p_2)...(1+p_r)|2^{r+5} \cdot 6p_1p_2...p_r.$$

It follows that $p_1 = 3$, $p_2 = 7$ and the above relation becomes

$$4 \cdot 8(1+p_3)...(1+p_r)|2^{r+6}p_3...p_r. \tag{10}$$

For $r = 2$, we obtain the solution $n = 2^5 \cdot 3 \cdot 7$.

For $r \geq 3$, we rewrite:

$$\frac{1+p_3}{2}\frac{1+p_4}{2}...\frac{1+p_r}{2}|2^3p_3...p_r,$$

and suppose that $p_3 < p_4 < ... < p_r$.

If d is a prime divisor of the integer $\frac{1+p_3}{2}$, obviously smaller than $p_3 > 5$, then $d = 2$. Then, for a prime divisor d' of $\frac{1+p_3}{2^2}$ we also obtain $d' = 2$. Once again, we take a prime divisor d'' of $\frac{1+p_3}{2^3}$ and there is only solution $d'' = 2$. Moreover, $\frac{1+p_3}{2^3}$ doesn't have another divisor. Hence, p_3 could be only 3, 7 or 15, which is impossible.

It results that the unique solution is $n = 2^5 \cdot 3 \cdot 7$. □

Proposition 5. *There are only three bi-unitary harmonic numbers of the form* (4) *with* $k = 6$*:* $2^6 \cdot 3 \cdot 7 \cdot 17$, $2^6 \cdot 3 \cdot 7 \cdot 17 \cdot 31$, $2^6 \cdot 3 \cdot 7 \cdot 17 \cdot 31 \cdot 61$.

Proof. Let $n = 2^6p_1p_2...p_r$ be a bi-unitary harmonic number and we compute

$$\sigma^{**}(n) = (1+2+2^2)(1+2^4)(1+p_1)(1+p_2)...(1+p_r), \quad \tau^{**}(n) = 2^r \cdot 6.$$

From Definition 1 we have

$$7 \cdot 17(1+p_1)(1+p_2)...(1+p_r)|2^{r+7} \cdot 3p_1p_2...p_r.$$

It follows that $p_1 = 7$, $p_2 = 17$ and the above relation becomes

$$8 \cdot 18(1+p_3)...(1+p_r)|2^{r+7} \cdot 3p_3...p_r.$$

Then $p_3 = 3$ and we have

$$4(1+p_4)...(1+p_r)|2^{r+3} \cdot p_4...p_r. \tag{11}$$

For $r = 3$, we obtain the solution $n = 2^6 \cdot 3 \cdot 7 \cdot 17$.

For $r \geq 4$, we rewrite:

$$\frac{1+p_4}{2}\frac{1+p_5}{2}...\frac{1+p_r}{2}|2^4p_4...p_r,$$

and suppose that $p_4 < p_5 < ... < p_r$. We obtain for $\frac{1+p_4}{2}$ the possible values $2, 4, 8, 16$ and there is only a prime solution for p_4, 31.

Hence, for $r = 4$ we have $n = 2^6 \cdot 3 \cdot 7 \cdot 17 \cdot 31$.

For $r \geq 5$, the relation

$$\frac{1+p_5}{2}\frac{1+p_6}{2}...\frac{1+p_r}{2}|31p_5...p_r,$$

gives us $p_5 = 61$ and another solution, $n = 2^6 \cdot 3 \cdot 7 \cdot 17 \cdot 31 \cdot 61$.

For $r \geq 6$, it results

$$\frac{1+p_6}{2}\frac{1+p_7}{2}...\frac{1+p_r}{2}|61p_6...p_r,$$

and a prime divisor d of $\frac{1+p_6}{2}$ could be only 61, because $d \leq \frac{1+p_6}{2} < p_6 < p_7 < ... < p_r$. But the solution $p_6 = 121$ isn't a prime number, so there are not bi-unitary harmonic numbers of the form $2^6p_1p_2...p_r$ with $r > 5$. $\square$

Proposition 6. *There are* 17 *bi-unitary harmonic numbers of the form* (4) *with* $k = 10$ *and* $r \leq 7$.

Proof. Let $n = 2^{10}p_1p_2...p_r$ be a bi-unitary harmonic number and we compute

$$\sigma^{**}(n) = (1+2+2^2+2^3+2^4)(1+2^6)(1+p_1)(1+p_2)...(1+p_r), \quad \tau^{**}(n) = 2^{r+1}\cdot 5.$$

From Definition 1 we have

$$31 \cdot 5 \cdot 13(1+p_1)(1+p_2)...(1+p_r)|2^{r+11} \cdot 5p_1p_2...p_r.$$

It follows that $p_1 = 13$, $p_2 = 31$ and the above relation becomes

$$14 \cdot 32(1+p_3)...(1+p_r)|2^{r+11} \cdot p_3...p_r.$$

Then $p_3 = 7$ and we have

$$8(1+p_4)...(1+p_r)|2^{r+5} \cdot p_4...p_r. \tag{12}$$

For $r = 3$, we obtain the solution $n = 2^{10} \cdot 7 \cdot 13 \cdot 31$.

For $r \geq 4$, we rewrite:

$$\frac{1+p_4}{2}\frac{1+p_5}{2}...\frac{1+p_r}{2}|2^5p_4...p_r,$$

and suppose that $p_4 < p_5 < ... < p_r$. We obtain for $\frac{1+p_4}{2}$ the possible values $2, 4, 8, 16, 32$ and there is only a prime solution for p_4, different from p_1, p_2, p_3, that is $p_4 = 3$.

Hence, for $r = 4$ we have $n = 2^{10} \cdot 3 \cdot 7 \cdot 13 \cdot 31$.

For $r \geq 5$, the relation

$$\frac{1+p_5}{2}\frac{1+p_6}{2}...\frac{1+p_r}{2}|2^4 \cdot 3p_5...p_r,$$

gives us $p_5 \in \{5, 11, 23, 47\}$ and, for $r = 5$, four other solutions: $2^{10} \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 31$, $2^{10} \cdot 3 \cdot 7 \cdot 11 \cdot 13 \cdot 31$, $2^{10} \cdot 3 \cdot 7 \cdot 13 \cdot 23 \cdot 31$, $2^{10} \cdot 3 \cdot 7 \cdot 13 \cdot 31 \cdot 47$.

Remark 2. *All bi-unitary harmonic numbers of the form $2^kp_1p_2...p_r$ found until now are smaller than 10^9 so they are also given in [12].*

For $r \geq 6$ and $p_5 = 5$, it results

$$\frac{1+p_6}{2}\frac{1+p_7}{2}...\frac{1+p_r}{2}|2^4 \cdot 5p_6...p_r,$$

and we find $p_6 \in \{19, 79, 179\}$ and we have three new solutions, from which only the first one is smaller than 10^9: $2^{10} \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 19 \cdot 31$, $2^{10} \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 31 \cdot 79$, $2^{10} \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 31 \cdot 179$.

For $r \geq 6$ and $p_5 = 11$, it results

$$\frac{1+p_6}{2}\frac{1+p_7}{2}...\frac{1+p_r}{2}|2^3 \cdot 11p_6...p_r,$$

and we find $p_6 \in \{43, 87\}$ and we have two new solutions, both greater than 10^9: $2^{10} \cdot 3 \cdot 7 \cdot 11 \cdot 13 \cdot 31 \cdot 43$, $2^{10} \cdot 3 \cdot 7 \cdot 11 \cdot 13 \cdot 31 \cdot 87$.

For $r \geq 6$ and $p_5 = 23$ or $p_5 = 47$ we didn't find solutions.

For $r \geq 7$, $p_5 = 5$ and $p_6 = 19$, we have

$$\frac{1+p_7}{2}\frac{1+p_8}{2}...\frac{1+p_r}{2}|2^3 \cdot 19p_7...p_r.$$

Hence $p_7 \in \{37, 151\}$ and the new solutions are: $2^{10} \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 19 \cdot 31 \cdot 37$, $2^{10} \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 19 \cdot 31 \cdot 151$.

For $r \geq 7$, $p_5 = 5$ and $p_6 = 79$, we have

$$\frac{1+p_7}{2}\frac{1+p_8}{2}...\frac{1+p_r}{2}|2 \cdot 79p_7...p_r.$$

Hence $p_7 \in \{157, 317\}$ and two new solutions are: $2^{10} \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 31 \cdot 79 \cdot 157$, $2^{10} \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 31 \cdot 79 \cdot 317$.

For $r \geq 7$, $p_5 = 5$ and $p_6 = 179$, we have

$$\frac{1+p_7}{2}\frac{1+p_8}{2}...\frac{1+p_r}{2}|179p_7...p_r,$$

and no solution for p_7.

For $r \geq 7$, $p_5 = 11$, and $p_6 = 43$, it results

$$\frac{1+p_7}{2}\frac{1+p_8}{2}...\frac{1+p_r}{2}|4 \cdot 43p_7...p_r,$$

with no solution for p_7.

Finally, for $r \geq 7$, $p_5 = 11$, and $p_6 = 87$, it results

$$\frac{1+p_7}{2}\frac{1+p_8}{2}...\frac{1+p_r}{2}|2 \cdot 87p_7...p_r.$$

We find $p_7 \in \{173, 347\}$, so other bi-unitary harmonic numbers are $2^{10} \cdot 3 \cdot 7 \cdot 11 \cdot 13 \cdot 31 \cdot 87 \cdot 173$, $2^{10} \cdot 3 \cdot 7 \cdot 11 \cdot 13 \cdot 31 \cdot 87 \cdot 347$. $\square$

In conclusion, we find some new bi-unitary harmonic numbers, different from those in [12], of the form $2^k p_1 p_2 ... p_r$, with $k \leq 10$ and $r \leq 7$. These new numbers are:

$$\begin{gathered} n = 2^{10} \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 31 \cdot 79, \\ n = 2^{10} \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 31 \cdot 179, \\ n = 2^{10} \cdot 3 \cdot 7 \cdot 11 \cdot 13 \cdot 31 \cdot 43, \\ n = 2^{10} \cdot 3 \cdot 7 \cdot 11 \cdot 13 \cdot 31 \cdot 87, \\ n = 2^{10} \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 19 \cdot 31 \cdot 37, \\ n = 2^{10} \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 19 \cdot 31 \cdot 151, \\ n = 2^{10} \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 31 \cdot 79 \cdot 157, \\ n = 2^{10} \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 31 \cdot 79 \cdot 317 \\ n = 2^{10} \cdot 3 \cdot 7 \cdot 11 \cdot 13 \cdot 31 \cdot 87 \cdot 173, \\ n = 2^{10} \cdot 3 \cdot 7 \cdot 11 \cdot 13 \cdot 31 \cdot 87 \cdot 347. \end{gathered}$$

4 Some bi-unitary harmonic numbers of particular forms

In this section we search for bi-unitary harmonic numbers with the prime factorization of some particular forms.

O. Ore shows in [8], that if n is a perfect number, then this is harmonic. Below we show that the only even number that is also a perfect number and bi-unitary harmonic number is 6.

Proposition 7. *If an even number n is at the same time a perfect and a bi-unitary harmonic number then $n = 6$.*

Proof. According to Euler-Euclid theorem [6, 10], an even number n is perfect if and only if $n = 2^k(2^{k+1} - 1)$, where $k \geq 1$ and $p = 2^{k+1} - 1$ is prime. For $k = 1$, we obtain $n = 6$, so 6 is a perfect number, and

$$H^{**}(6) = \frac{6\tau^{**}(6)}{\sigma^{**}(6)} = 2,$$

so 6 is bi-unitary harmonic number.

Let $k \geq 2$. As p is a prime number greater than or equal to 5, then $k + 1$ is odd, it follows k is even and we write $k = 2m$. Therefore, we rewrite n as:

$$n = 2^{2m}(2^{2m+1} - 1),$$

with $m \geq 1$. Assume that n is a bi-unitary harmonic number, so $\sigma^{**}(n) | n\tau^{**}(n)$ and

$$\sigma^{**}(2^{2m})\sigma^{**}(2^{2m+1} - 1) | 2^{2m}(2^{2m+1} - 1)\tau^{**}(2^{2m})\tau^{**}(2^{2m+1} - 1)$$

which is equivalent to

$$(2^m - 1)(2^{2m+1} - 1)2^{2m+1} | 2^{2m+2}(2^{2m+1} - 1) \cdot m.$$

But

$$((2^m - 1), 2(2^{2m+1} - 1)) = 1,$$

we deduce that $(2^m - 1)|m$, which is false, because $2^m - 1 > m$, for $m \geq 2$. For $m = 1$, we obtain $n = 2^2 \cdot 7$, which is not bi-unitary harmonic, because

$$H^{**}(2^2 \cdot 7) = \frac{2^2 \cdot 7 \cdot 2^2}{5 \cdot 2^3} = \frac{14}{5},$$

is not an integer number. In conclusion, the only even number that is also a perfect number and a bi-unitary harmonic number is 6. $\square$

Proposition 8. *If a number n is bi-unitary harmonic with the form pqt^2, then $n = 60$ or $n = 90$.*

Proof. Let $n = pqt^2$, where p, q, t are different prime numbers, we have

$$\sigma^{**}(n) = (1 + p)(1 + q)(1 + t^2),$$

and $n\tau^{**}(n) = 2^3 pqt^2$. If n is a bi-unitary harmonic number, then $\sigma^{**}(n)|n\tau^{**}(n)$, which implies

$$(1 + p)(1 + q)(1 + t^2)|2^3 pqt^2 \tag{13}$$

I. If number n is even, then one of p, q or t is equal to 2.

I.1. Let $p = 2$. From relation (13), we deduce that $3|2^4 qt^2$, so $q = 3$ or $t = 3$. For $q = 3$, we have $3 \cdot 4 \cdot (1 + t^2)|2^4 \cdot 3 \cdot t^2$, $(1 + t^2)|2^2 \cdot t^2$, so $(1 + t^2)|2^2$ which is false. For $t = 3$, we have $2 \cdot 3 \cdot 5 \cdot (1 + q)|2^4 \cdot 3^2 \cdot q$, $5 \cdot (1 + q)|2^3 \cdot 3 \cdot q$, so $5|q$, we deduce $q = 5$. Consequently, we obtain the solution $n = 2 \cdot 3^2 \cdot 5 = 90$.

I.2. Consider the case $t = 2$ and relation (13) becomes $5(1+p)(1+q)|2^5 \cdot pq$, so $5|pq$ that $p = 5$ or $q = 5$. For $p = 5$, it follows $2 \cdot 3 \cdot 5 \cdot (1 + q)|2^5 \cdot 5 \cdot q$, that $3|q$, therefore $q = 3$, we deduce the solution $n = 2^2 \cdot 3 \cdot 5 = 60$.

II. If number n is odd, then p, q and t are odd prime. Let $p < q$ and p' be the greatest prime divisor of $p + 1$. As follows $(1 + p)(1 + q)(1 + t^2)|2^3 pqt^2$, $p'|2^3 pqt^2$ we have the following situations:

II.1. If $p' = 2$, then $p + 1 = 2^2$ or $p + 1 = 2^3$ that $p = 3$ or $p = 7$. This implies a contradiction, because it would mean that $2|qt^2$ which is false.

II.2. If $p' \neq 2$, then $p'|pqt^2$, which means $p'|t^2$, that $p' = t$. So $p + 1 = 2t^a v$. Assume that $a \geq 3$ result $t|pq$, which is false. For $a = 2$, we have the relation $v(1 + q)(1 + t^2)|2^2 pq$. If number v has a prime divisor, then it must be p or q which is impossible, so $v = 1$.

II.2.1. For $p + 1 = 2t^2$, we obtain $(1 + q)(1 + t^2)|2^2 pq$. Let q' be the greatest prime divisor of $q + 1$, $q' = 2$ that $q = 3$ which is false, or $q'|p$ so $q' = p$. Therefore $1 + q = 2p$, and it is impossible for $q + 1$ to have another prime divisor.

As discussed before, we deduce that $(1 + t^2)|2q$, and $1 + t^2 = 2q$. So we have, $1 + t^2 = 2q = 4p - 2 = 8t^2 - 6$, and $t = 1$, which is false.

II.2.2. For $p + 1 = 2t$, we obtain $(1 + q)(1 + t^2)|2^2 pq$. If q' is the greatest prime divisor of $q + 1$, we have $q' = 2$ that $q = 3$ which is false, or $q'|pt$ so $q' = p$ or $q' = t$. Therefore $1 + q = 2p$, and it is impossible for $q + 1$ to have another prime divisor. Version $1 + q = 2t$, does not agree because it would mean that $p = q$, which is a contradiction. Therefore, we conclude that $(1 + t^2)|2qt$, and $1 + t^2 = 2q$ and as above we arrive at a false conclusion. $\square$

Remark 3. *In the following the remark that for a prime p we could not have $\frac{1+p}{2} = p \cdot a$, for any positive integer a will be usefull. Indeed, if the above relation holds, then from $a \geq 1$ we have $\frac{1+p}{2} \geq p$, hence $p \leq 1$, which is false.*

Proposition 9. *If the bi-unitary harmonic number n is of the form p^2q^2t, then it is $5^2 \cdot 7^2 \cdot 13$.*

Proof. Let $n = p^2q^2t$ be a bi-unitary harmonic number (obviously, p, q, t are different primes). We compute

$$\sigma^{**}(n) = (1+p^2)(1+q^2)(1+t), \quad \tau^{**}(n) = 8,$$

and, by definition 1, we have

$$(1+p^2)(1+q^2)(1+t)|8p^2q^2t. \tag{14}$$

I. If n is an even number, then $p = 2$ or $q = 2$ or $t = 2$. Because the cases $p = 2$ and $q = 2$ are the same, we shall study only $p = 2$ and $t = 2$.

I.1. For $p = 2$, relation (14) becomes

$$5(1+q^2)(1+t)|8 \cdot 4q^2t,$$

which means $q = 5$ or $t = 5$.

I.1.1. If $q = 5$, we have $26(1+t)|32 \cdot 5 \cdot t$, so $t = 13$. But for $t = 13$, the above relation is $26 \cdot 14|32 \cdot 5 \cdot 13$ which is false. Hence, $p = 2$ and $q = 5$ is not a solution.

I.1.2. If $t = 5$, we have $(1+q^2) \cdot 6|32q^2$, so $q = 3$. But the above relation is now $10 \cdot 6|32 \cdot 9$ which is not true.

I.2. For $t = 2$, relation (14) becomes

$$3(1+p^2)(1+q^2)|16p^2q^2,$$

which implies $p = 3$ (or, the same, $q = 3$). Then we obtain $3 \cdot 10 \cdot (1+q^2)|16 \cdot 9 \cdot q^2$, so $q = 5$. But $30 \cdot 26|16 \cdot 9 \cdot 25$ is false, so there is no solution in this case.

That means that there are not even bi-unitary harmonic numbers of the form p^2q^2t.

II. If n is an odd number, then primes p, q, t are also odd. In this case relation (14) could be written:

$$\frac{1+p^2}{2}\frac{1+q^2}{2}\frac{1+t}{2}|p^2q^2t. \tag{15}$$

Taking into account Remark 3, $\frac{1+p^2}{2} \in \{q, q^2, t, qt, q^2t\}$.

II.1. If $\frac{1+p^2}{2} = q$, then

$$\frac{1+q^2}{2}\frac{1+t}{2}|p^2qt,$$

hence, from Remark 3, $\frac{1+q^2}{2} \in \{p, p^2, t, pt, p^2t\}$.

II.1.1. For $\frac{1+q^2}{2} = p$ we obtain the equation $p^4 + 2p^2 - 8p + 5 = 0$, with no prime solution.

II.1.2. For $\frac{1+q^2}{2} = p^2$ we obtain the equation $p^4 - 6p^2 + 5 = 0$, with no prime solution.

II.1.3. For $\frac{1+q^2}{2} = pt$ we obtain

$$\frac{1+t}{2} | pq,$$

so $1 + t \in \{2p, 2q, 2pq\}$.

II.1.3.a) If $1 + t = 2p$, from $1 + q^2 = 2pt$ and $1 + p^2 = 2q$ results $1 + q^2 = t(1 + t)$ and $4 + (1 + t)^2 = 8q$. Then we obtain the equation $t^4 + 4t^3 - 50t^2 - 44t + 89 = 0$, with no prime solution.

II.1.3.b) If $1 + t = 2q$, from $1 + q^2 = 2pt$ and $1 + p^2 = 2q$ it results $p^2 = t$ which means that t is not prime (false).

II.1.3.c) If $1 + t = 2pq$, from $1 + q^2 = 2pt$ and $1 + p^2 = 2q$ it results $4 + (1 + p^2)^2 = 8pt$. Then we obtain the equation $7p^4 + 6p^2 - 8p - 5 = 0$, with no prime solution.

II.1.4 For $\frac{1+q^2}{2} = t$ we obtain

$$\frac{1+t}{2} | p^2q,$$

so $1 + t \in \{2p, 2p^2, 2q, 2pq, 2p^2q\}$.

II.1.4.a) If $1 + t = 2p$, then, from $1 + q^2 = 2t$ and $1 + p^2 = 2q$ we have the equation $p^4 + 2p^2 - 16p + 13 = 0$, with no prime solution.

II.1.4.b) If $1 + t = 2p^2$, then, from $1 + q^2 = 2t$ and $1 + p^2 = 2q$ we have the equation $63q^2 - 160q + 99 = 0$, with no prime solution.

II.1.4.c) If $1 + t = 2q$, then, from $1 + q^2 = 2t$ and $1 + p^2 = 2q$ we obtain $t = p^2$ which is false because t is prime.

II.1.4.d) If $1 + t = 2pq$, then, from $1 + q^2 = 2t$ and $1 + p^2 = 2q$ we obtain the equation $p^4 - 8p^3 + 2p^2 - 8p + 13 = 0$, with no prime solution.

II.1.4.e) If $1 + t = 2p^2q$, then, from $1 + q^2 = 2t$ and $1 + p^2 = 2q$ we obtain the equation $8p^4 + 6p^2 - 13 = 0$, with no prime solution.

II.1.5. For $\frac{1+q^2}{2} = p^2t$ we obtain

$$\frac{1+t}{2} | q,$$

so $1 + t = 2q$. But we also have $1 + p^2 = 2q$, so $t = p^2$, which is false because t is prime.

II.2. For $1 + p^2 = 2q^2$, relation (15) becomes

$$\frac{1+q^2}{2} \frac{1+t}{2} | p^2t,$$

hence, from Remark 3, $\frac{1+t}{2} \in \{p, p^2\}$.

II.2.1. For $1+t=2p$ we have

$$\frac{1+q^2}{2}|pt,$$

hence $1+q^2 \in \{2p, 2t, 2pt\}$.

II.2.1.a) If $1+q^2=2p$, then $t=q^2$, which is false because t is prime.

II.2.1.b) If $1+q^2=2t$, then, from $1+t=2p$ and $1+p^2=2q^2$ we obtain the equation $t^2-14t+13=0$, with the prime root $t=13$.

It results $p=7$, $q=5$, hence we have a bi-unitary harmonic number $7^2 \cdot 5^2 \cdot 13$.

II.2.1.c) If $1+q^2=2pt$, then, from $1+t=2p$ and $1+p^2=2q^2$ we obtain $3+p^2=4pt$. Then $p|3$, so $p=3$, which gives $q^2=5$, with no integer solution.

II.2.2. For $1+t=2p^2$ we have

$$\frac{1+q^2}{2}|t,$$

hence $1+q^2=2t$. We obtain $t=1$ which is not a solution.

II.3. For $1+p^2=2qt$, relation (15) becomes

$$\frac{1+q^2}{2}\frac{1+t}{2}|p^2q,$$

hence, from Remark 3, $\frac{1+q^2}{2} \in \{p, p^2\}$.

II.3.1. For $1+q^2=2p$ we have

$$\frac{1+t}{2}|pq,$$

hence $1+t \in \{2p, 2q, 2pq\}$.

II.3.1.a) If $1+t=2p$, then, from $1+q^2=2p$, it results $t=q^2$ which is false because t is prime.

II.3.1.b) If $1+t=2q$, then, from $1+q^2=2p$ and $1+p^2=2qt$, the following equation $q^4-14q^2+8q+5=0$ results, with no prime solution.

II.3.1.c) If $1+t=2pq$, then, from $1+q^2=2p$ and $1+p^2=2qt$, the following equation $7q^4+6q^2-8q-5=0$ results, with no prime solution.

II.3.2. For $1+q^2=2p^2$ we have

$$\frac{1+t}{2}|q,$$

hence $1+t=2q$. It follows $7q^2-4q-3=0$ which has no prime root.

II.4. For $1+p^2=2q^2t$, relation (15) is

$$\frac{1+q^2}{2}\frac{1+t}{2}|p^2,$$

hence, since $\frac{1+q^2}{2} \neq 1$, $\frac{1+t}{2} \neq 1$, we have $1+q^2=1+t=2p$. But $t=q^2$ is impossible because t is prime.

II.5. For $1+p^2=2t$, relation (15) becomes

$$\frac{1+q^2}{2}\frac{1+t}{2}|p^2q^2,$$

hence, from Remark 3, $\frac{1+q^2}{2}\in\{p,p^2\}$.

II.5.1. For $1+q^2=2p$, we obtain

$$\frac{1+t}{2}|pq^2,$$

so $1+t\in\{2p,2q,2q^2,2pq,2pq^2\}$.

II.5.1.a) If $1+t=2p$, then, from $1+q^2=2p$ it results $t=q^2$ which is false because t is prime.

II.5.1.b) If $1+t=2q$, then, from $1+q^2=2p$ and $1+p^2=2t$ the equation $q^4+2q^2-16q+13=0$ results, with no prime root.

II.5.1.c) If $1+t=2q^2$, then, from $1+q^2=2p$ and $1+p^2=2t$ the equation $p^2-8p+7=0$ results, with $p=7$ a prime root. We obtain $t=25$ which is not prime.

II.5.1.d) If $1+t=2pq$, then, from $1+q^2=2p$ and $1+p^2=2t$ the equation $q^4-8q^3+2q^2-8q+13=0$ results, with no prime root.

II.5.1.e) If $1+t=2pq^2$, then, from $1+q^2=2p$ and $1+p^2=2t$ the equation $7q^4+6q^2-13=0$ results, with no prime root.

II.5.2. For $1+q^2=2p^2$, we obtain

$$\frac{1+t}{2}|q^2,$$

so $1+t\in\{2q,2q^2\}$.

II.5.2.a) If $1+t=2q$, then, from $1+q^2=2p^2$ and $1+p^2=2t$ the equation $q^2-8q+7=0$ results, with $q=7$ prime root. We obtain $p=5$, $t=13$ and the already found solution $5^2\cdot7^2\cdot13$.

II.5.2.b) If $1+t=2q^2$, then, from $1+q^2=2p^2$ and $1+p^2=2t$ it results $t=1$, so we don't have a solution in this case.

We investigated all the values possible, so we can conclude that there is only one bi-unitary harmonic number of the form p^2q^2t and this is $5^2\cdot7^2\cdot13$.

□

Other types of integer harmonic numbers may be entered using the exponential and infinitary divisors [4].

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