Bulletin of the *Transilvania* University of Braşov • Vol 10(59), No. 2 - 2017
Series III: Mathematics, Informatics, Physics, 129-142

INTERIOR INVERSE PROBLEM FOR STURM-LIOUVILLE OPERATOR WITH EIGENPARAMETER DEPENDENT BOUNDARY CONDITIONS

Murat SAT[1]

Abstract

In this paper, the inverse spectral problem for Sturm-Liouville operator with eigenparameter dependent boundary conditions is studied and also uniqueness theorems are proved.

2000 *Mathematics Subject Classification:* 34B24; 34L05; 47E05.
Key words: inverse problem, internal point, eigenfunction, eigenvalue.

1 Introduction

The greatest success in spectral theory in general, in inverse spectral problems has been achieved for the Sturm-Liouville operator $Ly = -y'' + q(x)y$ which is also called the one dimensional Schrödinger operator. The first studies on the spectral theory of such operators were performed (see Refs. $[1-5]$).

Fulton (see Refs. [10]) studied the Sturm-Liouville problem with common conditions and got the eigenfunction expansion and asymptotic estimates of eigenvalues. Binding, Browne and Seddighi (see Refs. [7]) considered the Sturm-Liouville operator l satisfying

$$ly \equiv -\left(py'\right)' + qy = \lambda ry$$

with boundary conditions dependent on the spectral parameter. They obtained oscillation and comparison results as well as the asymptotic estimates of eigenvalues, which can be considered as an extension of Fulton's results. Browne and Sleeman (see Refs. [8]) searched the inverse nodal problem for the Sturm-Liouville problem with common conditions and showed that a dense set of nodal points of eigenfunctions for this problem is sufficient to determine the potential $q(x)$ and coefficient h of the boundary condition. Guliyev [12] discussed the regularized trace problem for the Sturm-Liouville equation with spectral parameter in the boundary conditions and obtained the trace formulae. Recently, operators

[1]Department of Mathematics, Faculty of Science and Art, Erzincan University, Erzincan, 24100, Turkey, e-mail: murat_sat24@hotmail.com

with boundary conditions dependent on the spectral parameter have been studied by authors (see Refs. $[6-18]$).

Mochizuki and Trooshin (see Refs. $[19, 20]$) discussed the inverse problem for interior spectral data of the Sturm-Liouville and Dirac operator and showed that the potential function can be uniquely determined by a set of eigenvalues of eigenfunctions at some internal point and one spectrum. Using Mochizuki and Trooshin's method, Yang (see Refs. $[21-24]$) investigated the interior inverse problem for the Sturm-Liouville operator with discontinuous conditions and with eigenparameter-dependent boundary conditions, differential pencils and Dirac operator with eigenparameter-dependent boundary conditions on the finite interval $[a, b]$. However, we are motivated by interior inverse spectral problems for Sturm Liouville operators with spectral parameter in boundary conditions. Interior inverse problems for operators have been studied recently by the authors (see Refs. $[25-30]$).

As far as we know, interior inverse spectral problems for Sturm Liouville operators with spectral parameter in boundary conditions have not been considered yet.

The main goal of the present work is to study the inverse problem of reconstructing the Sturm-Liouville with eigenparameter dependent boundary conditions on the basis of spectral data of a kind: one spectrum and some information on eigenfunctions at the internal point. The techniques used here will be adopted from (see Refs. $[19, 31, 32]$).

Consider the following Sturm-Liouville operator with eigenparameter dependent boundary conditions L satisfying

$$Ly \equiv -y'' + q(x)y = \lambda y, \quad x \in [0,1] \tag{1.1}$$

with boundary conditions,

$$y(0,\lambda)\cos\alpha + y'(0,\lambda)\sin\alpha = 0, \ \alpha \neq 0 \tag{1.2}$$

$$(z\lambda + e)\, y(1,\lambda) - (c\lambda + d)\, y'(1,\lambda) = 0, \tag{1.3}$$

where $zd - ec > 0$, $c \neq 0$, $q(x)$ (potential function) is a real-valued function, $q(x) \in L_2(0,1)$ and λ spectral parameter. The operator L is self adjoint on the $L_2(0,1)$ and has a discrete spectrum $\{\lambda_n\}$.

Let us introduce the second perturbed Sturm-Liouville operator with eigenparameter dependent boundary conditions $\widetilde{L}$ satisfying

$$\widetilde{L}\widetilde{y} \equiv -\widetilde{y}'' + \widetilde{q}(x)\widetilde{y} = \lambda\widetilde{y}, \quad x \in [0,1] \tag{1.4}$$

$$\widetilde{y}(0,\lambda)\cos\widetilde{\alpha} + \widetilde{y}'(0,\lambda)\sin\widetilde{\alpha} = 0, \ \widetilde{\alpha} \neq 0 \tag{1.5}$$

$$(\widetilde{z}\lambda + \widetilde{e})\, y(1,\lambda) - \left(\widetilde{c}\lambda + \widetilde{d}\right) y'(1,\lambda) = 0, \tag{1.6}$$

where $\widetilde{q}(x)$ is a real-valued function and $\widetilde{q}(x) \in L_2(0,1)$. The operator $\widetilde{L}$ is self adjoint on the $L_2(0,1)$ and has a discrete spectrum $\left\{\widetilde{\lambda}_n\right\}$.

2 Main results

The spectrum of the (1.1)-(1.3) consist of eigenvalue λ_n, $n \in \mathbb{N}$, and the sequence $\{\lambda_n, n \in \mathbb{N}\}$ satisfies asymptotic formula (see Refs. [7, 32]) :

$$\lambda_n = (n-1)^2\,\pi^2 - 2\frac{z}{c} - 2\cot\alpha + \int\limits_0^1 q(x)dx + o\left(\frac{1}{n}\right). \tag{2.1}$$

Let $y(x,\lambda)$ be the solution of equation $Ly = \lambda y$ satisfying the initial conditions $y(0,\lambda) = \sin\alpha$ and $y'(0,\lambda) = -\cos\alpha$, then

$$y\,(x,\lambda) = \sin\alpha\left[\cos\left(\sqrt{\lambda}x\right) + \int\limits_0^x K\,(x,t)\cos\left(\sqrt{\lambda}t\right)dt\right],$$

where the kernel $K\,(x,t)$ is the solution of the following equation

$$\frac{\partial^2 K\,(x,t)}{\partial x^2} - \frac{\partial^2 K\,(x,t)}{\partial t^2} - q(x)K\,(x,t) = 0.$$

When $b = \frac{1}{2}$, we get the following uniqueness Theorem 2.1.

Theorem 2.1. *If for every $n \in \mathbb{N}$ and $\frac{e}{c} = \frac{\widetilde{e}}{\widetilde{c}}$ and $\frac{d}{c} = \frac{\widetilde{d}}{\widetilde{c}}$, we have*

$$\lambda_n = \widetilde{\lambda}_n \text{ and } \frac{y_n'(\frac{1}{2},\lambda_n)}{y_n(\frac{1}{2},\lambda_n)} = \frac{\widetilde{y}_n'(\frac{1}{2},\widetilde{\lambda}_n)}{\widetilde{y}_n(\frac{1}{2},\widetilde{\lambda}_n)}, \tag{2.2}$$

then

$$q(x) = \widetilde{q}(x) \text{ a.e on the interval } x \in [0,1]\,,$$

and

$$\alpha = \widetilde{\alpha} \text{ and } \frac{z}{c} = \frac{\widetilde{z}}{\widetilde{c}}.$$

In the case $b \neq \frac{1}{2}$, the uniqueness theorem of $q(x)$ can be proved if we require the knowledge of a part of the second spectrum.

Let $\{m(n)\}$ be a sequence of natural numbers

$$m(n) = \frac{n}{\sigma}(1+\varepsilon_n),\ 0 < \sigma \leq 1, \varepsilon_n \to 0. \tag{2.3}$$

Lemma 2.1. (1) *Let $\{m(n)\}$ be a sequence of natural numbers satisfying (2.3) and $b \in (0, \frac{1}{2})$ are so chosen that $\sigma > 2b$. If for any $n \in \mathbb{N}$ and $\frac{e}{c} = \frac{\widetilde{e}}{\widetilde{c}}$,*

$$\lambda_{m(n)} = \widetilde{\lambda}_{m(n)},\ \frac{y_{m(n)}'(b,\lambda_{m(n)})}{y_{m(n)}(b,\lambda_{m(n)})} = \frac{\widetilde{y}_{m(n)}'(b,\widetilde{\lambda}_{m(n)})}{\widetilde{y}_{m(n)}(b,\widetilde{\lambda}_{m(n)})} \tag{2.4}$$

then

$$q(x) = \widetilde{q}(x) \text{ a.e on the interval } [0,b]\,,$$

and

$$\alpha = \widetilde{\alpha}.$$

(2) *Let* $\{m(n)\}$ *be a sequence of natural numbers satisfying* (2.3) *and* $b \in (\frac{1}{2}, 1)$ *are so chosen that* $\sigma > 2 - 2b$. *If for any* $n \in \mathbb{N}$ *and* $\frac{d}{c} = \frac{\widetilde{d}}{\widetilde{c}}$,

$$\lambda_{m(n)} = \widetilde{\lambda}_{m(n)},\ \frac{y'_{m(n)}(b, \lambda_{m(n)})}{y_{m(n)}(b, \lambda_{m(n)})} = \frac{\widetilde{y}'_{m(n)}(b, \widetilde{\lambda}_{m(n)})}{\widetilde{y}_{m(n)}(b, \widetilde{\lambda}_{m(n)})} \tag{2.5}$$

then

$$q(x) = \widetilde{q}(x) \text{ a.e on the interval } [b, 1],$$

and

$$\frac{z}{c} = \frac{\widetilde{z}}{\widetilde{c}}.$$

Let $\{l(n)\}_{n\in\mathbb{N}}$ and $\{r(n)\}_{n\in\mathbb{N}}$ be a sequence of natural numbers such that

$$l(n) = \frac{n}{\sigma_1}(1 + \varepsilon_{1,n}),\ 0 < \sigma_1 \le 1, \varepsilon_{1,n} \to 0, \tag{2.6}$$

$$r(n) = \frac{n}{\sigma_2}(1 + \varepsilon_{2,n}),\ 0 < \sigma_2 \le 1, \varepsilon_{2,n} \to 0, \tag{2.7}$$

and let μ_n be the eigenvalues of the problem (1.1), (1.2) and (2.8) and $\widetilde{\mu}_n$ be the eigenvalues of the problem (1.4), (1.5) and (2.8)

$$(a_1\lambda + e_1)\, y(1, \lambda) - (c_1\lambda + d_1)\, y'(1, \lambda) = 0, \tag{2.8}$$

where $a_1 d_1 - e_1 c_1 > 0$, $c_1 \neq 0$.

Using Mochizuki and Trooshin's method from Theorem 2.1 and Lemma 2.1, we will prove that the following Theorem 2.2 holds.

Theorem 2.2. *Let* $\{l(n)\}$ *and* $\{r(n)\}$ *be a sequence of natural numbers satisfying (2.6) and (2.7), and* $b \in \left(\frac{1}{2}, 1\right)$ *are so chosen that* $\sigma_1 > 2b - 1, \sigma_2 > 2 - 2b$. *If for any* $n \in \mathbb{N}, \frac{d}{c} = \frac{\widetilde{d}}{\widetilde{c}}$ *and* $\frac{e}{c} = \frac{\widetilde{e}}{\widetilde{c}}$, *we have*

$$\lambda_n = \widetilde{\lambda}_n,\ \mu_{l(n)} = \widetilde{\mu}_{l(n)} \text{ and } \frac{y'_{r(n)}(b, \lambda_{r(n)})}{y_{r(n)}(b, \lambda_{r(n)})} = \frac{\widetilde{y}'_{r(n)}(b, \widetilde{\lambda}_{r(n)})}{\widetilde{y}_{r(n)}(b, \widetilde{\lambda}_{r(n)})}, \tag{2.9}$$

then

$$q(x) = \widetilde{q}(x) \text{ a.e on the interval } [0, 1]$$

and

$$\alpha = \widetilde{\alpha} \text{ and } \frac{z}{c} = \frac{\widetilde{z}}{\widetilde{c}}.$$

3 Proof of the main results

Let $y(x,\lambda)$ be the solution to equation

$$-y''(x)+q(x)y(x)=\lambda y(x) \tag{3.1}$$

with the initial conditions $y(0,\lambda)=\sin\alpha$ and $y'(0,\lambda)=-\cos\alpha$, then we get (see Refs. [32]),

$$\begin{aligned} y(x,\lambda) &= \sin\alpha\left[\cos\left(\sqrt{\lambda}x\right)+\int_0^x K(x,t)\cos\left(\sqrt{\lambda}t\right)dt\right] \\ &= \sin\alpha\cos\left(\sqrt{\lambda}x\right)+O\left(\frac{e^{\tau x}}{\sqrt{\lambda}}\right), \end{aligned} \tag{3.2}$$

where $\tau=\left|\operatorname{Im}\sqrt{\lambda}\right|$.

Moreover, we have

$$\begin{aligned} y'(x,\lambda) &= \sin\alpha\left[-\sqrt{\lambda}\sin\sqrt{\lambda}x+K(x,x)\cos\left(\sqrt{\lambda}x\right)+\int_0^x K'(x,t)\cos\left(\sqrt{\lambda}t\right)dt\right] \\ &= \left(-\sqrt{\lambda}\sin\alpha\right)\sin\left(\sqrt{\lambda}x\right)+O\left(e^{\tau x}\right). \end{aligned}$$

Simple calculations show that the characteristics equation of problem (1.1) can be reduced to $\varphi(\lambda)=(z\lambda+e)\,y(1,\lambda)-(c\lambda+d)\,y'(1,\lambda)=0$, where

$$\varphi(\lambda)=c\sin z\lambda\sqrt{\lambda}\sin\sqrt{\lambda}+O\left(\lambda e^{\tau}\right). \tag{3.3}$$

Similarly, for the solution $\widetilde{y}(x,\lambda)$ of equation

$$-\widetilde{y}''(x)+\widetilde{q}(x)\widetilde{y}(x)=\lambda\widetilde{y}(x) \tag{3.4}$$

with the initial conditions $\widetilde{y}(0,\lambda)=\sin\widetilde{\alpha}$ and $\widetilde{y}'(0,\lambda)=-\cos\widetilde{\alpha}$, we have the following analogous results:

$$\begin{aligned} \widetilde{y}(x,\lambda) &= \sin\widetilde{\alpha}\left[\cos\left(\sqrt{\lambda}x\right)+\int_0^x \widetilde{K}(x,t)\cos\left(\sqrt{\lambda}t\right)dt\right], \\ &= \sin\widetilde{\alpha}\cos\left(\sqrt{\lambda}x\right)+O\left(\frac{e^{\tau x}}{\sqrt{\lambda}}\right), \end{aligned} \tag{3.5}$$

and

$$\widetilde{y}'(x,\lambda)=\left(-\sqrt{\lambda}\sin\widetilde{\alpha}\right)\sin\left(\sqrt{\lambda}x\right)+O\left(e^{\tau x}\right).$$

The characteristic equation of problem (1.4)-(1.6) can be reduced $\widetilde{\varphi}(\lambda)=(\widetilde{z}\lambda+\widetilde{e})\,\widetilde{y}(1,\lambda)-\left(\widetilde{c}\lambda+\widetilde{d}\right)\widetilde{y}'(1,\lambda)=0$, where

$$\widetilde{\varphi}(\lambda)=\widetilde{c}\sin\widetilde{z}\lambda\sqrt{\lambda}\sin\sqrt{\lambda}+O\left(\lambda e^{\tau}\right).$$

The eigenvalue set $\{\lambda_n, n\in\mathbb{N}\}$ of L coincides with zeros of $\varphi(\lambda)$.

Next, using equations (3.2) and (3.5), we obtain that

$$y(x,\lambda)\,\widetilde{y}(x,\lambda)=\frac{\sin\alpha\sin\widetilde{\alpha}}{2}\left[1+\cos\left(2\sqrt{\lambda}x\right)+\int_0^x\widehat{K}(x,\tau)\cos\left(2\sqrt{\lambda}\tau\right)d\tau\right] \tag{3.6}$$

where

$$\begin{aligned}\frac{1}{2}\widehat{K}(x,\tau) &= K(x,x-2\tau)+\widetilde{K}(x,x-2\tau)\\ &+\int_{-x+2\tau}^{x}K(x,s)\,\widetilde{K}(x,s-2\tau)\,ds\\ &+\int_{-x}^{x-2\tau}K(x,s)\,\widetilde{K}(x,s+2\tau)\,ds.\end{aligned}$$

Proof of Theorem 2.1 If we multiply (3.1) by $\widetilde{y}(x,\lambda)$ and (3.4) by $y(x,\lambda)$ respectively, then subtract them and after integrating on $\left[0,\frac{1}{2}\right]$, we obtain

$$\left[\widetilde{y}(x,\lambda)y'(x,\lambda)-y(x,\lambda)\widetilde{y}'(x,\lambda)\right]\Big|_0^{\frac{1}{2}}+\int_0^{\frac{1}{2}}\left[\widetilde{q}(x)-q(x)\right]y(x,\lambda)\,\widetilde{y}(x,\lambda)dx=0. \tag{3.7}$$

Together with the initial conditions at 0, then it yields

$$\begin{aligned}&\left[\widetilde{y}(\tfrac{1}{2},\lambda)y'(\tfrac{1}{2},\lambda)-y(\tfrac{1}{2},\lambda)\widetilde{y}'(\tfrac{1}{2},\lambda)\right]+\sin(\widetilde{\alpha}-\alpha)\\ &+\int_0^{\frac{1}{2}}\left[\widetilde{q}(x)-q(x)\right]y(x,\lambda)\,\widetilde{y}(x,\lambda)dx=0.\end{aligned} \tag{3.8}$$

Denote

$$Q(x)=\widetilde{q}(x)-q(x). \tag{3.9}$$

For $\lambda=\lambda_n$, by given assumption it follows that

$$\left[\widetilde{y}_n(\tfrac{1}{2},\widetilde{\lambda}_n)y_n'(\tfrac{1}{2},\lambda_n)-y_n(\tfrac{1}{2},\lambda_n)\widetilde{y}_n'(\tfrac{1}{2},\widetilde{\lambda}_n)\right]=0.$$

If we substitute (3.6) into (3.8), then we have

$$\begin{aligned}0 &= -2(\cot\widetilde{\alpha}-\cot\alpha)+\int_0^{\frac{1}{2}}Q(x)\,dx+\int_0^{\frac{1}{2}}Q(x)\cos\left(2\sqrt{\lambda}x\right)dx\\ &+\int_0^{\frac{1}{2}}Q(x)\left[\int_0^x\widehat{K}(x,\tau)\cos\left(2\sqrt{\lambda}\tau\right)d\tau\right]dx.\end{aligned} \tag{3.10}$$

Before we continue the proof of this theorem, we should write the following Lemma.

Lemma 3.1. *Let* (a,b) *be a finite interval and* $f(x) \in L(a,b)$. *Then*

$$\lim_{|\lambda|\to\infty} \int_a^b f(x)\cos\lambda x dx = 0.$$

This Lemma is well known and are usually called the Riemann-Lebesgue Lemmas. (see Refs. [34]).

By using of the Riemann-Lebesque Lemma letting $\lambda = \lambda_n \to +\infty$ in the equation (3.10), then we obtain

$$-2\left(\cot\widetilde{\alpha} - \cot\alpha\right) + \int_0^{\frac{1}{2}} Q(x)\,dx = 0. \tag{3.11}$$

Denote

$$H(\lambda) = \int_0^{\frac{1}{2}} Q(x)\cos\left(2\sqrt{\lambda}x\right)dx + \int_0^{\frac{1}{2}} Q(x)\left[\int_0^x \widehat{K}(x,\tau)\cos\left(2\sqrt{\lambda}\tau\right)d\tau\right]dx. \tag{3.12}$$

By given assumption, (3.10) and (3.11) we have

$$H(\lambda_n) = 0.$$

From (3.12), we have for all complex λ

$$|H(\lambda)| \leq C_1 e^{\tau}, \tag{3.13}$$

for some positive constant C_1. Define

$$\Phi(\lambda) = \frac{H(\lambda)}{\varphi(\lambda)}, \tag{3.14}$$

which is the entire function from the above arguments and it follows from (3.3) and (3.13) that

$$\Phi(\lambda) = O(1)$$

for $|\lambda|$ large enough. Thus, by Liouville's theorem, we obtain for all λ

$$\Phi(\lambda) = C.$$

where C is constant.

Let us show that the constant $C=0$. We can write the equation $H(\lambda) = C\varphi(\lambda)$ in the form

$$\begin{aligned}&\int_0^{\frac{1}{2}} Q(x)\cos\left(2\sqrt{\lambda}x\right)dx + \int_0^{\frac{1}{2}} Q(x)\left[\int_0^x \widehat{K}(x,\tau)\cos\left(2\sqrt{\lambda}\tau\right)d\tau\right]dx\\ =\;& C\left[c\sin\alpha\lambda\sqrt{\lambda}\sin\sqrt{\lambda} + O\left(\lambda e^{\tau}\right)\right]\end{aligned}$$

that is

$$\begin{aligned}&\frac{1}{c\sin\alpha\lambda\sqrt{\lambda}}\left\{\int_0^{\frac{1}{2}} Q(x)\cos\left(2\sqrt{\lambda}x\right)dx\right.\\ &\left.+\int_0^{\frac{1}{2}} Q(x)\left[\int_0^x \widehat{K}(x,\tau)\cos\left(2\sqrt{\lambda}\tau\right)d\tau\right]dx\right\}\\ =\;& C\left[\sin\sqrt{\lambda} + O\left(\frac{1}{\sqrt{\lambda}}e^{\tau}\right)\right]\end{aligned}$$

Since the limit of the left side of the above equality is zero as real $\lambda\to\infty$, we obtain that $C=0$. Thus

$$H(\lambda) = 0 \text{ for all } \lambda.$$

Using the change of variables, after some reordering, it can be written as

$$\int_0^{\frac{1}{2}} \cos\left(2\sqrt{\lambda}\tau\right)\left[Q(\tau) + \int_\tau^{\frac{1}{2}} Q(x)\widehat{K}(x,\tau)\,d\tau\right]dx = 0 \tag{3.15}$$

which implies that from the completeness of the functions $\cos\left(2\sqrt{\lambda}\tau\right)$, $\lambda\in R$ in $L_2\left(0,\frac{1}{2}\right)$

$$Q(\tau) + \int_\tau^{\frac{1}{2}} Q(x)\widehat{K}(x,\tau)\,dx = 0,\; 0<\tau<\frac{1}{2}. \tag{3.16}$$

But this equation is a homogeneous Volterra integral equation and has only the zero solution. Thus $Q(x)=0$ for almost everywhere on $0<x<\frac{1}{2}$, that is $q(x)=\widetilde{q}(x)$ for almost everywhere $x\in\left[0,\frac{1}{2}\right]$.

From (3.11) and $Q(x)=0$, the equality $\alpha=\widetilde{\alpha}$.

We can prove $q(x)=\widetilde{q}(x)$ almost everywhere on $\left[\frac{1}{2},1\right]$ similarly. Then we obtain $q(x)=\widetilde{q}(x)$ almost everywhere on $\left[\frac{1}{2},1\right]$ and $\frac{z}{c}=\frac{\widetilde{z}}{\widetilde{c}}$.

Therefore, Theorem 2.1 is proved.

Next, we show that Lemma 2.1 holds.

Proof of Lemma 2.1. (1) Let $y\,(x,\lambda)$ be the solution to

$$-y''(x)+q(x)y(x)=\lambda y(x) \tag{3.17}$$

with the initial conditions $y(0,\lambda)=\sin\alpha$ and $y'(0,\lambda)=-\cos\alpha$. Similarly let $\widetilde{y}\,(x,\lambda)$ be the solution of

$$-\widetilde{y}''(x)+\widetilde{q}(x)\widetilde{y}(x)=\lambda\widetilde{y}(x) \tag{3.18}$$

with the initial conditions $\widetilde{y}(0,\lambda)=\sin\widetilde{\alpha}$ and $\widetilde{y}'(0,\lambda)=-\cos\widetilde{\alpha}$.

If we multiply (3.17) by $\widetilde{y}\,(x,\lambda)$ and (3.18) by $y\,(x,\lambda)$, and subtract after integrating on $[0,b]$, we obtain

$$\begin{aligned} G(\lambda) &= \int_0^b \left[\widetilde{q}\,(x)-q\,(x)\right] y\,(x,\lambda)\,\widetilde{y}(x,\lambda)dx+\sin\left(\widetilde{\alpha}-\alpha\right) \\ &= \left[y(x,\lambda)\widetilde{y}'(x,\lambda)-\widetilde{y}(x,\lambda)y'(x,\lambda)\right]|_{x=b} \end{aligned} \tag{3.19}$$

From the assumption

$$\frac{y'_{m(n)}(b,\lambda_{m(n)})}{y_{m(n)}(b,\lambda_{m(n)})}=\frac{\widetilde{y}'_{m(n)}(b,\lambda_{m(n)})}{\widetilde{y}_{m(n)}(b,\lambda_{m(n)})},$$

it follows that,

$$G(\lambda_{m(n)})=0,\ \ n\in\mathbb{N}.$$

Next, we will show that $G(\lambda)=0$ on the whole complex plane.

From (3.19) we see that the entire function $G(\lambda)$ is a function of exponential type $\le 2b$. One has

$$|G\,(\lambda)|\le C_2e^{2br|\sin\theta|} \tag{3.20}$$

for some positive constant C_2.

Let us define the indicator of function $G(\lambda)$ by the formula

$$h(\theta)=\lim_{r\to\infty}\sup\frac{\ln\left|G(re^{i\theta})\right|}{r}. \tag{3.21}$$

Since $|Im\lambda|=r\,|\sin\theta|$, $\theta=\arg\lambda$ from (3.20) and (3.21) it follows that

$$h(\theta)\le 2b\,|\sin\theta|\,. \tag{3.22}$$

According to a known property (see Refs. [33]) the set of zeros of every entire function of the exponential type, not identically zero, satisfies the inequality:

$$\lim_{r\to\infty}\inf\frac{n(r)}{r}\le\frac{1}{2\pi}\int_0^{2\pi}h(\theta)d\theta \tag{3.23}$$

where $n(r)$ is the number of zeros of $G(\lambda)$ in the disk $|\lambda| \leq r$. From (3.22), one gets

$$\frac{1}{2\pi}\int\limits_0^{2\pi} h(\theta)d\theta \leq \frac{b}{\pi}\int\limits_0^{2\pi} |\sin\theta|\, d\theta = \frac{4b}{\pi}. \tag{3.24}$$

From the assumption (2.3) and the known asymptotic expression (2.1) of the eigenvalues $\sqrt{\lambda_n}$ we obtain;

$$n(r) \geq 2 \sum_{\frac{\pi(n-1)}{\sigma}\left[1+O\left(\frac{1}{n}\right)\right]<r} 1 = \frac{2}{\pi}\sigma r(1+o\,(1)),\ r \to \infty. \tag{3.25}$$

For the case $\sigma > 2b$,

$$\lim_{r\to\infty} \frac{n(r)}{r} \geq \frac{2}{\pi}\sigma > \frac{4b}{\pi} = 2b\int\limits_0^{2\pi} |\sin\theta|\, d\theta \geq \frac{1}{2\pi}\int\limits_0^{2\pi} h(\theta)d\theta. \tag{3.26}$$

The inequalities (3.23) and (3.26) imply that $G(\lambda) = 0$ on the whole λ plane. As we already mentioned, if $G(\lambda) = 0$, then the conclusion of Lemma 2.1 is true.

We can prove $q\,(x) = \widetilde{q}\,(x)$ almost everywhere on $[b, 1]$ similarly.

This completes the proof of Lemma 2.1.

Now we prove that Theorem 2.2 is valid.

Proof of Theorem 2.2. Since

$$\lambda_{r(n)} = \widetilde{\lambda}_{r(n)},\ \frac{y'_{r(n)}(b, \lambda_{r(n)})}{y_{r(n)}(b, \lambda_{r(n)})} = \frac{\widetilde{y}'_{r(n)}(b, \lambda_{r(n)})}{\widetilde{y}_{r(n)}(b, \lambda_{r(n)})}$$

where $\{r(n)\}$ satisfies (2.7) and $\sigma_2 > 2 - 2b$, by Lemma 2.1, we obtain that $q\,(x) = \widetilde{q}\,(x)$ a.e. on $[b, 1]$ and $\frac{z}{c} = \frac{\widetilde{z}}{\widetilde{c}}$.

Thus, we only need to prove that $q\,(x) = \widetilde{q}\,(x)$ a.e. on $[0, b]$ and $\alpha = \widetilde{\alpha}$.

Similar to (3.19), in the case $b \in \left(\frac{1}{2}, 1\right)$, we have

$$\begin{aligned} G(\lambda) &= \int\limits_0^b [\widetilde{q}\,(x) - q\,(x)]\, y\,(x, \lambda)\, \widetilde{y}(x, \lambda)dx + \sin\,(\widetilde{\alpha} - \alpha) \\ &= \left[y(x, \lambda)\widetilde{y}'(x, \lambda) - \widetilde{y}(x, \lambda)y'(x, \lambda)\right]|_{x=b} \end{aligned} \tag{3.27}$$

Let us show that $G(\lambda) = 0$ on the whole λ plane.

Eigenfunctions $y_n(x)$ and $\widetilde{y}_n(x)$ satisfy the same boundary condition at 1 and $q\,(x) = \widetilde{q}\,(x)$ for almost everywhere on $[b, 1]$. This means that

$$y_n(x) = \xi_n \widetilde{y}_n(x) \tag{3.28}$$

on $[b, 1]$ for any $n \in \mathbb{N}$ where ξ_n are constants.

Let $\rho_n = \sqrt{\lambda_n}$ and $s_n = \sqrt{\mu_n}$. From (3.27) and (3.28) we obtain

$$G(\rho_n) = 0,\ n \in \mathbb{N}$$

and in the same way that

$$G(s_{l_n}) = 0, \ n \in \mathbb{N}.$$

Note that eigenvalues ρ_n and s_n possess the asymptotic expression (2.1). We can count the number of ρ_n and s_n located inside radius r, we get $1+\frac{2}{\pi}r\left[1+O(\frac{1}{n})\right]$ of λ_n's and $1+\frac{2}{\pi}r\sigma_1\left[1+O(\frac{1}{n})\right]$ of μ_n's. Thus, the total number of ρ_n's and s_n's

$$n(r) = 2 + \frac{2}{\pi}\left[r(\sigma_1+1)+O(\frac{1}{n})\right]$$

and

$$\lim_{r\to\infty}\frac{n(r)}{r} = \frac{2}{\pi}(\sigma_1+1).$$

Repeating the last part in the proof of Lemma 2.1, we can show that $G(\lambda)=0$ on the whole λ-plane. This implies that $q\,(x)=\widetilde{q}\,(x)$ a.e. on $[0,b]$ consequently $q\,(x)=\widetilde{q}\,(x)$ a.e. on $[0,\pi]$ and $\alpha=\widetilde{\alpha}$.

Hence the proof of Theorem 2.2 is completed.

References

[1] Borg, G., *Eine umkehrung der Sturm-Liouvilleschen eigenwertaufgabe*, Acta Math. **78** (1946), no.1, 1-96.

[2] Ambarzumyan, V.A., *Über eine frage der eigenwerttheorie*, Zeitschrift für Physik **53** (1929), 690-695.

[3] Levinson, N., *The inverse Sturm-Liouville problem,* Matematisk Tidsskrift. B (1949), 25-30.

[4] Levitan, B. M., *On the determination of the Sturm-Liouville operator from one and two spectra*, Izvestiya: Math. **12** (1978), no.1, 179-193.

[5] Hochstadt, H., *The inverse Sturm-Liouville problem,* Communications on Pure and Applied Math. **26** (1973), no.5, 715-729.

[6] Bauer, W. F., *Modified Sturm-Liouville systems,* Quarterly of Applied Math. **11** (1953), no.3 273-283.

[7] Binding, P. A., Patrick, J. B., Seddighi, K., *Sturm–Liouville problems with eigenparameter dependent boundary conditions,* Proceedings of the Edinburgh Mathematical Society **37** (1994), no.1, 57-72.

[8] Browne, P. J., Sleeman, B. D., *Inverse nodal problems for Sturm-Liouville equations with eigenparameter dependent boundary conditions,* Inverse problems **12** (1996), no.4, 377.

[9] Feller, W., *The parabolic differential equations and the associated semi-groups of transforms*, Ann. of Math. **55** (1952), 468–519.

[10] Fulton, C. T., *Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions*, Proc. Roy. Soc. Edinburgh, **77A** (1977), no.3, 293–308.

[11] Gaskell, R. E., *A problem in heat conduction and an expansion theorem*, Amer. J. Math. **64** (1942), no.1, 447–455.

[12] Guliyev, N. J., *The regularized trace formula for the Sturm-Liouville equation with spectral parameter in the boundary conditions*, Proc. Inst. Math. Natl. Alad. Sci. Azerb. **22** (2005), 99–102.

[13] Wang, Y. P., Yang, C. F., Huang, Z. Y., *Half inverse problem for Sturm-Liouville operators with boundary conditions dependent on the spectral parameter*, Turk. J. Math. **37** (2013), no.3, 445-454.

[14] Ozkan, A. S., Keskin, B., *Inverse nodal problems for Sturm–Liouville equation with eigenparameter-dependent boundary and jump conditions*, Inverse Problems in Science and Engineering, **23** (2015), no.8, 1306-1312.

[15] Wang, Y. P., *Inverse problems for Sturm–Liouville operators with interior discontinuities and boundary conditions dependent on the spectral parameter*, Mathematical Methods in the Applied Sciences, **36** (2013), no.7, 857-868.

[16] Wang, Y. P., Shieh, C. T., *Inverse Problems for Sturm–Liouville Equations with Boundary Conditions Linearly Dependent on the Spectral Parameter from Partial Information*, Results in Math. **65** (2014), no.1, 105-119.

[17] Yang, C. F., *New trace formula for the matrix Sturm-Liouville equation with eigenparameter dependent boundary conditions*, Turkish Journal of Math. **37** (2013), no.2, 278-285.

[18] Buterin, S., *On half inverse problem for differential pencils with the spectral parameter in boundary conditions*, Tamkang journal of Math. **42** (2011), no.3, 355-364.

[19] Mochizuki, K., Trooshin, I., *Inverse problem for interior spectral data of Sturm-Liouville operator*, J. Inverse Ill-Posed Problems, **9** (2001), no.4, 425-433.

[20] Mochizuki K., Trooshin, I., *Inverse problem for interior spectral data of the Dirac operator*, Communications-Korean Mathematics Society, **16** (2001), no.3, 437-444.

[21] Yang, C. F., Yang, X. P., *An interior inverse problem for the Sturm–Liouville operator with discontinuous conditions*, Applied Mathematics Letters, **22** (2009), no.9, 1315-1319.

[22] Yang, C. F., *An interior inverse problem for discontinuous boundary-value problems*, Integral Equations and Operator Theory, **65** (2009), no.4, 593-604.

[23] Yang, C. F., Guo, Y. X., *Determination of a differential pencil from interior spectral data*, Journal of Mathematical Analysis and Applications, **375** (2011), no.1, 284-293.

[24] Yang, C. F., *Determination of Dirac operator with eigenparameter-dependent boundary conditions from interior spectral data*, Inverse Problems in Science and Engineering, **20** (2012), no.3, 351-369.

[25] Wang, Y. P., *An interior inverse problem for Sturm-Liouville operators with eigenparameter dependent boundary conditions*, Tamkang Journal of Math. **42** (2011), no.3, 395-403.

[26] Wang, Y. P., *The inverse problem for differential pencils with eigenparameter dependent boundary conditions from interior spectral data*, Applied Mathematics Letters **25** (2012), no.7, 1061-1067.

[27] Sat, M., Panakhov, E., *A uniqueness theorem for Bessel operator from interior spectral data*, Abstract and Applied Analysis, **2013**. Hindawi Publishing Corporation, 2013.

[28] Panakhov, E., Sat, M., I*nverse problem for the interior spectral data of the equation of hydrogen atom*, Ukrainian Mathematical Journal, **64** (2013), no.11.

[29] Yang, C. F., Buterin, S., *Uniqueness of the interior transmission problem with partial information on the potential and eigenvalues*, Journal of Differential Equations, **260** (2016), no.6, 4871-4887.

[30] Wang, Y. P., Shieh, C. T., Ma, Y. T., *Inverse spectral problems for Sturm–Liouville operators with partial information*, Applied Mathematics Letters, **26** (2013): no.12, 1175-1181.

[31] Hochstadt, H., Lieberman, B., *An inverse Sturm-Liouville problem with mixed given data,* SIAM J. Appl. Math.*,* **34** (1978)**,** no.4, 676-680.

[32] Yang, C. F., Huang, Z. Y., *A half-inverse problem with eigenparameter dependent boundary conditions,*Numerical functional analysis and optimization, **31** (2010), no.6, 754-762.

[33] Levin, B. J., *Distribution of zeros of entire functions*, AMS Transl. **5** Providence, 1964.

[34] Levitan, B. M., Sargsian, I. S., *Introduction to spectral theory: selfadjoint ordinary differential operators: Selfadjoint Ordinary Differential Operators*, American Mathematical Soc. no.39, 1975.