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$\mathbb{R}-$ COMPLEX CARTAN SPACES

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Abstract

In this paper, we investigate the $\mathbb{R}-$ complex Cartan spaces highlighting some classes of $\mathbb{R}-$ complex Hermitian Cartan spaces.In this study we use Chern-Cartan and canonical connections. Some classes of $\mathbb{R}-$ complex Hermitian Cartan spaces are introduced(weakly Kähler-Cartan, Kähler-Cartan,strongly Kähler-Cartan). The conditions for a R - complex Hermitian Cartan space to be weakly Berwald-Cartan or Berwald-Cartan are obtained. In the last section we emphasize the necessary and sufficient conditions under which an $\mathbb{R}-$ complex Hermitian Cartan space with a Randers metric is Berwald-Cartan.We come with some explicit examples to illustrate the interest for this classes of spaces.

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1 Introduction

Hamilton real space, and in particular Cartan spaces, were a frequent subject of study in recent years [7],[8]. The study of these classes of spaces was initiated by R. Miron in [17], which offered a geometric approach to the concepts used in mechanical and physical. His study was inspired by the well-known duality between the Lagrangian mechanics and Hamiltonian by Legendre transformation. This link between Lagrangian and Hamiltonian given by Legendre transformation had a great influence on the study of this type of geometry and brought a multitude of applications.

Many of these results from real Hamilton geometry were then translated into complex Hamilton geometry and after that into real and Finsler geometry [19].

This section arises from the need to extend recent results from $\mathbb{R}-$ complex Finsler spaces geometry [3],[18],[19].

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R- complex Cartan spaces study was initiated by M. Purcaru in [20]. In the first section we reproduce some of the results of [20], which then develop in the following sections, highlighting some classes of R- complex Hermitian Cartan spaces .

2 $\mathbb{R}-$complex Cartan spaces: definition, properties

Let M be an n-dimensional complex manifold and $\left(z^k\right)$ be local coordinates in a local chart.The complexified of the real tangent bundle $T_C M$ splits into the sum of the holomorfic tangent bundle $T'M$ and its conjugate $T''M$.The bundle $T'M$ has complex manifold structure and the coordinates in a local chart are $u = (z^k, \eta^k)$.These are changed by the rules $z'^k = z'^k(z)$; $\eta'^k = \frac{\partial z'^k}{\partial z^j}\eta^j$. The dual of $T'M$ is denoted by T'^*M. On the manifold T'^*M, in a local chart, a point u^* is characterized by the coordinates $u^* = (z^k, \zeta_k), k = \overline{1,n}$ and a change of these has the form

$$z'^k = z'^k(z)\ ;\ \zeta'_k = \frac{\partial z^j}{\partial z'^k}\zeta_j,\ rank\left(\frac{\partial z^j}{\partial z'^k}\right) = n \tag{1}$$

Definition 1. *[20] A $\mathbb{R}-$complex Cartan metric on M is a continuous function $\mathcal{C} : T'^*M \setminus \{0\} \longrightarrow \mathbb{R}_+$, satisfying: i) $H = \mathcal{C}^2$ is smooth on T'^*M except for the 0 section ;*

ii) $\mathcal{C}(z,\zeta) \geq 0$, *the equality holds if and only if* $\zeta = 0$;

iii) $\mathcal{C}(z, \lambda\zeta, \bar{z}, \lambda\bar{\zeta}) = \lambda^2\mathcal{C}(z,\zeta,\bar{z},\bar{\zeta}), \forall\lambda \in \mathbb{R}$.

Let us set the following metric tensors:

$$h^{ij} := \frac{\partial^2 H}{\partial\zeta_i\partial\zeta_j}\ ;\ h^{\bar{j}i} := \frac{\partial^2 H}{\partial\zeta_i\partial\bar{\zeta}_j};\ h^{\bar{i}\bar{j}} := \frac{\partial^2 H}{\partial\bar{\zeta}_i\partial\bar{\zeta}_j}. \tag{2}$$

Proposition 1. *[20] The $\mathbb{R}-$complex Cartan metric satisfies the conditions:*

i) $\frac{\partial H}{\partial\zeta_j}\zeta_j + \frac{\partial H}{\partial\bar{\zeta}_j}\bar{\zeta}_j = 2H$; *ii)* $h^{ij}\zeta_j + h^{\bar{j}i}\,\bar{\zeta}_j = \frac{\partial H}{\partial\zeta_i}$;

iii) $2H = h^{ij}\zeta_i\zeta_j + 2h^{\bar{j}i}\zeta_i\bar{\zeta}_j + h^{ij}\bar{\zeta}_i\bar{\zeta}_j$;

iv) $\frac{\partial h^{ik}}{\partial\zeta_j}\zeta_j + \frac{h^{ik}}{\partial\bar{\zeta}_j}\bar{\zeta}_j = 0$; $\frac{\partial h^{\bar{k}i}}{\partial\zeta_j}\zeta_j + \frac{h^{\bar{k}i}}{\partial\bar{\zeta}_j}\bar{\zeta}_j = 0$.

An immediat consequence of the homogenity condition leads us to the following Cartan type compex tensors:

$$C^{ijk} = -\frac{\partial h^{ij}}{\partial\zeta_k};\ C^{ij\bar{k}} = -\frac{\partial h^{ij}}{\partial\bar{\zeta}_k};\ C^{i\bar{j}\bar{k}} = -\frac{\partial h^{\bar{j}i}}{\partial\bar{\zeta}_k}, \tag{3}$$

and their conjugates.We denote with 0 and $\bar{0}$ the contracting of the Cartan tensors with ζ_k or $\bar{\zeta}_k$, respectively.

The Cartan complex tensors are symmetric in the indices of the same type:

$$\overline{C^{ij\bar{k}}} = C^{\bar{i}\bar{j}k} = C^{k\bar{i}\bar{j}}; \ \ \overline{C^{i\bar{j}\bar{k}}} = C^{\bar{i}jk} = C^{jk\bar{i}}, \tag{4}$$

$$C^{ij0} + C^{ij\bar{0}} = 0; \ \ C^{\bar{j}i0} + C^{\bar{j}i\bar{0}} = 0; \ \ C^{0ij} + C^{\bar{0}ij} = 0. \tag{5}$$

In complex Cartan geometry a strongly pseudoconvex requirement is assumed, that is the metric tensor $h^{\bar{j}i}$ which defines a positive-definite quadratic form, and then the $\left(h^{\bar{j}i}\right)$ matrix is invertible.

We remark that by restricting the homogenity of Cartan function in the definition of the complex Cartan space, the associated Hamiltonian H acquires a more general form than in the complex Cartan geometry([19]).

Firstly, we notice that if there is a set of local charts where $h^{ij} = 0$, then we have $C^{ijk} = C^{ij\bar{k}} = C^{\bar{k}ij} = 0$, and $h^{\bar{j}i}$ depends only on position z, or in [19] terminology is a purely Hermitian Cartan space. We have a very important question: which of the two metric tensors $h^{\bar{j}i}$ or h^{ij} must be invertible? Although in Hermitian geometry the compulsory request is that $h^{\bar{j}i}$ has to be invertible, in terms of physical notions it seems more important that h^{ij} has to be invertible.

These considerations lead us to a new class of spaces:

Definition 2. *[20] An ℝ− complex Hermitian Cartan space is the pair $(M, \mathcal{C})$, where H is an ℝ−complex Cartan metric , satisfying:*

$$h^{\bar{j}i} = \frac{\partial^2 H}{\partial \zeta_i \partial \bar{\zeta}_j}$$

is nondegenerated, i.e. $\det(h^{\bar{j}i}) \neq 0$ *in any point of* $T'^*M \backslash \{0\}$, *and determines a Hermitian metric structure .*

The most important problem in the study of T'^*M is that of the existence of a complex nonlinear connection, depending only on the Hamiltonian H.

In a complex Cartan space a special derivative law is usually considered, namely the Chern-Cartan connection [19]. Similar reasons lead us to a ℝ− complex Hermitian Cartan spaces.

Theorem 1. *[20] A complex nonlinear connection for the ℝ−complex Hermitian Cartan space $(M, \mathcal{C})$, called the Chern-Cartan (c.n.c.), is given by:*

$$\overset{CC}{N}_{ji} = -h_{j\bar{k}} \left(\frac{\partial h^{\bar{k}\bar{m}}}{\partial z^i} \partial \bar{\zeta}_m + \frac{\partial h^{\bar{k}l}}{\partial z^i} \partial \zeta_l \right). \tag{6}$$

Remark 1. *Moreover,the functions* $N_{ij} = \frac{1}{2} \left(\overset{CC}{N}_{ji} + \overset{CC}{N}_{ij} \right)$ *and*

$$\overset{c}{N}_{ij} = \frac{\partial \overset{CC}{N}_{ji}}{\partial \zeta_m} \zeta_m = - \left[\frac{\partial h_{j\bar{l}}}{\partial \zeta_m} \left(\frac{\partial h^{\bar{k}\bar{r}}}{\partial z^i} \partial \bar{\zeta}_r + \frac{\partial h^{\bar{k}l}}{\partial z^i} \partial \zeta_l \right) + h_{j\bar{l}} \frac{\partial h^{\bar{l}m}}{\partial z^i} \right] \zeta_m \tag{7}$$

both determine (c.n.c.) on T'^*M.

Now, having this (*c.n.c.*), our aim is to obtain an $N-(c.l.c.)$, of Chern-Cartan type. Let us consider the metric structure acting on $T_C(T'^*M)$:

$$\mathbf{H} = h_{i\bar{j}} dz^i \otimes d\bar{z}^j + h^{\bar{j}i} \delta\zeta_i \otimes \delta\bar{\zeta}_j.$$

called the $\overset{CC}{N}-$lift of the Hermitian metric $h_{i\bar{j}}$

A $N-(c.l.c.)$ D is said to be metrical if $D_X H = 0$, that is

$$(D_X H)(Y, Z) = X(H(Y, Z)) - \mathbf{H}(D_X Y, Z) - \mathbf{H}(Y, D_X Z) = 0,$$

for any $X, Y, Z \in \Gamma T_C(T'^*M)$.Then, in the adapted frame of Chern-Cartan (*c.n.c.*) , $\left\{\delta_i, \dot{\partial}^i, \delta_{\bar{\imath}}, \dot{\partial}^{\bar{\imath}}\right\}$ we have :

Theorem 2. *[20] In an $\mathbb{R}-$ complex Hermitian Cartan space, an $\overset{CC}{N}-$complex liniar connexion, which is metrical of $(1,0)-$type is given by*

$$\overset{CC}{H}{}^i_{jk} = h^{\bar{m}i} \frac{\delta h_{j\bar{m}}}{\delta z^k};\ \overset{CC}{V}{}_j^{ik} = h^{\bar{m}i} \frac{\partial h_{j\bar{m}}}{\partial \zeta_k};\ \overset{CC}{H}{}^{\bar{\imath}}_{\bar{j}k} = \overset{CC}{V}{}_{\bar{j}}^{\bar{\imath}k} = 0. \tag{8}$$

This $\overset{CC}{N}-(c.l.c.)$, will be called Chern-Cartan connection of an $\mathbb{R}-$complex Hermitian Cartan space. We have relations:

$$\overset{CC}{H}{}^i_{jk} = \dot{\partial}^i \overset{CC}{N}_{ji} \text{şi} \overset{CC}{H}{}^i_{jk} \zeta_i = \overset{CC}{N}_{ji}. \tag{9}$$

and the following nonzero torsions:

$$\overset{*}{h}T(\delta_k, \delta_j) = \left(\overset{CC}{H}{}^i_{jk} - \overset{CC}{H}{}^i_{kj}\right)\delta_i\ ;\ \overset{*}{v}T(\delta_{\bar{k}}, \delta_j) = -\delta_{\bar{k}}\left(\overset{CC}{N}_{ij}\right)\dot{\partial}^i \tag{10}$$

$$\overset{*}{h}T\left(\dot{\partial}^k, \delta_j\right) = \overset{CC}{V}{}_j^{ik}\delta_i\ ;\ \overset{*}{v}T\left(\dot{\partial}^{\bar{k}}, \delta_j\right) = -\dot{\partial}^{\bar{k}}\left(\overset{CC}{N}_{ij}\right)\dot{\partial}^i$$

Using the correspondence between the various geometrical objects on an $\mathbb{R}-$complex Finsler space and those of an $\mathbb{R}-$complex Cartan space, via complex Legendre transformation, it is proved:

Proposition 2. *The $\mathcal{L}-$ dual of the complex spray of Chern-Finsler (c.n.c.) is given by*

$$\frac{\partial \zeta_i}{dt} - \overset{CC}{N}_{ji} \frac{d^* \bar{z}^j}{dt} = 0,$$

and it is called the complex spray of the $\mathcal{L}-$dual of $\mathbb{R}-$complex Hermitian

3 Classes of $\mathbb{R}$ -complex Hermitian Cartan spaces

We begin with the study of connections of an $\mathbb{R}$ - complex Cartan space and the first one we consider, is an $\mathbb{R}$ - complex Cartan space (*c.n.c.*) with the following coefficients:

$$\overset{CC}{N}_{ji} = -h_{j\bar{k}} \left(\frac{\partial h^{\bar{k}\bar{m}}}{\partial z^i} \partial \bar{\zeta}_m + \frac{\partial h^{\bar{k}l}}{\partial z^i} \partial \zeta_l \right) \tag{11}$$

Therefore, because H is $\mathbb{R}$ -homogenous of degree 2 in the fibre variables and denoting by H_i the complex spray coefficients , we have:

$$(\dot{\partial}^j H_i)\zeta_j + (\dot{\partial}^{\bar{j}} H_i)\bar{\zeta}_j \ = 2H_i, \tag{12}$$

and this shows us that H_i are $\mathbb{R}$ - homogenous of degree 2 with respect to ζ. After that, we obtain also:

$$(\dot{\partial}^j \overset{CC}{N}_{li})\zeta_j + (\dot{\partial}^{\bar{j}} \overset{CC}{N}_{li})\bar{\zeta}_j \ = \overset{CC}{N}_{li} \tag{13}$$

this proves that $\overset{CC}{N}_{li}$ are $\mathbb{R}$ - homogenous of degree 1 with respect to ζ.

Based on this relation and taking into account $\overset{CC}{H^i_{jk}} = \dot{\partial}^i \overset{CC}{N}_{ji}$ we obtain:

$$(\dot{\partial}^j H^i_{lk})\zeta_j + (\dot{\partial}^{\bar{j}} H^i_{lk})\bar{\zeta}_j \ = 0 \tag{14}$$

i.e. H^i_{lk} are $\mathbb{R}$ -homogenous of degree 0 with respect to ζ.

The second important nonlinear connection is the canonical one (*c.n.c.*). The local coefficients of this connection on $(M, \mathcal{C})$ are given by relation $\overset{c}{N}_{ji} = (\dot{\partial}^k N_{ji})\zeta_k$ and they are $\mathbb{R}$ - homogenous of degree 1. We associate to Chern-Cartan (*c.n.c.*) another connection of Berwald-type:

$$B\Gamma := \left(N_{ji}, \ B^i_{jk} := \dot{\partial}^k N_{jk}, \ B^i_{j\bar{k}} := \dot{\partial}^{\bar{k}} \overset{c}{N}_{ji}, 0, 0 \right) \tag{15}$$

$B\Gamma$ neither h^*- nor v^*- metrical, but we have the following properties:

$$B^i_{jk}\zeta_i = \overset{c}{N}_{jk} + (\dot{\partial}^{\bar{r}} \overset{c}{N}_{jk})\bar{\zeta}_r \ , \ B^i_{jk} = B^i_{kj} \tag{16}$$

We set the connection form

$$\omega^i_j(z, \zeta) = H^i_{jk} dz^k + V^{ik}_j \delta\zeta_k \tag{17}$$

which satisfies the following structure equations

$$d(dz^i) - dz^k \wedge \omega^i_k = \theta^i \ ; \ \ d(\delta\zeta_k) + \delta\zeta_k \wedge \omega^i_k = \tau^i, \tag{18}$$

and their complex conjugates, where d is exterior differential with respect to the Chern-Cartan (*c.n.c.*). We have

$d(\delta\zeta_k) = -dN_{kj} \wedge dz^k = -(\delta_{\overline{h}} N_{kj}) d\bar{z}^h \wedge dz^j - H^s_{kj}\delta\zeta_s \wedge dz^j - (\dot{\partial}^{\overline{h}} N_{kj})\delta\bar{\zeta}_h \wedge dz^j$
And next we determine torsion forms :

$$\theta^i = -\frac{1}{2} T^{*i}_{jk} dz^j \wedge dz^k - V^{ik}_j dz^j \wedge \delta\zeta_k \; ; \qquad (19)$$
$$\tau^i = -(\delta_{\overline{h}} N_{ik}) d\bar{z}^h \wedge dz^k - (\dot{\partial}^{\overline{h}} N_{kj})\delta\bar{\zeta}_h \wedge dz^j,$$

where $T^{*i}_{jk} = H^i_{jk} - H^i_{kj}$, and the mixted part of the torsion form θ^i vanishes in the purely Hermitian case (i.e.$V^{ik}_j = 0$). Also from $\theta^i = 0$ we obtain $\frac{\partial h^{j\bar{m}}}{\partial z^i} = \frac{\partial h^{i\bar{m}}}{\partial z^j}$.
We check that: $\overset{c}{N}_{ik} = N_{ik} - \frac{1}{2}[T^{*i}_{jk}\zeta_l + (\dot{\partial}^{\bar{r}} N_{ik})\bar{\zeta}_r]$, and its differentiation with respect to ζ leads us to:

$$B^i_{jk} = H^i_{jk} - \frac{1}{2}[\dot{\partial}^l (T^{*i}_{jk}\zeta_l) + (\dot{\partial}^{\bar{r}} H^i_{jk})\bar{\zeta}_r] \qquad (20)$$

In $\mathbb{R}$ -complex Finsler space geometry we have three kinds of Kahler properties. According to this we introduce similar notions on $\mathbb{R}-$complex Hermitian Cartan space.

Definition 3. *Let $(M, \mathcal{C})$ be an $\mathbb{R}-$complex Hermitian Cartan space. $(M, \mathcal{C})$ is calling*
i) strongly Kähler-Cartan if $T^{*i}_{jk} = 0$;
ii) Kähler-Cartan if $T^{*i}_{jk}\zeta_j = 0$;
iii) weakly Kähler-Cartan if $h^{i\bar{m}} T^{*i}_{jk}\zeta_j\bar{\zeta}_j = 0$.

In complex Finsler geometry Kähler and Kähler notions coincide, here, similar to $\mathbb{R}$ -complex Finsler space case, this does not happen.

If H is Kähler it results that $B^i_{jk} = H^i_{jk} - \frac{1}{2}(\dot{\partial}^{\bar{r}} H^i_{jk})\bar{\zeta}_r$ and taking into account H^i_{jk} is $\mathbb{R}-$ homogeneous of degree 0 with respect to ζ we have $B^i_{jk} = H^i_{jk}$.But $B^i_{jk} = B^i_{kj}$, and from this we obtain $H^i_{jk} = H^i_{kj}$.Therefore, we deduced:

Theorem 3. *Let $(M, \mathcal{C})$ be an $\mathbb{R}-$complex Hermitian Cartan space. Then, $\mathcal{C}$ is Kähler-Cartan and the coefficients H^i_{jk} are $(0, 0)-$ homogeneous with respect to ζ if and only if $\mathcal{C}$ is strongly Kähler-Cartan and $B^i_{jk} = H^i_{jk}$.*

If Chern-Cartan (*c.n.c.*) comes from a complex spray we have $N_{ik} = \overset{c}{N}_{ik}$ and taking into account $\overset{c}{N}_{ik} = N_{ik} - \frac{1}{2}[T^{*i}_{jk}\zeta_l + (\dot{\partial}^{\bar{r}} N_{ik})\bar{\zeta}_r]$ it results $T^{*i}_{jk}\zeta_l + (\dot{\partial}^{\bar{r}} N_{ik})\bar{\zeta}_r = 0$. By contracting with ζ_k and differentiating, after a few computations we obtain the following outcome:

Theorem 4. *Let $(M, \mathcal{C})$ be an $\mathbb{R}-$complex Hermitian Cartan space . Then,Chern-Cartan (c.n.c.) comes from a complex spray if and only if its local coeficients are $(1, 0)-$ homogeneous with respect to ζ and $\mathcal{C}$ is Kähler-Cartan. Moreover, in this case $\mathcal{C}$ is stongly Kähler-Cartan and $N_{ik} = \overset{c}{N}_{ik}$.*

In addition, we can go further and

$$\overset{c}{\delta}_k = \delta_k - (\overset{c}{N}_{ik} - N_{ik})\dot{\partial}^i \tag{21}$$

We notice that if $\overset{c}{\delta}_k = \delta_k$ then we have $\overset{c}{N}_{ik} = N_{ik}$, and from the above theorem, we obtain that Chern- Cartan($c.n.c.$) comes from a complex spray.

Now, we consider the relation $\overset{c}{N}_{ik} = N_{ik} - \frac{1}{2}[T^{*i}_{jk}\zeta_l + (\dot{\partial}^{\bar{r}} N_{ik})\bar{\zeta}_r]$and contract it with $h^{i\bar{m}}\bar{\zeta}_m$.

$\overset{c}{N}_{ik}h^{i\bar{m}}\bar{\zeta}_m = N_{ik}h^{i\bar{m}}\bar{\zeta}_m - \frac{1}{2}[T^{*i}_{jk}\zeta_l + (\dot{\partial}^{\bar{r}} N_{ik})\bar{\zeta}_r]h^{i\bar{m}}\bar{\zeta}_m$, i.e. ($\overset{c}{N}_{ik} - N_{ik})h^{i\bar{m}}\bar{\zeta}_m = -\frac{1}{2}h^{i\bar{m}}\bar{\zeta}_m[T^{*i}_{jk}\zeta_l + (\dot{\partial}^{\bar{r}} N_{ik})\bar{\zeta}_r]$

But $\overset{c}{\delta}_k(\dot{\partial}^{\bar{r}} H)\bar{\zeta}_r = \delta_k(\dot{\partial}^{\bar{r}} H)\bar{\zeta}_r - (\overset{c}{N}_{ik} - N_{ik})\dot{\partial}^i(\dot{\partial}^{\bar{r}} H)\bar{\zeta}_r$ and we obtain $\overset{c}{\delta}_k(\dot{\partial}^{\bar{r}} H)\bar{\zeta}_r = -(\overset{c}{N}_{ik} - N_{ik})h^{i\bar{r}}\bar{\zeta}_r$ and after that $2\overset{c}{\delta}_k(\dot{\partial}^{\bar{r}} H)\bar{\zeta}_r - h^{i\bar{m}}\bar{\zeta}_m(\dot{\partial}^{\bar{r}} N_{ik})\bar{\zeta}_r = 0$. And so, we have following Theorem:

Theorem 5. *Let $(M, \mathcal{C})$ be an $\mathbb{R}-$complex Hermitian Cartan space. Then, $\mathcal{C}$ is weakly Kähler- Cartan if and only if*

$$[2\overset{c}{\delta}_k(\dot{\partial}^{\bar{r}} H) - h^{i\bar{m}}\bar{\zeta}_m(\dot{\partial}^{\bar{r}} N_{ik})]\bar{\zeta}_r = 0 \tag{22}$$

Our next purpose is to introduce the Berwald-Cartan notions and to obtain some relations between Berwald-Cartan and Kähler- Cartan spaces.

Definition 4. *Let $(M, \mathcal{C})$ be a $\mathbb{R}-$complex Hermitian Cartan space .*

i) $(M, \mathcal{C})$ is weakly Berwald-Cartan if the local coefficients B^i_{jk} depend only on position z

ii) $(M, \mathcal{C})$ is Berwald-Cartan if the local coefficients H^i_{jk} depend only on position z.

Whereas $B^i_{jk} = H^i_{jk} - \frac{1}{2}[\dot{\partial}^l(T^{*i}_{jk}\zeta_l) + (\dot{\partial}^{\bar{r}} H^i_{jk})\bar{\zeta}_r]$ we can conclude that any $\mathbb{R}$ -complex Hermitian Cartan space which is Berwald-Cartan is weakly Berwald-Cartan. But the converse is not true. Because $(M, \mathcal{C})$ is Berwald-Cartan we have

$$B^i_{jk}(z) = \frac{1}{2}[H^i_{jk}(z) + H^i_{kj}(z)] \tag{23}$$

Theorem 6. *Let $(M, \mathcal{C})$ be an $\mathbb{R}-$complex Hermitian Cartan space. Then, $(M, \mathcal{C})$ is a weakly Berwald-Cartan space if and only if $B^i_{j\bar{k}}$ depend only on the position z. Furthermore, in this case*

$$\overset{c}{N}_{ik} = B^i_{jk}(z)\zeta_j + B^i_{k\bar{h}}(z)\bar{\zeta}_h \tag{24}$$

Proof. Assuming that $B^i_{jk} = B^i_{jk}(z)$ we have $\dot{\partial}^{\bar{r}} B^i_{jk} = 0$, and on the other hand $\dot{\partial}^{\bar{r}} B^i_{jk} = \dot{\partial}^{\bar{r}}(\dot{\partial}^k \overset{c}{N}_{ij}) = \dot{\partial}^k(\dot{\partial}^{\bar{r}} \overset{c}{N}_{ij}) = \dot{\partial}^k B^i_{j\bar{r}}$, so $\dot{\partial}^k B^i_{j\bar{r}} = 0$. After the complex conjugation we obtain $\dot{\partial}^{\bar{k}} B^{\bar{i}}_{\bar{j}r} = 0$, which means that $B^{\bar{i}}_{\bar{j}r}$ are holomorphic and due to the strong maximum principle $B^{\bar{i}}_{\bar{j}r}$ depend only on position z, and from here $B^i_{j\bar{r}}$ depend only on z. Conversely, from $B^i_{k\bar{h}} = B^i_{k\bar{h}}(z)$ it results $\dot{\partial}^l B^i_{j\bar{k}} = 0$, $\dot{\partial}^l B^i_{j\bar{k}} = \dot{\partial}^l(\dot{\partial}^{\bar{k}} \overset{c}{N}_{ij}) = \dot{\partial}^{\bar{k}}(\dot{\partial}^l \overset{c}{N}_{ij}) = \dot{\partial}^{\bar{k}} B^i_{lj}$, so B^i_{lj} are holomorphic and $B^i_{jk} = B^i_{jk}(z)$. □

We also can prove that:

Theorem 7. *Let $(M, \mathcal{C})$ be an $\mathbb{R}-$complex Hermitian Cartan space . Then, $(M, \mathcal{C})$ is a Berwald-Cartan space if and only if $\dot{\partial}^{\bar{r}} N^i_{ik}$ depend only on position z. Moreover, in this case,*

$$N_{ik} = H^i_{jk}(z)\zeta_j + (\dot{\partial}^{\bar{h}} N_{ij})(z)\bar{\zeta}_h \tag{25}$$

and

$$\dot{\partial}^{\bar{k}} \overset{c}{N}_{ij} = \frac{1}{2}\dot{\partial}^{\bar{k}} N_{ij}$$

We call strongly Berwald-Cartan space, an $\mathbb{R}-$ complex Hermitian Cartan space which is at the same time Berwald-Cartan and Kähler-Cartan.

Returning to $\mathcal{L}$- dual process and the correspondents between diferent geometric notions which are generated we can get some conclusions.

First and foremost, we obtain:

Theorem 8. *Let (M, F) be an $\mathbb{R}-$ complex Hermitian Finsler space. If (M, F) is Kähler then its $\mathcal{L}$- dual $(M, \mathcal{C})$ is Kähler-Cartan.*

We move forward and assume that (M, F) is an $\mathbb{R}-$ Berwald space. It means that $L^i_{jk} = L^i_{jk}(z)$ and from here it results that $\dot{\partial}_h L^i_{jk} = \dot{\partial}_h L^i_{jk}(z) = 0$. Taking into account $\mathcal{L}$-dual process we obtain $\dot{\partial}^{\bar{h}} H^i_{jk} = 0$. Therefore the H^i_{jk} functions are holomorphic with respect to ζ.

But we know that H^i_{jk} are $\mathbb{R}-$homogeneous of degree 0 with respect to ζ , and due to the strong maximum principle we have $H^i_{jk} = H^i_{jk}(z)$.

Moreover if we assume that (M, F) is an $\mathbb{R}-$complex Kähler space it results:

Theorem 9. *Let (M, F) be an $\mathbb{R}-$ complex Hermitian Finsler space. If (M, F) is strongly Berwald then its $\mathcal{L}$- dual $(M, \mathcal{C})$ is strongly Berwald-Cartan.*

First example of $\mathbb{R}-$complex Hermitian Cartan spaces which are Berwald is given by the class of purely Hermitian spaces. So, considering a purely Hermitian metric, (i.e. $h^{i\bar{j}} = h^{i\bar{j}}(z)$ and $h^{\bar{r}\bar{m}}(z)$) with $h^{i\bar{j}}$ invertible),we have

$$N_{ik} = -h_{\bar{m}i}(\frac{\partial h^{\bar{r}\bar{m}}}{\partial z^k}\bar{\zeta}_r + \frac{\partial h^{s\bar{m}}}{\partial z^k}\zeta_s), \tag{26}$$

and from here it results $H^i_{sk} = h_{\bar{m}i}\frac{\partial h^{s\bar{m}}}{\partial z^k}$ depend only on z. Thus, all purely Hermitian spaces, with $h^{i\bar{j}}$ invertible, are Berwald-Cartan.

The second example of Berwald-Cartan space is given by function:

$$H(z,w,\zeta,v)=e^{2\sigma}\sqrt{\left(\zeta+\bar{\zeta}\right)^4+(v+\bar{v})^4},\ \text{with}\ \zeta,v\neq 0 \tag{27}$$

on $\mathbb{C}^2$, where $\sigma(z,w)$ is a real valued function. We relabeled the usual local coordinates z^1, z^2,ζ_1 , ζ_2 as z, w,ζ, v respectively. Now, we can compute the coefficients:

$$H_1=(\frac{\partial\sigma}{\partial z}\zeta+\frac{\partial\sigma}{\partial w}v)\,(\zeta+\zeta)\ \ ;\ H_2=(\frac{\partial\sigma}{\partial z}\zeta+\frac{\partial\sigma}{\partial w}v)\,(v+\bar{v}) \tag{28}$$

From

$$N_{1i}=-2\left(\zeta+\bar{\zeta}\right)\frac{\partial\sigma}{\partial z^i};\ N_{2i}=-2\,(v+\bar{v})\,\frac{\partial\sigma}{\partial z^i},\ i=1,2, \tag{29}$$

we find the horizontal coefficients of Chern-Cartan connection:

$$H^1_{1i}=H^2_{2i}=-2\frac{\partial\sigma}{\partial z^i};\ H^1_{2i}=H^2_{1i}=0,\ i=1,2, \tag{30}$$

which depend only on z^i, $i=1,2$.

4 $\mathbb{R}$-complex Hermitian Berwald spaces with Randers-Cartan metrics

We consider $z\in M$, $\zeta\in T^{*\prime}_zM$, $\zeta=\zeta_i\frac{\partial}{\partial z^i}$. An $\mathbb{R}-$complex Cartan space $(M,\mathcal{C})$is called Randers-Cartan if

$$\mathcal{C}=\alpha+\beta, \tag{31}$$

where

$$\alpha^2(z,\zeta,\bar{z},\bar{\zeta}):=Re\{a^{ij}\zeta_i\zeta_j\}+a^{i\bar{j}}\zeta_i\bar{\zeta}_j;$$
$$\beta(z,\zeta,\bar{z},\bar{\zeta}):=Re\{b^i\zeta_i\}.$$

with $a^{ij}=a^{ij}\,(z)$,$a^{i\bar{j}}=a^{i\bar{j}}\,(z)$, and $b=b_i(z)dz^i$ is a $(1,0)-$ differential form. The Randers function produces two tensor fields h^{ij} and $h^{i\bar{j}}$. For the study of $\mathbb{R}-$complex Hermitian Cartan spaces with Randers metric, we assume that $a^{ij}=0$. Then, only $h^{i\bar{j}}$ tensor field is invertible and it has the following properties:

Proposition 3. *For an $\mathbb{R}-$complex Hermitian Randers-Cartan space, with $a^{ij}=0$, we have*

i) $h^{i\bar{j}}=\frac{\mathcal{C}}{\alpha}a^{i\bar{j}}-\frac{\beta}{2\alpha^3}\zeta^i\bar{\zeta}^j+\frac{1}{2}b^ib^{\bar{j}}+\frac{1}{2\alpha}(b^{\bar{j}}\zeta^i+b^i\zeta^{\bar{j}})$ *şi*

$h^{ij}=-\frac{\beta}{2\alpha^3}\zeta^i\zeta^j+\frac{1}{2}b^ib^j+\frac{1}{2\alpha}(b^j\zeta^i+b^i\zeta^j)$;

ii) $h_{i\bar{j}}=\frac{\alpha}{\mathcal{C}}a_{\bar{j}i}+\frac{2\beta+\alpha\omega}{\mathcal{C}}\zeta_i\bar{\zeta}_j-\frac{\alpha^3}{T\mathcal{C}}b_i\bar{b}_j-\frac{\alpha}{T\mathcal{C}}\left[\left(\bar{\varepsilon}+2\alpha\right)\zeta^i\bar{b}_j+\left(\varepsilon+2\alpha\right)b_i\bar{\zeta}^j\right]$;

iii) $\det\left(h^{i\bar{j}}\right)=\left(\frac{\mathcal{C}}{\alpha}\right)^n\frac{T}{4\alpha\mathcal{C}}\det\left(a^{i\bar{j}}\right)$,

where

$$\alpha^2 = a^{j\bar{k}}\zeta_i\bar{\zeta}_j;\ \zeta^i = a^{i\bar{j}}\bar{\zeta}_j;\ b_j = a_{j\bar{k}}b^{\bar{k}};\ b^l = b_{\bar{k}}a^{l\bar{k}};\ b^{\bar{k}} := \bar{b}^k; \tag{32}$$

$$\varepsilon := b^j\zeta_j;\ \omega := b_j b^j = \bar{\omega};\ \varepsilon + \bar{\varepsilon} = 2\beta;$$

$$T := \alpha(4\mathcal{C} + 2\beta + \alpha\omega) + \varepsilon\bar{\varepsilon} > 0.$$

After a technical computation, we obtain:

$$N_{ij} = \overset{a}{N}_{ij} + \frac{2}{T}[(2\mathcal{C} - \varepsilon)\zeta_i + \alpha^2 b^i](\overset{a}{\delta}_j\beta) + \mathcal{C}h_{\bar{r}i}\frac{\partial b^{\bar{r}}}{\partial z^j} \tag{33}$$

where $\overset{a}{N}_{kj} := a_{\bar{m}i}\frac{\partial^2\alpha^2}{\partial z^k\partial\zeta_m} = a_{\bar{m}i}\frac{\partial a^{s\bar{m}}}{\partial z^k}\zeta_s$ and $2(\overset{a}{\delta}_j\beta) := \frac{\partial\beta}{\partial z^j} - \overset{a}{N}_{kj}(\dot{\partial}_k\beta) = \frac{\partial\bar{b}_r}{\partial z^j}\zeta^{\bar{r}} + \frac{\partial b^{\bar{r}}}{\partial z^j}\bar{\zeta}_r$.

Proceeding as in the case of $\mathbb{R}-$complex Finsler spaces we can prove:

Lemma 1. *The functions b^i and b_i are holomorphic if and only if $\overset{a}{\delta}_j\beta = 0$.*

Theorem 10. *Let $(M, \mathcal{C})$ be an $\mathbb{R}-$complex Hermitian Randers-Cartan space, with $a_{ij} = 0$. If $\overset{a}{\delta}_j\beta = 0$, then it is Berwald-Cartan space and $N_{ij} = \overset{a}{N}_{ij}$. Moreover, if α is Kähler-Cartan, then $\mathcal{C}$ is strongly Kähler-Cartan.*

If $\overset{a}{\delta}_j\beta = 0$ it results that $N_{ij} = \overset{a}{N}_{ij} + \mathcal{C}h_{\bar{r}i}\frac{\partial b^{\bar{r}}}{\partial z^j}$ and using the previous Lemma we have $N_{ij} = \overset{a}{N}_{ij}$. Since α is a purely Hermitian metric, thus it is Berwald- Cartan, we obtain that $\mathcal{C} = \alpha + \beta$ is also Berwald-Cartan and then $H^i_{jk}(z) = \overset{a}{H}{}^i_{jk}(z)$. Now, we assume that α is Kähler-Cartan and we have $T^{*i}_{jk} = \overset{a}{H}{}^i_{jk} - \overset{a}{H}{}^i_{kj} = 0$, which show us that $\mathcal{C}$ is strongly Kähler-Cartan.

An example is given by function:

$$\mathcal{C} = \sqrt{e^{z^1+\bar{z}^1}\,|\zeta_1|^2 + e^{z^2+\bar{z}^2}\,|\zeta_2|^2} + \frac{1}{2}(e^{z^2}\zeta^2 + e^{\bar{z}^2}\bar{\zeta}^2)$$

which is a Hermitian Randers-Cartan metric having
$\det(h^{i\bar{j}}) = \left(\frac{\mathcal{C}}{\alpha}\right)^2\frac{T}{4\alpha\mathcal{C}}\det(a^{i\bar{j}}) = \frac{\mathcal{C}T}{4\alpha^3}e^{z^1+\bar{z}^1+z^2+\bar{z}^2} > 0$, $(i, j = 1, 2)$, and
$T = \alpha(5\mathcal{C} + \beta) + \varepsilon\bar{\varepsilon} > 0$.

A direct computation gives
$2(\overset{a}{\delta}_j\beta) = \frac{\partial\bar{b}_2}{\partial z^j}\zeta^{\bar{2}} + \frac{\partial b^{\bar{2}}}{\partial z^j}\bar{\zeta}_2 = 0$,and $\frac{\partial b^{\bar{m}}}{\partial z^j} = 0$, $(j, m = 1, 2)$. Substituting these relations into coefficients formula, we have: $N_{11} = \overset{a}{N}_{11} = -\zeta_1$; $N_{12} = \overset{a}{N}_{12} = N_{21} = \overset{a}{N}_{21} = 0$; $N_{22} = \overset{a}{N}_{22} = -\zeta_2$ Therefore, our metric is Berwald-Cartan. Due to the coefficients form we deduce that it is Kähler-Cartan too, and together with the previous theorem this leads us to the strongly Kähler- Cartan property.

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