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# ON ALMOST CONTACT METRIC 2-HYPERSURFACES IN KÄHLERIAN MANIFOLDS

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Dedicated to Professor Vadim Feodorovich Kirichenko on his $70^{th}$ birthday

**Abstract**

It is proved that 2-hypersurfaces in a Kählerian manifold admit non-cosymplectic and non-Kenmotsu almost contact metric structures of cosymplectic type.

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*Key words:* almost contact metric structure, hypersurface, Kählerian manifold, type number, structure of cosymplectic type.

## 1 Introduction

The classification on first order differential-geometrical invariants of the almost Hermitian structures can be rightfully attributed to the most significant results obtained by the outstanding American mathematician Alfred Gray and his Spanish colleague Luis M. Hervella. In accordance with this classification, all the almost Hermitian structures are divided into sixteen classes. Analytical criteria for each concrete structure to belong to one or another class have been obtained [12].

The class of Kählerian structures is the most important and best studied Gray–Hervella class. It belongs to each of sixteen Gray–Hervella classes of almost Hermitian structures. That is why every result on the geometry of almost Hermitian manifolds of any class is also relevant to Kählerian manifolds. Note that Kählerian geometry is currently an intensively developing area of modern differential geometry. This topic has undoubtedly a rich inner content and close ties with other parts of geometry as well as various areas of modern theoretical physics.

This paper is devoted to the question about the form of an almost contact metric structure on a hypersurface in a Kählerian manifold. It is known that a

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cosymplectic structure is induced on a totally geodesic hypersurface (i.e. on a hypersurface with type number zero) in a Kählerian manifold. It is difficult to determine who was the first to establish such an important property. This fact immediately follows from the constructions of outstanding American geometer David E. Blair [11] as well as from Lidia V. Stepanova's works [19], [20] and from the papers of many authors. The problem of an almost contact metric structure on a hypersurface with type number 1 (i.e. 1-hypersurface) in a Kählerian manifold was studied in [7]. It turned out that the almost contact metric structure on a 1-hypersurface in a Kählerian manifold is also cosymplectic. Moreover, it was proved that the condition that the type number is at most one is necessary and sufficient for the almost contact metric structure on a hypersurface of a Kählerian manifold to be cosymplectic [7].

The goal of the present note is to consider almost contact metric structures on 2-hypersurfaces in a Kählerian manifold. It is essentially a more complicated problem. We remark also that the present article is a continuation of the authors study on almost contact metric hypersurfaces in almost Hermitian manifolds, mainly six-dimensional (see, for example, [2], [3], [5], [8], [9], [21]).

## 2 Preliminaries

As it is well known, an almost Hermitian ($AH$-) structure on an even-dimensional manifold $M^{2n}$ is a pair $(J,\, g)$ , where $J$ is an almost complex structure and $g = \langle \cdot,\, \cdot \rangle$ is a Riemannian metric $g = \langle \cdot,\, \cdot \rangle$ on this manifold. Moreover, the following condition must hold:

$$\langle JX,\ JY \rangle = \langle X,\ Y \rangle\,, \quad X, Y \in \aleph(M^{2n}),$$

where $\aleph(M^{2n})$ is the module of the smooth vector fields on $M^{2n}$ [12], [15]. A manifold with a fixed almost Hermitian structure is called an almost Hermitian ($AH$-) manifold. All considered manifolds, tensor fields and similar objects are assumed to be of the class $C^\infty$.

The specification of an almost Hermitian structure on a manifold is equivalent to the setting of a $G$-structure, where $G$ is the unitary group $U(n)$ [8], [15]. Its elements are the frames adapted to the structure (or $A$-frames). These frames look as follows:

$$(p,\ \varepsilon_1, \ldots, \varepsilon_n,\ \varepsilon_{\hat{1}}, \ldots, \varepsilon_{\hat{n}}),$$

where $\varepsilon_a$ are the eigenvectors that correspond to the eigenvalue $i = \sqrt{-1}$ , and $\varepsilon_{\hat{a}}$ are the eigenvectors that correspond to the eigenvalue $-i$ . Here the index $a$ ranges from 1 to $n$, and we state $\hat{a} = a + n$.

Therefore, the matrices of the operator of the almost complex structure and of the Riemannian metric written in an $A$-frame look as follows, respectively:

$$\left( J_j^k \right) = \left( \begin{array}{c|c} iI_n & 0 \\ \hline 0 & -iI_n \end{array} \right); \ \left( g_{kj} \right) = \left( \begin{array}{c|c} 0 & I_n \\ \hline I_n & 0 \end{array} \right),$$

where $I_n$ is the identity matrix; $k, j = 1, \ldots, 2n$.

We recall that the fundamental (or Kählerian) form of an almost Hermitian manifold is determined by the relation

$$F(X,\ Y) = \langle X,\ JY \rangle\,, \quad X, Y \in \aleph(M^{2n}).$$

By direct computing it is easy to obtain that in an A-frame the fundamental form matrix looks as follows:

$$\left( F_{kj} \right) = \left( \begin{array}{c|c} 0 & iI_n \\ \hline -iI_n & 0 \end{array} \right).$$

An almost Hermitian manifold is called Hermitian, if its structure is integrable. The following Gray–Hervella identity characterizes the Hermitian structure [12], [15]:

$$\nabla_X(F)(Y, Z) - \nabla_{JX}(F)(JY, Z) = 0,$$

where $X, Y, Z \in \aleph(M^{2n})$. A Kählerian structure must comply with the condition $\nabla F = 0$, where $\nabla$ is the Levi–Civita connection of the metric $g = \langle \cdot, \cdot \rangle$.

The first group of the Cartan structural equations of a Hermitian manifold written in an A-frame looks as follows:

$$d\omega^a = \omega_b^a \wedge \omega^b + B_c^{ab}\omega^c \wedge \omega_b,$$

$$d\omega_a = -\omega_\alpha^b \wedge \omega_b + B_{ab}^c \omega_c \wedge \omega^b,$$

where $\left\{ B^{ab}{}_c \right\}$ and $\{ B_{ab}{}^c \}$ are the components of the Kirichenko tensors of $M^{2n}$ [1], [4]; $a, b, c = 1, ..., n$.

Let $N$ be an oriented hypersurface in a Hermitian manifold $M^{2n}$ and let $\sigma$ be the second fundamental form of the immersion of $N$ into $M^{2n}$. As it is well-known [2], [3], [5], [10], the almost Hermitian structure on $M^{2n}$ induces an almost contact metric structure on $N$. We recall also that an almost contact metric structure on an odd-dimensional manifold $N$ is defined by the system of tensor fields $\{\Phi,\ \xi,\ \eta,\ g\}$ on this manifold, where $\xi$ is a vector field, $\eta$ is a covector field, $\Phi$ is a tensor of the type (1, 1) and $\langle \cdot, \cdot \rangle$ is the Riemannian metric [11], [15], [18]. Moreover, the following conditions are fulfilled:

$$\eta(\xi) = 1,\ \Phi(\xi) = 0,\ \eta \circ \Phi = 0,\ \Phi^2 = -id + \xi \otimes \eta,$$

$$\langle \Phi X, \Phi Y \rangle = \langle X, Y \rangle - \eta(X)\eta(Y), \\ X, Y \in \aleph(N),$$

where $\aleph(N)$ is the module of the smooth vector fields on $N$. As an example of an almost contact metric structure we can consider the cosymplectic structure that is characterized by the following conditions [11]:

$$\nabla\eta = 0,\ \nabla\Phi = 0.$$

It has been proved that the manifold, admitting the cosymplectic structure, is locally equivalent to the product $M \times R$, where $M$ is a Kählerian manifold [14]. We note that the cosymplectic structures have many remarkable properties and play a fundamental role in contact geometry and mathematical physics [13], [15].

As it was mentioned, the almost contact metric structures are closely connected to the almost Hermitian structures. For instance, if $(N, \{\Phi, \xi, \eta, g\})$ is an almost contact metric manifold, then an almost Hermitian structure is induced on the product $N \times R$ [11], [15]. If this almost Hermitian structure is integrable, then the input almost contact metric structure is called normal. As it is known, a normal contact metric structure is called Sasakian [15]. On the other hand, we can characterize the Sasakian structure by the following condition:

$$\nabla_X(\Phi)Y = \langle X, Y \rangle\,\xi - \eta(Y)X,\ X, Y \in \aleph(N). \tag{1}$$

For example, Sasakian structures are induced on totally umbilical hypersurfaces in a K ahlerian manifold [15]. As it is well known, the Sasakian structures have also many important properties. In 1972 Katsuei Kenmotsu has introduced a new class of almost contact metric structures, defined by the condition

$$\nabla_X(\Phi)Y = \langle \Phi X, Y \rangle\,\xi - \eta(Y)\Phi X,\ X, Y \in \aleph(N). \tag{2}$$

The Kenmotsu manifolds are normal and integrable, but they are not contact manifolds, consequently, they can not be Sasakian. In spite of the fact that conditions (1) and (2) are similar, the properties of Kenmotsu manifolds are to some extent antipodal to the Sasakian manifolds properties [15]. We remark that the fundamental monograph by Gh. Pitiş contains a detailed description of Kenmotsu manifolds as well as a collection of examples of such manifolds [18].

At the end of this section, note that when we give a Riemannian manifold and its submanifold (in particular, its hypersurface), the rank of determined second fundamental form is called the type number [17].

## 3 The main results

**Theorem 1.** *The Cartan structural equations of the almost contact metric structure induced on a hypersurface in a Kählerian manifold are the following:*

$$\begin{aligned} d\omega^\alpha &= \omega^\alpha_\beta \wedge \omega^\beta + i\sigma^\alpha_\beta \omega^\beta \wedge \omega + i\sigma^{\alpha\beta}\omega_\beta \wedge \omega; \\ d\omega_\alpha &= -\omega^\beta_\alpha \wedge \omega_\beta - i\sigma^\beta_\alpha \omega_\beta \wedge \omega - i\sigma_{\alpha\beta}\omega^\beta \wedge \omega; \\ d\omega &= -2i\sigma^\alpha_\beta\,\omega^\beta \wedge \omega_\alpha + i\sigma_{3\beta}\omega \wedge \omega^\beta - i\sigma^\beta_3 \omega \wedge \omega_\beta. \end{aligned} \tag{3}$$

*Proof.* Let us use the Cartan structural equations of the almost contact metric structure on a hypersurface $N^{2n-1}$ in a Hermitian manifold $M^{2n}$ [19], [20]:

$$d\omega^{\alpha} = \omega^{\alpha}_{\beta} \wedge \omega^{\beta} + B^{\alpha\beta}_{\gamma}\omega^{\gamma} \wedge \omega_{\beta} + \left(\sqrt{2}B^{\alpha 3}_{\beta} + i\sigma^{\alpha}_{\beta}\right)\omega^{\beta} \wedge \omega +$$
$$+ \left(-\frac{1}{\sqrt{2}}B^{\alpha\beta}{}_{3} + i\sigma^{\alpha\beta}\right)\omega_{\beta} \wedge \omega;$$

$$d\omega_{\alpha} = -\omega^{\beta}_{\alpha} \wedge \omega_{\beta} + B^{\gamma}_{\alpha\beta}\omega_{\gamma} \wedge \omega^{\beta} + \left(\sqrt{2}B_{\alpha 3}{}^{\beta} - i\sigma^{\beta}_{\alpha}\right)\omega_{\beta} \wedge \omega +$$
$$+ \left(-\frac{1}{\sqrt{2}}B_{\alpha\beta}{}^{3} - i\sigma_{\alpha\beta}\right)\omega^{\beta} \wedge \omega;$$

$$d\omega = \left(\sqrt{2}B^{3\alpha}{}_{\beta} - \sqrt{2}B_{3\beta}{}^{\alpha} - 2i\sigma^{\alpha}_{\beta}\right)\omega^{\beta} \wedge \omega_{\alpha} + \left(B_{3\beta}{}^{3} + i\sigma_{3\beta}\right)\omega \wedge \omega^{\beta} +$$
$$+ \left(B^{3\beta}{}_{3} - i\sigma^{\beta}_{3}\right)\omega \wedge \omega_{\beta},$$

where

$$B^{ab}{}_{c} = -\frac{i}{2}J^{a}_{\hat{b},c}, \quad B_{ab}{}^{c} = \frac{i}{2}J^{\hat{a}}_{b,\hat{c}}.$$

We note that $\left\{J^{j}_{k,m}\right\}$ are the components of $\nabla J$ . We also note that the systems of functions $\left\{B^{ab}{}_{c}\right\}$ and $\left\{B_{ab}{}^{c}\right\}$ are the components of the virtual Kirichenko tensors [1] of the AH-structure on the manifold $M^{2n}$. Here $\alpha, \beta, \gamma = 1, \ldots, n-1$; $a, b, c = 1, \ldots, n$; $\hat{a} = a + n$; $\sigma$ is the second fundamental form of the immersion of the hypersurface $N^{2n-1}$ into $M^{2n}$.

It is known that the Hermitian manifold is Kählerian if and only if its Kirichenko tensors vanish [4], [6]. That is why we obtain the structural equations (3). □

Suppose that the type number of a hypersurface in a Kählerian manifold is equal to 2, i.e. the rank of the matrix $(\sigma_{ps})$ is equal to 2. The simplest matrix $(\sigma_{ps})$ of rank two is the following:

$$(\sigma_{ps}) = \left(\begin{array}{c|c|c}
\begin{pmatrix} \sigma_{11} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix} & \begin{matrix} 0 \\ \dots \\ 0 \end{matrix} & 0 \\
\hline
0 \dots 0 & 0 & 0 \dots 0 \\
\hline
0 & \begin{matrix} 0 \\ \dots \\ 0 \end{matrix} & \begin{pmatrix} \sigma_{\hat{1}\hat{1}} & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix}
\end{array}\right),$$

$$p, s = 1, \ldots, 2n-1.$$

Evidently $\sigma_{11} \neq 0$, $\sigma_{\hat{1}\hat{1}} \neq 0$, otherwise the type number is zero. In this case we can rewrite the structural equations (3) as follows:

$$\begin{aligned} &d\omega^1 = \omega^1_\beta \wedge \omega^\beta + i\sigma^{11}\omega_1 \wedge \omega, \\ &d\omega^2 = \omega^2_\beta \wedge \omega^\beta, \\ &d\omega_1 = -\omega^\beta_1 \wedge \omega_\beta - i\sigma_{11}\omega^1 \wedge \omega, \\ &d\omega_2 = -\omega^\beta_2 \wedge \omega_\beta, \\ &d\omega = 0. \end{aligned} \tag{4}$$

Considering the structural equations (4), we can find the kind of the almost contact metric structure induced on a 2-hypersurface in a Kählerian manifold. In [16] V.F. Kirichenko and I.V. Uskorev have introduced a new class of almost contact metric structure. Namely, they have defined the almost contact metric structure with the close contact form as the structures of *cosymplectic type*. The condition

$$d\omega = 0$$

is necessary and sufficient for an almost contact metric structure to be of cosymplectic type [16].

Evidently, a trivial example of structure of cosymplectic type is the cosymplectic structure and another example is the Kenmotsu structure. But the paper by V.F. Kirichenko and I.V. Uskorev does not contain examples of non-cosymplectic and non-Kenmotsu structure of cosymplectic type. So we obtain the following result.

**Theorem 2.** *2-hypersurfaces in a Kählerian manifold admit almost contact metric structures of cosymplectic type.*

Let us compare the structural equations (4) with the Cartan equations of the most important classes of almost contact metric structures [15], [20]:

1. Cosymplectic structure:

$$\begin{aligned} &d\omega^\alpha = \omega^\alpha_\beta \wedge \omega^\beta, \\ &d\omega_\alpha = -\omega^\beta_\alpha \wedge \omega_\beta, \\ &d\omega = 0. \end{aligned} \tag{5}$$

2. Sasakian structure:

$$\begin{aligned} &d\omega^\alpha = \omega^\alpha_\beta \wedge \omega^\beta - i\omega \wedge \omega^\alpha, \\ &d\omega_\alpha = -\omega^\beta_\alpha \wedge \omega_\beta + i\omega \wedge \omega_\alpha, \\ &d\omega = -2i\omega^\alpha \wedge \omega_\alpha. \end{aligned} \tag{6}$$

3. Kenmotsu structure:

$$\begin{aligned} &d\omega^\alpha = \omega^\alpha_\beta \wedge \omega^\beta + \omega \wedge \omega^\alpha, \\ &d\omega_\alpha = -\omega^\beta_\alpha \wedge \omega_\beta + \omega \wedge \omega_\alpha, \\ &d\omega = 0. \end{aligned} \tag{7}$$

4. Quasi-Sasakian structure:

$$\begin{aligned} d\omega^\alpha &= \omega^\alpha_\beta \wedge \omega^\beta + B^\alpha_\beta \omega^\beta \wedge \omega, \\ d\omega_\alpha &= -\omega^\beta_\alpha \wedge \omega_\beta - B^\beta_\alpha \omega_\beta \wedge \omega, \\ d\omega &= 2\, B^\alpha_\beta\, \omega^\beta \wedge \omega_\alpha. \end{aligned} \tag{8}$$

It is easy to see that the structural equations (4) do not determine an almost contact metric structure of these kinds. We only remark that the fact that the almost contact metric structure on a 2-hypersurface in a Kählerian manifold is not cosymplectic follows from the above mentioned results [7]. So, we have proved the following statement.

**Theorem 3.** 2-*hypersurfaces in a Kählerian manifold admit non-Sasakian and non-Kenmotsu almost contact metric structures.*

We recall also that a conformal transformation of an almost contact metric structure $\{\Phi, \xi, \eta, g\}$ on the manifold $N$ is a transition to the almost contact metric structure $\{\tilde{\Phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g}\}$, where $\tilde{\Phi} = \Phi$, $\tilde{\xi} = e^f \xi$, $\tilde{\eta} = e^{-f}\eta$ and $\tilde{g} = e^{-2f} g$. Here $f$ is a smooth function on the manifold $N$ [11], [16].

Applying the result that the structure of cosymplectic type is invariant under canonical conformal transformations [16], we obtain the following property of the almost contact metric structure defined by (4).

**Corollary 1.** *The almost contact metric structure determined by (4) on* 2-*hypersurface of a Kählerian manifold is invariant under canonical conformal transformations.*

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