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ON HOLOMORPHIC LIE ALGEBROIDS

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Abstract

The geometry of holomorphic Lie algebroids is analyzed in this paper. Specific notions such as the anchor map or vertical and complete lifts are studied globally and locally, as well as classical concepts such as the differential, nonlinear connection or semisprays and sprays, characterized in the context of holomorphic Lie algebroids.

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Introduction

Lie algebroids are a generalization of Lie algebras and integrable distributions. They are anchored vector bundles with a Lie bracket defined on the modules of sections. Lie algebroids provide a natural setting in which one can develop the theory of differential operators such as the exterior derivative of forms and the Lie derivative with respect to a vector field. This setting is slightly more general than that of the tangent and cotangent bundles of a smooth manifold and their exterior powers.

Lie algebroids represent an active domain of research, with applications in many areas of mathematics and physics. A well-known example is the work of A. Weinstein [13] in the area of Mechanics, who developed a generalized theory of Lagrangians on Lie algebroids and obtained the Euler-Lagrange equations using the structure of the dual of Lie algebroids and Legendre transformations associated with a regular Lagrangian. E. Martinez [6, 7] developed the Klein's formalism on Lie algebroids using the notion of prolongation of Lie algebroid over a smooth map, and has proposed a modified version of symplectic formalism, in which the bundles tangent to E and E^* are replaced by their prolongations, $\mathcal{T}E$ and $\mathcal{T}E^*$. More recently, Lie algebroids have been investigated by M. Anastasiei [2, 3] and L. Popescu [11, 12].

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In complex geometry, some properties of complex and holomorphic Lie algebroids have been studied in [4], [5]. The present paper analyzes specific notions from real Lie algebroids theory in the case of holomorphic Lie algebroids, where all the geometrical objects considered are holomorphic. The first section gives basic definitions of a holomorphic anchor map, holomorphic Lie algebroid, Lie bracket on such an algebroid, which are locally characterized in the second section, where the Lie algebroid is also complexified. The third section introduces a linear connection on the holomorphic Lie algebroid, completed with its torsion and curvature. The fourth section introduces the notions of vertical and complete lifts of holomorphic vector fields on holomorphic Lie algebroids, completed with local expressions. The classical notions of semisprays and sprays are defined in this case in the last section following the construction of M. Anastasiei. An important new result of this last section is the obtaining of a complex spray from the variational problem.

1 Basic concepts

Let M be a complex n-dimensional manifold and E a holomorphic vector bundle of rank m over M. Denote by $\pi : E \to M$ the holomorphic bundle projection, by $\Gamma(E)$ the module of holomorphic sections of π and let $T_{\mathbb{C}}M = T'M \oplus T''M$ be the complexified tangent bundle of M, split into the holomorphic and antiholomorphic tangent bundles.

On a vector bundle (E, π, M) the definition of a derivative law is $D : \chi(M) \times \Gamma(E) \to \Gamma(E)$, $D_X s$, such that $D_{fX}s = fD_X s$ and $D_X(fs) = fD_X s + X(f)$. While these notions make sense on the fibers of E, the Lie bracket $[s_1, s_2]f$, where $s_1, s_2 \in \Gamma(E)$, has no mathematical meaning. Hence the notion of Lie algebroids.

Definition 1.1. *The holomorphic vector bundle E over M is called anchored if there exists a holomorphic vector bundle morphism $\rho : E \to T'M$, called anchor map.*

Denote by $\Gamma(T'M)$ the module of holomorphic sections of $T'M$, that is, the holomorphic vector fields on M, and by $\mathcal{H}(M)$ the ring of holomorphic functions on M.

Using the anchor map, we can define a Lie bracket on E from the Lie bracket on $T'M$ by

$$\rho_E([s_1, s_2]_E) = [\rho_E(s_1), \rho_E(s_2)]_{T'M}, \tag{1.1}$$

$s_1, s_2 \in \Gamma(E)$. For any $f \in \mathcal{H}(M)$,

$$\begin{aligned}\rho_E[s_1, fs_2]_E = [\rho_E(s_1), \rho_E(fs_2)]_{T'M} = [\rho_E(s_1), f\rho_E(s_2)]_{T'M} = \\ = f[\rho_E(s_1), \rho_E(s_2)]_{T'M} + \rho_E(s_1)(f)\rho_E(s_2).\end{aligned}$$

These considerations lead to the following definition ([13, 5, 4, 6]):

Definition 1.2. *A holomorphic Lie algebroid over M is a triple $(E,[\cdot,\cdot]_E,\rho_E)$, where E is a holomorphic vector bundle anchored over M, $[\cdot,\cdot]_E$ is a Lie bracket on $\Gamma(E)$ and $\rho_E:\Gamma(E)\to\Gamma(T'M)$ is the homomorphism of complex modules induced by the anchor map ρ such that*

$$[s_1,fs_2]_E=f[s_1,s_2]_E+\rho_E(s_1)(f)s_2 \tag{1.2}$$

for all $s_1,s_2\in\Gamma(E)$ and all $f\in\mathcal{H}(M)$.

Note that (1.1) means that $\rho_E:(\Gamma(E),[\cdot,\cdot]_E)\to(\Gamma(T'M),[\cdot,\cdot])$ is a complex Lie algebra homomorphism.

Also, the Lie bracket $[\cdot,\cdot]_E$ satisfies the Jacobi identity

$$[s_1,[s_2,s_3]_E]_E+[s_2,[s_3,s_1]_E]_E+[s_3,[s_1,s_2]_E]_E=0. \tag{1.3}$$

On a holomorphic Lie algebroid E, a differential $d_E:\Gamma(\wedge^kE^*)\to\Gamma(\wedge^{k+1}E^*)$ can be introduced in a classical manner, by

$$\begin{aligned} d_E\varphi(s_0,\dots,s_k)=&\sum_{i=0}^{k}(-1)^i\rho_E(s_i)(\varphi(s_0,\dots,\widehat{s}_i,\dots,s_k))+\\ &+\sum_{i<j}(-1)^{i+j}\varphi([s_i,s_j]_E,s_0,\dots,\widehat{s}_i,\dots,\widehat{s}_j,\dots,s_k), \end{aligned} \tag{1.4}$$

where $\varphi\in\Gamma(\wedge^kE^*)$ and $s_i\in\Gamma(E)$, $i=\overline{1,k}$.

2 Local expressions

If $(z^k)_{k=\overline{1,n}}$ is a local complex coordinate system on $U\subset M$ and $\{e_\alpha\}_{\alpha=\overline{1,m}}$ is a local frame of sections of E on U, then (z^k,u^α) are local complex coordinates on $\pi^{-1}(U)\subset E$, where $e=u^\alpha e_\alpha(z)$, $e\in E$.

Let $g_{UV}:U\cap V\to GL(m,\mathbb{C})$ be the holomorphic transition functions of E. In $z\in U\cap V$, $g_{UV}(z)$ is represented by the complex matrix of holomorphic functions $\left(M^\alpha_\beta(z)\right)$, such that, if $(\widetilde{z}^k,\widetilde{u}^\alpha)$ are local coordinates on $\pi^{-1}(V)$, then these change by the rules

$$\widetilde{z}^k=\widetilde{z}^k(z),\qquad \widetilde{u}^\alpha=M^\alpha_\beta(z)u^\beta. \tag{2.1}$$

The Jacobi matrix of the transformation laws (2.1) is

$$\begin{pmatrix} \dfrac{\partial\widetilde{z}^k}{\partial z^h} & 0\\[2ex] \dfrac{\partial M^\alpha_\beta}{\partial z^h}u^\beta & M^\alpha_\beta \end{pmatrix} \tag{2.2}$$

Let $\left(W^\beta_\alpha\right)$ be the inverse matrix of $\left(M^\alpha_\beta\right)$, and $\{e_\alpha\}$ a base of sections on E, that is, $u=u^\alpha e_\alpha$ for any $u\in\Gamma(E)$. Then these change by the rules

$$\widetilde{e}_\alpha=W^\beta_\alpha e_\beta.$$

The action of the holomorphic anchor map ρ_E can locally be described by

$$\rho_E(e_\alpha) = \rho_\alpha^k \frac{\partial}{\partial z^k}, \tag{2.3}$$

while the Lie bracket $[\cdot,\cdot]_E$ is locally given by

$$[e_\alpha, e_\beta]_E = C_{\alpha\beta}^{\gamma} e_\gamma. \tag{2.4}$$

The holomorphic functions $\rho_\alpha^k = \rho_\alpha^k(z)$ and $C_{\alpha\beta}^{\gamma} = C_{\alpha\beta}^{\gamma}(z)$ on M are called *the holomorphic structure functions* of the Lie algebroid E. A change of local charts on E implies

$$\widetilde{\rho}_\alpha^k = W_\alpha^\beta \rho_\beta^h \frac{\partial \widetilde{z}^k}{\partial z^h}. \tag{2.5}$$

Since E is a holomorphic vector bundle, it has the structure of a complex manifold, and the natural complex structure acts on its sections by $J_E(e_\alpha) = ie_\alpha$ and $J_E(\bar{e}_\alpha) = -i\bar{e}_\alpha$. Hence, the complexified bundle $E_{\mathbb{C}}$ of E decomposes into $E_{\mathbb{C}} = E' \oplus E''$. The sections of $E_{\mathbb{C}}$ are given as usual by $\Gamma(E') = \{s - iJ_E s \mid s \in \Gamma(E)\}$ and $\Gamma(E'') = \{s + iJ_E s \mid s \in \Gamma(E)\}$, respectively. The local basis of sections of E' is $\{e_\alpha\}_{\alpha=1,m}$, while for E'', the basis is represented by their conjugates $\{\bar{e}_\alpha := e_{\bar{\alpha}}\}_{\alpha=1,m}$. Since $\rho_E : E \to T'M$ is a homomorphism of complex modules, it extends naturally to the complexified bundle by $\rho'(e_\alpha) = \rho_E(e_\alpha)$ and $\rho''(e_{\bar{\alpha}}) = \rho_E(e_{\bar{\alpha}})$. Thus, the anchor map can be decomposed into $\rho_E = \rho' \oplus \rho''$ on the complexified bundle, and since E is holomorphic, the functions $\rho(z)$ are holomorphic, hence $\rho_\alpha^{\bar{k}} = \rho_{\bar{\alpha}}^k = 0$ and $\rho_{\bar{\alpha}}^{\bar{k}} = \overline{\rho_\alpha^k}$. Thus, the anchored bundles $(E', \rho', T'M)$ and $(E'', \rho'', T''M)$ are complex Lie algebroids ([4]). The Lie brackets are defined as

$$[e_\alpha, e_\beta]' = [e_\alpha, e_\beta]_E = C_{\alpha\beta}^{\gamma} e_\gamma; \quad [e_{\bar{\alpha}}, e_{\bar{\beta}}]'' = \overline{[e_\alpha, e_\beta]}_E = C_{\bar{\alpha}\bar{\beta}}^{\bar{\gamma}} e_{\bar{\gamma}},$$

where $C_{\bar{\alpha}\bar{\beta}}^{\bar{\gamma}} = \overline{C_{\alpha\beta}^{\gamma}}$. On the complexified bundle $E_{\mathbb{C}}$, we have to consider also the Lie brackets

$$[e_\alpha, e_{\bar{\beta}}] = C_{\alpha\bar{\beta}}^{\gamma} e_\gamma + C_{\alpha\bar{\beta}}^{\bar{\gamma}} e_{\bar{\gamma}}; \quad [e_{\bar{\alpha}}, e_\beta] = C_{\bar{\alpha}\beta}^{\gamma} e_\gamma + C_{\bar{\alpha}\beta}^{\bar{\gamma}} s_{\bar{\gamma}}.$$

It is obvious that $\overline{[e_\alpha, e_{\bar{\beta}}]} = [e_{\bar{\alpha}}, e_\beta]$, hence $\overline{C_{\alpha\bar{\beta}}^{\bar{\gamma}}} = C_{\alpha\bar{\beta}}^{\bar{\gamma}}$ and $\overline{C_{\alpha\bar{\beta}}^{\gamma}} = C_{\bar{\alpha}\beta}^{\bar{\gamma}}$.

Proposition 2.1. *The structure functions of the complexified Lie algebroid* $(E_{\mathbb{C}}, [\cdot,\cdot], \rho_E)$ *satisfy the identities:*

$$\begin{aligned}
&\rho_\alpha^j \frac{\partial \rho_\beta^i}{\partial z^j} - \rho_\beta^j \frac{\partial \rho_\alpha^i}{\partial z^j} = \rho_\gamma^i C_{\alpha\beta}^{\gamma}, \quad \rho_\gamma^i C_{\alpha\bar{\beta}}^{\gamma} = -\rho_{\bar{\beta}}^{\bar{j}} \frac{\partial \rho_\alpha^i}{\partial \bar{z}^j}, \quad \rho_{\bar{\gamma}}^{\bar{i}} C_{\alpha\bar{\beta}}^{\bar{\gamma}} = \rho_\alpha^j \frac{\partial \rho_{\bar{\beta}}^{\bar{i}}}{\partial z^j}, \\
&\rho_{\bar{\alpha}}^{\bar{j}} \frac{\partial \rho_{\bar{\beta}}^{\bar{i}}}{\partial \bar{z}^j} - \rho_{\bar{\beta}}^{\bar{j}} \frac{\partial \rho_{\bar{\alpha}}^{\bar{i}}}{\partial \bar{z}^j} = \rho_{\bar{\gamma}}^{\bar{i}} C_{\bar{\alpha}\bar{\beta}}^{\bar{\gamma}}, \quad \rho_{\bar{\gamma}}^{\bar{i}} C_{\bar{\alpha}\beta}^{\bar{\gamma}} = -\rho_\beta^j \frac{\partial \rho_{\bar{\alpha}}^{\bar{i}}}{\partial z^j}, \quad \rho_\gamma^i C_{\bar{\alpha}\beta}^{\gamma} = \rho_{\bar{\alpha}}^{\bar{j}} \frac{\partial \rho_\beta^i}{\partial \bar{z}^j}.
\end{aligned}$$

Proof. The identities follow by direct computations using (1.1), (2.3) and (2.4). □

Next, we consider the dual algebroid E^* and its complexification $E^*_{\mathbb{C}} = E'^* \oplus E''^*$ according to the eigenvalues $\pm i$ of the dual complex structure defined by $J^*_E(e^\alpha) = ie^\alpha$ and $J^*_E(e^{\bar\alpha}) = -ie^{\bar\alpha}$, with the dual cobasis $\{e^\alpha, e_{\bar\alpha}\}$, that is, $e^\alpha(e_\beta) = \delta^\alpha_\beta$, $e^\alpha(e_{\bar\beta}) = 0$ and their conjugates. We have $\Gamma(E'^*) = \{\omega - iJ^*_E\omega \mid \omega \in \Gamma(E^*)\}$ and $\Gamma(E''^*) = \{\omega + iJ^*_E\omega \mid \omega \in \Gamma(E^*)\}$. Then, the differential operator (1.4) extends to the complexified forms $\Omega_{\mathbb{C}}(E) = \Gamma\big(\Omega^{p,q}(E_{\mathbb{C}})\big)$ by $d_E = \partial'_E + \partial_E + \bar\partial_E + \partial''_E$, where

$$\partial'_E : \Omega^{p,q}(E_{\mathbb{C}}) \to \Omega^{p+2,q-1}(E_{\mathbb{C}})\ ; \quad \partial_E : \Omega^{p,q}(E_{\mathbb{C}}) \to \Omega^{p+1,q}(E_{\mathbb{C}});$$
$$\bar\partial_E : \Omega^{p,q}(E_{\mathbb{C}}) \to \Omega^{p,q+1}(E_{\mathbb{C}})\ ; \quad \partial''_E : \Omega^{p,q}(E_{\mathbb{C}}) \to \Omega^{p-1,q+2}(E_{\mathbb{C}}).$$

The total space of the holomorphic Lie algebroid E has a structure of complex manifold, since the transition functions $M^\alpha_\beta(z)$ are holomorphic. Consider the complexified tangent bundle of E, $T_{\mathbb{C}}E = T'E \oplus T''E$, where $T'E$ is the holomorphic tangent bundle and $T''E = \overline{T'E}$.

On $T'E$, a natural frame of fields is $\left\{\dfrac{\partial}{\partial z^k}, \dfrac{\partial}{\partial u^\alpha}\right\}$, which, due to the (2.2) matrix, changes by the rules

$$\begin{aligned}\frac{\partial}{\partial z^h} &= \frac{\partial \widetilde{z}^k}{\partial z^h}\frac{\partial}{\partial \widetilde{z}^k} + \frac{\partial M^\alpha_\beta}{\partial z^h}u^\beta\frac{\partial}{\partial \widetilde{u}^\alpha},\\ \frac{\partial}{\partial u^\beta} &= M^\alpha_\beta\frac{\partial}{\partial \widetilde{u}^\alpha}.\end{aligned} \tag{2.6}$$

Since E is a complex manifold, it follows that $\left\{\dfrac{\partial}{\partial \bar z^k}, \dfrac{\partial}{\partial \bar u^\alpha}\right\}$ is a local frame on $T''E = \overline{T'E}$ and its rules of change are deduced from (2.6) by conjugation.

As a mapping between manifolds, the holomorphic anchor ρ induced by ρ_E maps (z^k, u^α) on E to (z^k, η^k) on $T'M$, where $\eta^k = u^\alpha\rho^k_\alpha(z)$. A change of local charts implies that $(\widetilde{z}^k, \widetilde{u}^\alpha)$ is mapped to (z'^k, η'^k), where

$$z'^k = \widetilde{z}^k(z) = z'^k(z)$$

and

$$\eta'^k = \widetilde{u}^\alpha\widetilde{\rho}^k_\alpha(\widetilde{z}) = u^\gamma\rho^h_\gamma\frac{\partial \widetilde{z}^k}{\partial z^h} = \eta^h\frac{\partial \widetilde{z}^k}{\partial z^h},$$

such that

$$z'^k = z'^k(z), \qquad \eta'^k = u^\gamma\rho^h_\gamma\frac{\partial z'^k}{\partial z^h} \tag{2.7}$$

are local coordinates in a new chart on $T'M$.

The local expression of the differential operator ∂_E for $f \in \mathcal{H}(M)$ is

$$\partial_E f = \frac{\partial f}{\partial z^k}\rho^k_\alpha e^\alpha$$

and, for $\omega \in \Gamma(E'*)$, $\omega = \omega_\alpha e^\alpha$,

$$\partial_E\omega = \left(\frac{\partial \omega_\beta}{\partial z^i}\rho^i_\alpha - \frac{1}{2}\omega_\gamma C^\gamma_{\alpha\beta}\right)e^\alpha \wedge e^\beta.$$

In particular,

$$\partial_E z^k = \rho^k_\alpha e^\alpha, \qquad \partial_E e^\alpha = -\frac{1}{2} C^{\ \alpha}_{\beta\gamma} e^\beta \wedge e^\gamma.$$

3 Linear connections on E

Definition 3.1. *A linear connection on the holomorphic Lie algebroid* $(E, \rho_E, [.,.]_E)$ *is a map* $\nabla : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ *such that*

1) ∇ *is* $\mathbb{R}$*-bilinear;*

2) $\nabla_{fu} v = f\ \nabla_u v$ *for all* $f \in C^\infty(M)$ *and* $u, v \in \Gamma(E)$*;*

3) $\nabla_u f v = (\rho_E(u) f)\, v + f \nabla_u v$ *for all* $f \in C^\infty(M)$ *and* $u, v \in \Gamma(E)$*;*

4) $\overline{\nabla_u v} = \nabla_{\bar{u}} \bar{v}$*.*

A linear connection can be interpreted as a linear mapping $\nabla : \Gamma(E) \to \mathcal{A}^1(E)$ given by $\nabla : s \mapsto \nabla_s$, $\nabla_s(u) = \nabla_s u$ satisfying **3)** and **4)**. As on any holomorphic bundle, with respect to the holomorphic field of frames $\{s_\alpha\}$ we can consider the connection 1-forms $\theta^\beta_\alpha : \Gamma(E) \to \mathbb{C}$ on $(E, \rho_E, [.,.]_E)$ given by

$$\nabla_s e_\alpha = \theta^\beta_\alpha(s) e_\beta, \quad \alpha = \overline{1, m}. \tag{3.1}$$

By denoting $\nabla_{e_\alpha} e_\beta = \Gamma^{\ \gamma}_{\alpha\beta} e_\gamma$, then $\theta^\gamma_\beta(e_\alpha) = \Gamma^{\ \gamma}_{\alpha\beta} e_\gamma$. For $s = s^\alpha e_\alpha$ and $u = u^\beta e_\beta$, the action of the linear connection on $\Gamma(E)$ is

$$\nabla_s u = s^\alpha \left\{ \rho^k_\alpha \frac{\partial u^\gamma}{\partial z^k} + \Gamma^{\ \gamma}_{\alpha\beta} u^\beta \right\} e_\gamma, \tag{3.2}$$

which yields the expression of the 1-form $u \mapsto \nabla u$, $\nabla u(s) = \nabla_s u$ as

$$\nabla u = \left(\rho \circ d_E u^\gamma + \theta^\gamma_\beta \circ u^\beta\right) e_\gamma. \tag{3.3}$$

By considering the natural complex structure on E, we obtain by $\mathbb{C}-$linearity a linear connection on $(E_{\mathbb{C}}, \rho_E, [.,.]_E)$ acting on $\Gamma(E')$ and $\Gamma(E'')$ as follows:

$$\begin{aligned} \nabla_{e_\alpha} e_\beta &= \Gamma^\gamma_{\alpha\beta} e_\gamma + \Gamma^{\bar\gamma}_{\alpha\beta} e_{\bar\gamma}\ ; \quad \nabla_{e_\alpha} e_{\bar\beta} = \Gamma^\gamma_{\alpha\bar\beta} e_\gamma + \Gamma^{\bar\gamma}_{\alpha\bar\beta} e_{\bar\gamma}\ ; \\ \nabla_{e_{\bar\alpha}} e_\beta &= \Gamma^\gamma_{\bar\alpha\beta} e_\gamma + \Gamma^{\bar\gamma}_{\bar\alpha\beta} e_{\bar\gamma}\ ; \quad \nabla_{e_{\bar\alpha}} e_{\bar\beta} = \Gamma^\gamma_{\bar\alpha\bar\beta} e_\gamma + \Gamma^{\bar\gamma}_{\bar\alpha\bar\beta} e_{\bar\gamma}\ . \end{aligned} \tag{3.4}$$

The connection ∇ satisfies **4)**, hence $\overline{\Gamma^{\ \gamma}_{\alpha\beta}} = \Gamma^{\bar\gamma}_{\bar\alpha\bar\beta}$, $\overline{\Gamma^{\ \bar\gamma}_{\alpha\beta}} = \Gamma^{\gamma}_{\bar\alpha\bar\beta}$ *etc.*

The connection forms θ^β_α extended to $E_{\mathbb{C}}$ are called of $(1,0)$-type if $\theta^\beta_\alpha(s) = 0$ for all $s \in \Gamma(E'')$, that is, $\nabla_{e_\alpha} e_{\bar\beta} = \nabla_{e_{\bar\alpha}} e_\beta = 0$.

Moreover, we assume that ∇ is a *complex connection* ([4]) with respect to J_E, i.e. $(\nabla_u J_E)(v) = J_E(\nabla_u v)$. Then, $\Gamma^{\bar\gamma}_{\alpha\beta} = \Gamma^{\ \gamma}_{\alpha\bar\beta} = \Gamma^{\bar\gamma}_{\bar\alpha\beta} = \Gamma^{\ \gamma}_{\bar\alpha\bar\beta} = 0$ in (3.4), hence ∇ preserves the distributions $\Gamma(E')$ and $\Gamma(E'')$, respectively.

Consider on $(E, \rho_E, [.,.]_E)$ a Hermitian scalar product $g : \Gamma(E) \times \Gamma(E) \to \mathbb{C}$, which means that besides the linearity of the first term we have $g(u, v) = \overline{g(v, u)}$. The connection ∇ is called *metric with respect to* g if $\nabla g = 0$, where

$$(\nabla_s g)(u, v) = \rho(s)\,(g(u, v)) - g\,(\nabla_s u, v) - g(u, \nabla_s v), \quad \forall s, u, v \in \Gamma(E). \tag{3.5}$$

As in the case of Hermitian bundles, there is a unique linear connection, metric with respect to g, of $(1,0)$-type. Indeed, from

$$0 = (\nabla_{e_\alpha} g)(e_\beta, e_\gamma) = \rho(e_\alpha)\,(g(e_\beta, e_\gamma)) - g\,(\nabla_{e_\alpha} e_\beta, e_\gamma) - g(e_\beta, \nabla_{e_\alpha} e_\gamma),$$

by denoting $g(e_\beta, e_{\bar\gamma}) = g_{\beta\bar\gamma}$, we get $\rho^k_\alpha \dfrac{\partial g_{\beta\bar\gamma}}{\partial z^k} = \theta^\delta_\beta(e_\alpha) g_{\delta\bar\gamma} + \overline{\theta^\delta_\gamma}(e_\alpha) g_{\beta\bar\delta}$. Since the connection is of $(1,0)$-type, we have $\overline{\theta^\delta_\gamma}(e_\alpha) = \overline{\theta^\delta_\gamma(e_{\bar\alpha})} = 0$, hence

$$\theta^\delta_\beta(e_\alpha) = g^{\bar\gamma\delta} \rho^k_\alpha \frac{\partial g_{\beta\bar\gamma}}{\partial z^k}. \tag{3.6}$$

The torsion of a complex linear connection ∇ on E is defined as usual by $T(u, v) = \nabla_u v - \nabla_v u - [u, v]_E$. Its extension to $\Gamma(E_\mathbb{C})$ has the local components $T^{\ \gamma}_{\alpha\beta} e_\gamma := T(e_\alpha, e_\beta)$, etc. given by

$$\begin{aligned} T^{\ \gamma}_{\alpha\beta} &= \Gamma^{\ \gamma}_{\alpha\beta} - \Gamma^{\ \gamma}_{\beta\alpha} - C^{\ \gamma}_{\alpha\beta} \\ T^{\ \gamma}_{\alpha\bar\beta} &= \Gamma^{\ \gamma}_{\alpha\bar\beta} - C^{\ \gamma}_{\alpha\bar\beta} \\ T^{\ \bar\gamma}_{\alpha\bar\beta} &= \Gamma^{\ \bar\gamma}_{\alpha\bar\beta} - C^{\ \bar\gamma}_{\alpha\bar\beta} \end{aligned} \tag{3.7}$$

and their conjugates.

Again in a classical manner, the curvature of the complex linear connection ∇ is $R(u, v)s = \nabla_u \nabla_v s - \nabla_v \nabla_u s - \nabla_{[u,v]_E} s$, where $u, v, s \in \Gamma(E)$. On the complexified bundle $E_\mathbb{C}$, its local components are $R(e_\alpha, e_\beta) e_\gamma = R^{\ \delta}_{\gamma\alpha\beta} e_\delta$, $R(e_\alpha, e_\beta) e_{\bar\gamma} = R^{\ \bar\delta}_{\bar\gamma\alpha\beta} e_{\bar\delta}$, etc., where

$$\begin{aligned} R^{\ \delta}_{\gamma\alpha\beta} &= \Gamma^{\ \sigma}_{\beta\gamma}\Gamma^{\ \delta}_{\alpha\sigma} - \Gamma^{\ \sigma}_{\alpha\gamma}\Gamma^{\ \delta}_{\beta\sigma} - C^{\ \sigma}_{\alpha\beta}\Gamma^{\ \delta}_{\sigma\gamma} + \rho^k_\alpha \frac{\partial \Gamma^{\ \delta}_{\beta\gamma}}{\partial z^k} - \rho^k_\beta \frac{\partial \Gamma^{\ \delta}_{\alpha\gamma}}{\partial z^k} \\ R^{\ \bar\delta}_{\bar\gamma\alpha\beta} &= \Gamma^{\ \bar\sigma}_{\beta\bar\gamma}\Gamma^{\ \bar\delta}_{\alpha\bar\sigma} - \Gamma^{\ \bar\sigma}_{\alpha\bar\gamma}\Gamma^{\ \bar\delta}_{\beta\bar\sigma} - C^{\ \sigma}_{\alpha\beta}\Gamma^{\ \bar\delta}_{\sigma\bar\gamma} + \rho^k_\alpha \frac{\partial \Gamma^{\ \bar\delta}_{\beta\bar\gamma}}{\partial z^k} - \rho^k_\beta \frac{\partial \Gamma^{\ \bar\delta}_{\alpha\bar\gamma}}{\partial z^k} \\ R^{\ \delta}_{\gamma\alpha\bar\beta} &= \Gamma^{\ \sigma}_{\bar\beta\gamma}\Gamma^{\ \delta}_{\alpha\sigma} - \Gamma^{\ \sigma}_{\alpha\gamma}\Gamma^{\ \delta}_{\bar\beta\sigma} - C^{\ \sigma}_{\alpha\bar\beta}\Gamma^{\ \delta}_{\sigma\gamma} - C^{\ \bar\sigma}_{\alpha\bar\beta}\Gamma^{\ \delta}_{\bar\sigma\gamma} + \rho^k_\alpha \frac{\partial \Gamma^{\ \delta}_{\bar\beta\gamma}}{\partial z^k} - \rho^{\bar k}_{\bar\beta} \frac{\partial \Gamma^{\ \delta}_{\alpha\gamma}}{\partial z^{\bar k}} \\ R^{\ \bar\delta}_{\bar\gamma\alpha\bar\beta} &= \Gamma^{\ \bar\sigma}_{\bar\beta\bar\gamma}\Gamma^{\ \bar\delta}_{\alpha\bar\sigma} - \Gamma^{\ \bar\sigma}_{\alpha\bar\gamma}\Gamma^{\ \bar\delta}_{\bar\beta\bar\sigma} - C^{\ \sigma}_{\alpha\bar\beta}\Gamma^{\ \bar\delta}_{\sigma\bar\gamma} - C^{\ \bar\sigma}_{\alpha\bar\beta}\Gamma^{\ \bar\delta}_{\bar\sigma\bar\gamma} + \rho^k_\alpha \frac{\partial \Gamma^{\ \bar\delta}_{\bar\beta\gamma}}{\partial z^k} - \rho^{\bar k}_{\bar\beta} \frac{\partial \Gamma^{\ \bar\delta}_{\alpha\bar\gamma}}{\partial z^{\bar k}} \end{aligned} \tag{3.8}$$

and $R^{\ \delta}_{\gamma\bar\alpha\bar\beta} = \overline{R^{\ \bar\delta}_{\bar\gamma\alpha\beta}}$, etc.

By writing the curvature of ∇ as $R = \nabla \circ \nabla$, then the structure equations can be written in terms of the curvature 2-form R^α_β of ∇, that is,

$$R^\alpha_\beta = d\theta^\alpha_\beta + \theta^\alpha_\gamma \wedge \theta^\gamma_\beta,$$

such that

$$d_E R^\alpha_\beta = R^\alpha_\gamma \wedge \theta^\gamma_\beta - \theta^\alpha_\gamma \wedge R^\gamma_\beta$$

are the Bianchi identities.

The linear connection ∇ on E decomposes into ([8])

$$\nabla = \nabla' + \nabla'',$$

where $\nabla' : \Omega^{p,q}(E_{\mathbb{C}}) \to \Omega^{p+1,q}(E_{\mathbb{C}})$ and $\nabla'' : \Omega^{p,q}(E_{\mathbb{C}}) \to \Omega^{p,q+1}(E_{\mathbb{C}})$. The decomposition of the curvature is

$$R = \nabla' \circ \nabla' + \nabla' \circ \nabla'' + \nabla'' \circ \nabla' + \nabla'' \circ \nabla'',$$

such that the connection and curvature forms can be written as

$$\theta = \theta^{1,0} + \theta^{0,1}, \quad R = R^{2,0} + R^{1,1} + R^{0,2}.$$

For $u \in \Gamma(E_{\mathbb{C}})$ and $f \in \mathcal{H}(M)$, the identity (3.3) yields

$$\nabla'(fu) = \rho(u)\partial_E f + f\nabla' u, \quad \nabla''(fu) = \rho(u)\bar{\partial}_E f + f\nabla'' u,$$

such that, if the connection form θ is of $(1,0)$-type, that is, $\theta^{(0,1)} = 0$, then ∇ is of $(1,0)$-type and $\nabla'' = \bar{\partial}_E$. The converse holds as well.

4 Vertical and complete lifts

Since all the objects considered are holomorphic, the construction from this section is similar to the real case (see, for instance, [6, 10]). Let f be a holomorphic function on M. Its vertical lift f^v on E is defined by $f^v(e) = f(\pi(e))$, $e \in E$. The vertical lift of a section $Z \in \Gamma_{hol}(E)$, $Z = Z^\alpha s_\alpha$, is a vector field on E given by

$$Z^v(z,u) = Z^\alpha(z)\frac{\partial}{\partial u^\alpha}. \tag{4.1}$$

In particular, $s^v_\alpha = \dfrac{\partial}{\partial u^\alpha}$.

Some basic properties of the vertical lift are given in the following

Lemma 4.1. *If Z, W are holomorphic sections of E and f is a holomorphic function on M, then*

$$(Z+W)^v = Z^v + W^v, \qquad (fZ)^v = f^v Z^v, \quad Z^v f^v = 0.$$

The complete lift of a holomorphic function f on M is the holomorphic function f^c on E given by $f^c(e) = \partial^E f(e) = \rho_E(e) f$. Its local expression is

$$f^c(e) = u^\alpha \rho^k_\alpha \frac{\partial f}{\partial z^k}. \tag{4.2}$$

Lemma 4.2. *If Z is a holomorphic section on E and f, g are holomorphic functions on M, then*

$$(f+g)^c = f^c + g^c, \qquad (fg)^c = f^c g^v + f^v g^c, \qquad Z^v f^c = (\rho_E(Z)f)^v.$$

The complete lift Z^c of a section $Z \in \Gamma_{hol}(E)$ is a vector field on E defined by

$$Z^c(z,u) = Z^\alpha \rho_\alpha^k \frac{\partial}{\partial z^k} + \left(\rho_\beta^k \frac{\partial Z^\alpha}{\partial z^k} - Z^\gamma C_{\gamma\beta}^{\alpha}\right) u^\beta \frac{\partial}{\partial u^\alpha}. \tag{4.3}$$

In particular, $s_\alpha^c = \rho_\alpha^k \dfrac{\partial}{\partial z^k} - C_{\alpha\beta}^{\gamma} u^\beta \dfrac{\partial}{\partial u^\gamma}$.

Lemma 4.3. *If Z is a holomorphic section on E and f is a holomorphic function on M, then*

$$Z^c f^c = (\rho_E(Z)f)^c, \qquad Z^c f^v = (\rho_E(Z)f)^v.$$

Proof. The first property is proven by a straightforward computation using (4.1), (4.3) and Proposition 2.1.

The second property is a consequence of the first. Indeed, using also Lemma 4.2

$$\frac{1}{2} Z^c (f^2)^c = Z^c(f^c f^v) = (Z^c f^c) f^v + f^c (Z^c f^v) = (\rho_E(Z)f)^c f^v + f^c (Z^c f^v).$$

But the first identity gives, on the other hand,

$$\frac{1}{2} Z^c (f^2)^c = \frac{1}{2}(\rho_E(Z)f^2)^c = (f\rho_E(Z)f)^c = f^c(\rho_E(Z)f)^v + f^v(\rho_E(Z)f)^c$$

and these lead to $Z^c f^v = (\rho_E(Z)f)^v$. □

The Lie brackets of the complete and vertical lifts on E are given in the following

Lemma 4.4. *If Z and W are holomorphic sections on E, then*

$$[Z^c, W^c] = [Z,W]_E^c, \qquad [Z^c, W^v] = [Z,W]_E^v, \qquad [Z^v, W^v] = 0.$$

5 Semisprays and sprays for holomorphic anchored vector bundles

Following the steps from the real case ([2, 3]), semisprays can also be introduced on a holomorphic anchored vector bundle (E, π, M). Let ρ_E denote the anchor map, π_*, the tangent map of the projection π and $\tau_E : T'E \to E$, the holomorphic tangent bundle of E.

Definition 5.1. *A holomorphic section $S : E \to T'E$ is called semispray if*

i) $\tau_E \circ S = \mathrm{Id}_E$,

ii) $\pi_* \circ S = \rho_E$.

Let $c : I \to M,\ I \subset \mathbb{R}$ be a complex curve on M, $\widetilde{c} : I \to E$ a complex curve on E such that $\pi \circ \widetilde{c} = c$ and denote by $\dot{\widetilde{c}}$ the tangent vector field to curve $\widetilde{c}$.

Definition 5.2. *The vector field $\dot{\widetilde{c}}$ is called admissible if*

$$\pi_*(\dot{\widetilde{c}}) = \rho(\widetilde{c}). \tag{5.1}$$

Locally, $c(t) = (z^k(t))$, $\widetilde{c} = (z^k(t), u^\alpha(t))$ and $\dot{\widetilde{c}} = \dfrac{dz^k}{dt}\dfrac{\partial}{\partial z^k} + \dfrac{du^\alpha}{dt}\dfrac{\partial}{\partial u^\alpha}$, $t \in I$. Then, curve $\dot{\widetilde{c}}$ is admissible if and only if

$$\frac{dz^k}{dt}(t) = \rho^k_\alpha(z(t))u^\alpha(t),\ \forall t \in I.$$

If $S = Z^k \dfrac{\partial}{\partial z^k} + U^\alpha \dfrac{\partial}{\partial u^\alpha}$, then, using the definition, it follows that S is a semispray if and only if

$$Z^k(z, u) = \rho^k_\alpha(z)u^\alpha. \tag{5.2}$$

The coefficients $U^\alpha(z, u)$ are not determined, thus, for easier computations, let $U^\alpha = -2G^\alpha$, such that

$$S = \rho^k_\alpha u^\alpha \frac{\partial}{\partial z^k} - 2G^\alpha(z, u)\frac{\partial}{\partial u^\alpha}. \tag{5.3}$$

The rules of change for the coordinates of S are obtained using the (2.2) matrix:

$$\widetilde{Z}^k = \frac{\partial \widetilde{z}^k}{\partial z^h} Z^h \tag{5.4}$$

and

$$\widetilde{G}^\alpha = M^\alpha_\beta G^\beta - \frac{1}{2}\frac{\partial M^\alpha_\beta}{\partial z^k} u^\beta \rho^k_\gamma u^\gamma. \tag{5.5}$$

Moreover, due to (2.5), the coefficients $Z^k(z, u)$ given by (5.2) verify the (5.4) laws of change, which leads to the following result.

Proposition 5.1. *A vector field $S = \rho^k_\alpha u^\alpha \dfrac{\partial}{\partial z^k} - 2G^\alpha \dfrac{\partial}{\partial u^\alpha} \in \Gamma(T'E)$ is a semispray if and only if the coefficients G^α verify the* (5.5) *rules of transformation.*

A curve $c : t \mapsto (z^i(t), u^\alpha(t))$ on E is an integral curve of the semispray S if it satisfies the system of differential equations

$$\frac{dz^i}{dt} = \rho^k_\alpha(t)u^\alpha, \qquad \frac{du^\alpha}{dt} + 2G^\alpha(z, u) = 0. \tag{5.6}$$

A semispray can then be characterized also by

Proposition 5.2. *A vector field on E is a semispray if and only if all its integral curves are admissible.*

Another result similar to the real case is

Proposition 5.3. *Any two semisprays on E differ by a vertical vector field on E.*

Now, if $h_\lambda : E \to E$ is the complex homothety $h_\lambda : e \mapsto \lambda e,\ \lambda \in \mathbb{C},\ e \in E$, then a semispray S on E is called spray if

$$S \circ h_\lambda = \lambda h_{\lambda,*} \circ S. \tag{5.7}$$

Since the action of h_λ is locally described by $h_\lambda : (z^k, u^\alpha) \mapsto (z^k, \lambda u^\alpha)$, condition (5.7) becomes, equivalently,

$$G^\alpha(z, \lambda u) = \lambda^2 G^\alpha(z, u), \tag{5.8}$$

that is, the functions G^α are complex homogeneous of degree 2 in u.

Let $L = u^\alpha \dfrac{\partial}{\partial u^\alpha}$ be the complex Liouville vector field on E. Then, an even simpler formulation for the condition of spray can be obtained using Euler's theorem for homogeneous functions:

$$[L, S]_E = S. \tag{5.9}$$

We now try to obtain a complex spray from the variational problem. The first step is to express the Euler-Lagrange equations on the holomorphic Lie algebroid E. Since

$$\frac{d}{dt} = \frac{dz^k}{dt}\frac{\partial}{\partial z^k} + \frac{d\bar{z}^k}{dt}\frac{\partial}{\partial \bar{z}^k} + \frac{du^\alpha}{dt}\frac{\partial}{\partial u^\alpha} + \frac{d\bar{u}^\alpha}{dt}\frac{\partial}{\partial \bar{u}^\alpha},$$

from (5.6) it follows that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial u^\beta}\right) = \rho^k_\alpha u^\alpha \frac{\partial^2 L}{\partial z^k \partial u^\beta} + \rho^{\bar{k}}_{\bar{\alpha}} \bar{u}^\alpha \frac{\partial^2 L}{\partial \bar{z}^k \partial u^\beta} - 2G^\alpha \frac{\partial^2 L}{\partial u^\alpha \partial u^\beta} - 2\bar{G}^\alpha \frac{\partial^2 L}{\partial \bar{u}^\alpha \partial u^\beta}$$

or

$$\frac{d}{dt}\left(\frac{\partial L}{\partial u^\beta}\right) = \rho^k_\alpha u^\alpha \frac{\partial^2 L}{\partial z^k \partial u^\beta} + \rho^{\bar{k}}_{\bar{\alpha}} \bar{u}^\alpha \frac{\partial^2 L}{\partial \bar{z}^k \partial u^\beta} - 2G^\alpha g_{\alpha\beta} - 2\bar{G}^\alpha g_{\beta\bar{\alpha}}. \tag{5.10}$$

Following the ideas of Weinstein ([13]), the Euler-Lagrange equations on E are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial u^\beta}\right) = \rho^k_\beta \frac{\partial L}{\partial z^k} + \rho^{\bar{k}}_\beta \frac{\partial L}{\partial \bar{z}^k} + Q^\alpha_\beta \frac{\partial L}{\partial u^\alpha} + Q^{\bar{\alpha}}_\beta \frac{\partial L}{\partial \bar{u}^\alpha}, \tag{5.11}$$

where $\rho^{\bar{k}}_\beta = 0$ since E is holomorphic and Q^α_β and $Q^{\bar{\alpha}}_\beta$ must be determined.

From (5.10) and (5.11) we obtain

$$\left(\rho^k_\alpha u^\alpha \frac{\partial^2 L}{\partial z^k \partial u^\beta} - \rho^k_\beta \frac{\partial L}{\partial z^k} - Q^\alpha_\beta \frac{\partial L}{\partial u^\alpha} - 2G^\alpha g_{\alpha\beta}\right) + \left(\rho^{\bar{k}}_{\bar{\alpha}} \bar{u}^\alpha \frac{\partial^2 L}{\partial \bar{z}^k \partial u^\beta} - 2\bar{G}^\alpha g_{\beta\bar{\alpha}} - Q^{\bar{\alpha}}_\beta \frac{\partial L}{\partial \bar{u}^\alpha}\right) = 0,$$

which leads to

$$2\bar{G}^{\alpha}g_{\beta\bar{\alpha}}=\left(\rho_{\bar{\alpha}}^{\bar{k}}\bar{u}^{\alpha}\frac{\partial^2 L}{\partial\bar{z}^k\partial u^\beta}-Q_{\beta}^{\bar{\alpha}}\frac{\partial L}{\partial\bar{u}^\alpha}\right)+\left(\rho_{\alpha}^{k}u^{\alpha}\frac{\partial^2 L}{\partial z^k\partial u^\beta}-\rho_{\beta}^{k}\frac{\partial L}{\partial z^k}-Q_{\beta}^{\alpha}\frac{\partial L}{\partial u^\alpha}-2G^{\alpha}g_{\alpha\beta}\right). \quad (5.12)$$

If we denote by $-E_\beta$ the second paranthesis in (5.12), then

$$G^{\alpha}=\frac{1}{2}g^{\bar{\beta}\alpha}\left(\rho_{\gamma}^{k}u^{\gamma}\frac{\partial^2 L}{\partial z^k\partial\bar{u}^\beta}-Q_{\bar{\beta}}^{\gamma}\frac{\partial L}{\partial u^\gamma}\right)-\frac{1}{2}g^{\bar{\beta}\alpha}E_{\bar{\beta}}$$

or

$$G^{\alpha}=\frac{1}{2}g^{\bar{\beta}\alpha}\left(\rho_{\gamma}^{k}u^{\gamma}\frac{\partial^2 L}{\partial z^k\partial\bar{u}^\beta}-Q_{\bar{\beta}}^{\gamma}\frac{\partial L}{\partial u^\gamma}\right)-R^{\alpha}, \quad (5.13)$$

where $R^{\alpha}=\frac{1}{2}g^{\bar{\beta}\alpha}E_{\bar{\beta}}$. From the arbitrariness of Q_{β}^{α}, we can assume that R^α is a variable amount to be determined in the following.

Thus, we have to determine next $Q_{\bar{\beta}}^{\gamma}$ and R^α such that G^α be the coefficients of the spray satisfying (5.5). We have

$$\begin{aligned}\tilde{G}^{\alpha}&=\frac{1}{2}g^{\bar{\delta}\varepsilon}M_{\bar{\delta}}^{\bar{\beta}}M_{\varepsilon}^{\alpha}\left\{\tilde{\rho}_{\gamma}^{k}M_{\sigma}^{\gamma}u^{\sigma}\frac{\partial}{\partial\tilde{z}^k}\left(W_{\bar{\beta}}^{\bar{\theta}}\frac{\partial L}{\partial\bar{u}^\theta}\right)-\tilde{Q}_{\bar{\beta}}^{\gamma}W_{\gamma}^{\theta}\frac{\partial L}{\partial u^\theta}\right\}-\tilde{R}^{\alpha}\\&=\frac{1}{2}g^{\bar{\delta}\varepsilon}M_{\bar{\delta}}^{\bar{\beta}}M_{\varepsilon}^{\alpha}\left\{\rho_{\gamma}^{h}u^{\gamma}\frac{\partial\tilde{z}^k}{\partial z^h}\frac{\partial}{\partial\tilde{z}^k}\left(W_{\bar{\beta}}^{\bar{\theta}}\frac{\partial L}{\partial\bar{u}^\theta}\right)-\tilde{Q}_{\bar{\beta}}^{\gamma}W_{\gamma}^{\theta}\frac{\partial L}{\partial u^\theta}\right\}-\tilde{R}^{\alpha}\\&=\frac{1}{2}g^{\bar{\delta}\varepsilon}M_{\bar{\delta}}^{\bar{\beta}}M_{\varepsilon}^{\alpha}\left\{\rho_{\gamma}^{h}u^{\gamma}W_{\bar{\beta}}^{\bar{\theta}}\frac{\partial^2 L}{\partial z^k\partial\bar{u}^\theta}-\tilde{Q}_{\bar{\beta}}^{\gamma}W_{\gamma}^{\theta}\frac{\partial L}{\partial u^\theta}\right\}-\tilde{R}^{\alpha}\end{aligned}$$

or

$$\tilde{G}^{\alpha}=\frac{1}{2}g^{\bar{\delta}\varepsilon}M_{\varepsilon}^{\alpha}\rho_{\gamma}^{h}u^{\gamma}\frac{\partial^2 L}{\partial z^k\partial\bar{u}^\delta}-\frac{1}{2}\tilde{g}^{\bar{\beta}\alpha}\tilde{Q}_{\bar{\beta}}^{\gamma}W_{\gamma}^{\theta}\frac{\partial L}{\partial u^\theta}-\tilde{R}^{\alpha} \quad (5.14)$$

In order for $\tilde{G}^\alpha$ to verify (5.5), we write

$$\begin{aligned}\tilde{G}^{\alpha}=&M_{\varepsilon}^{\alpha}\left(\frac{1}{2}g^{\bar{\delta}\varepsilon}\left\{\rho_{\gamma}^{h}u^{\gamma}\frac{\partial^2 L}{\partial z^k\partial\bar{u}^\delta}-Q_{\bar{\beta}}^{\gamma}\frac{\partial L}{\partial u^\gamma}\right\}-R^{\varepsilon}\right)-\frac{1}{2}\frac{\partial M_{\beta}^{\alpha}}{\partial z^h}u^{\beta}\rho_{\gamma}^{h}u^{\gamma}\\&-\left(\frac{1}{2}\tilde{g}^{\bar{\beta}\alpha}\tilde{Q}_{\bar{\beta}}^{\gamma}W_{\gamma}^{\theta}\frac{\partial L}{\partial u^\theta}-\frac{1}{2}M_{\varepsilon}^{\alpha}g^{\bar{\delta}\varepsilon}Q_{\bar{\delta}}^{\gamma}\frac{\partial L}{\partial u^\gamma}\right)-\left(\tilde{R}^{\alpha}-M_{\varepsilon}^{\alpha}R^{\varepsilon}-\frac{1}{2}\frac{\partial M_{\beta}^{\alpha}}{\partial z^h}u^{\beta}\rho_{\gamma}^{h}u^{\gamma}\right).\end{aligned} \quad (5.15)$$

Comparing this with (5.5) it yields that the second row must vanish. Forcing things a bit, we ask that both brackets in this row be cancelled, which means that $Q_{\bar{\delta}}^{\gamma}$ and R^ε must satisfy

$$\tilde{g}^{\bar{\beta}\alpha}\tilde{Q}_{\bar{\beta}}^{\gamma}W_{\gamma}^{\theta}\frac{\partial L}{\partial u^\theta}=g^{\bar{\delta}\varepsilon}Q_{\bar{\delta}}^{\gamma}M_{\varepsilon}^{\alpha}\frac{\partial L}{\partial u^\gamma} \quad (5.16)$$

$$\tilde{R}^{\alpha}-M_{\varepsilon}^{\alpha}R^{\varepsilon}=\frac{1}{2}\frac{\partial M_{\beta}^{\alpha}}{\partial z^h}u^{\beta}\rho_{\gamma}^{h}u^{\gamma} \quad (5.17)$$

A simple computation shows that any (distinguished) tensor $Q^{\gamma}_{\bar{\delta}}$ on E verifies (5.16), so it would be best to choose $Q^{\gamma}_{\bar{\delta}} = 0$, and the spray will look like the canonical spray from Lagrange spaces ([8]). Following an idea from [2], another choice of d-tensor could be $Q^{\gamma}_{\bar{\beta}} = C^{\gamma}_{\alpha\bar{\beta}} u^{\alpha}$.

By taking

$$M^{\alpha}_{\varepsilon} R^{\varepsilon} = -\frac{1}{4} \frac{\partial M^{\alpha}_{\beta}}{\partial z^k} u^{\beta} \rho^k_{\gamma} u^{\gamma},$$

which means

$$R^{\varepsilon} = -\frac{1}{4} W^{\varepsilon}_{\alpha} \frac{\partial M^{\alpha}_{\beta}}{\partial z^k} u^{\beta} \rho^k_{\gamma} u^{\gamma}, \tag{5.18}$$

we have

$$\begin{aligned}
\tilde{R}^{\alpha} &= -\frac{1}{4} M^{\alpha}_{\varepsilon} \frac{\partial W^{\varepsilon}_{\beta}}{\partial \tilde{z}^k} \tilde{u}^{\beta} \tilde{\rho}^k_{\gamma} \tilde{u}^{\gamma} \\
&= -\frac{1}{4} M^{\alpha}_{\varepsilon} \frac{\partial W^{\varepsilon}_{\beta}}{\partial \tilde{z}^k} M^{\beta}_{\sigma} u^{\sigma} W^{\delta}_{\gamma} \rho^h_{\delta} \frac{\partial \tilde{z}^k}{\partial z^h} M^{\gamma}_{\theta} u^{\theta} \\
&= -\frac{1}{4} M^{\alpha}_{\varepsilon} \frac{\partial W^{\varepsilon}_{\beta}}{\partial \tilde{z}^k} M^{\beta}_{\sigma} u^{\sigma} \rho^h_{\delta} \frac{\partial \tilde{z}^k}{\partial z^h} u^{\delta} \\
&= \frac{1}{4} W^{\varepsilon}_{\beta} \frac{\partial M^{\alpha}_{\varepsilon}}{\partial z^h} M^{\beta}_{\sigma} u^{\sigma} \rho^h_{\delta} u^{\delta} \\
&= \frac{1}{4} \frac{\partial M^{\alpha}_{\sigma}}{\partial z^h} u^{\sigma} \rho^h_{\delta} u^{\delta}.
\end{aligned}$$

Hence R^{ε} from (5.18) satisfies (5.17).

Theorem 5.1. *On a holomorphic Lie algebroid E endowed with a regular Lagrangian $L(z,u)$ and a Hermitian metric tensor $g_{\bar{\alpha}\beta}$ with $\det(g_{\bar{\alpha}\beta}) \neq 0$, a complex canonical spray is given by*

$$G^{\alpha} = \frac{1}{2} \left(g^{\bar{\beta}\alpha} \frac{\partial^2 L}{\partial z^k \partial \bar{u}^{\beta}} + \frac{1}{2} W^{\alpha}_{\varepsilon} \frac{\partial M^{\varepsilon}_{\beta}}{\partial z^k} u^{\beta} \right) \rho^k_{\gamma} u^{\gamma} \tag{5.19}$$

Remark 5.1. *If the Lagrangian on E is complex homogeneous, then the spray is complex homogeneous of degree 2 in u.*

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