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FEKETE-SZEGÖ ESTIMATES FOR A CLASS OF ANALYTIC FUNCTIONS DEFINED BY QUASI-SUBORDINATION

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Abstract

In this paper, we obtain estimates for the Fekete-Szegö functional associated with the k-th root transform for a class of analytic and univalent functions defined by means of quasi-subordination. Connections with previously known results are also pointed out.

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1 Introduction

Let $\mathcal{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of univalent functions.

Suppose that f and g are two analytic functions in $\mathbb{D}$. The function f is *subordinate* to the function g, denoted by $f \prec g$, if there exists an analytic function w defined in $\mathbb{D}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z)),\ z \in \mathbb{D}$.

Ma and Minda [13] defined the following two classes of functions

$$S^*(\phi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \phi(z),\ z \in \mathbb{D} \right\} \tag{2}$$

$$C(\phi) = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z),\ z \in \mathbb{D} \right\} \tag{3}$$

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where ϕ is an analytic function with positive real part in $\mathbb{D}$ with $\phi(0) = 1,\ \phi'(0) > 0$ and such that $\phi(\mathbb{D})$ is a starlike region with respect to 1 and symmetric with respect to the real axis.

The classes $S^*(\phi)$ and $C(\phi)$ contain, as special cases, several well-known subclasses of starlike and convex functions.

Following Ma and Minda, many authors considered similar classes defined by subordination (see [1], [9], [12]).

Joining the notion of subordination and majorization, Robertson in [18] introduced the concept of quasi-subordination. Suppose that f and g are analytic functions in $\mathbb{D}$. Then, we say that f is *quasi-subordinate* to g, denoted by $f \prec_q g$, if there exist two analytic functions φ and w in $\mathbb{D}$ with $|\varphi(z)| \leq 1, w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = \varphi(z)g(w(z)), z \in \mathbb{D}$. If $\varphi(z) = 1$, then $f(z) = g(w(z))$ and so $f \prec g$ in $\mathbb{D}$. Also, if $w(z) = z$ then, $f(z) = \varphi(z)g(z)$ in which case we say that f is *majorized* by g in $\mathbb{D}$ and write $f \ll g$.

Several results related with quasi-subordination may be found in [2], [5], [11], [17].

Throughout this paper it is assumed that ϕ is an analytic function in $\mathbb{D}$ with $\phi(0) = 1$. Applying the notion of quasi-subordination we define the following class of functions as a linear combination between the quantities $f'(z)$ and $1 + \dfrac{zf''(z)}{f'(z)}$.

Definition 1. *Let $f \in \mathcal{S}$ and $\alpha \in [0, 1]$. We say that the function f is in the class $\mathcal{G}_q(\alpha, \phi)$ if*

$$(1-\alpha)f'(z) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) - 1 \prec_q \phi(z) - 1,\ z \in \mathbb{D}. \tag{4}$$

The class $\mathcal{G}_q(\alpha, \phi)$, in the form of quasi-subordination, is analogous to the class $\mathcal{G}_\alpha(\phi)$ considered in [12].

Setting $\alpha = 1$, the class $\mathcal{G}_q(\alpha, \phi)$ reduces to the class $C_q(\phi)$ defined by Mohad and Darus in [14]. Further, if $\alpha = 0$ the class $\mathcal{G}_q(\alpha, \phi)$ becomes the class $R_q(\phi)$ studied by the same authors in [14].

In [6] Fekete and Szegö found the maximum value of the coefficient functional $|a_3 - \mu a_2^2|, \mu \in \mathbb{R}$ for functions of the form (1) belonging to the class $\mathcal{S}$. The problem of finding the maximum value of $|a_3 - \mu a_2^2|$ is known as the Fekete-Szegö problem. Throughout the years many authors have considered this problem for various subclasses of $\mathcal{S}$ (see, for example, [4], [8], [16], [19]).

For a function $f \in \mathcal{S}$ of the form (1), the k-th root transform is defined by

$$F(z) = z\left[\frac{f(z^k)}{z^k}\right]^{1/k} = z + \sum_{n=1}^{\infty} b_{kn+1} z^{kn+1},\ z \in \mathbb{D}. \tag{5}$$

Motivated by [7] and [14], in this paper we obtain estimates for the Fekete-Szegö functional associated with the k-th root transform of a function f belonging to the class $\mathcal{G}_q(\alpha, \phi)$. Connections with previous results are also pointed out.

Let $\mathcal{B}$ be the class of analytic functions w, normalized by $w(0) = 0$ and such that $|w(z)| < 1$ in $\mathbb{D}$.

In order to prove our results, the next two lemmas are needed.

Lemma 1. *([10]) Let $w(z) = w_1 z + w_2 z^2 + \dots$ be in the class $\mathcal{B}$. Then, for any complex number t*

$$|w_2 - t w_1^2| \leq \max\{1; |t|\}. \tag{6}$$

The result is sharp for the function $w(z) = z^2$ or $w(z) = z$.

Lemma 2. *([15]) Let $\varphi(z) = C_0 + C_1 z + C_2 z^2 + \dots$ be an analytic function in $\mathbb{D}$ such that $|\varphi(z)| \leq 1$. Then, $|C_0| \leq 1$ and*

$$|C_n| \leq 1 - |C_0|^2 \leq 1, \; n \in \{1, 2, \dots\}. \tag{7}$$

2 Main results

Unless otherwise mentioned, we assume throughout this section that

$$w(z) = w_1 z + w_2 z^2 + \dots, \; \varphi(z) = C_0 + C_1 z + C_2 z^2 + \dots$$

and

$$\phi(z) = 1 + B_1 z + B_2 z^2 + \dots, \; B_1 \in \mathbb{R}, \; B_1 > 0.$$

Theorem 1. *Let f, given by (1), be in the class $\mathcal{G}_q(\alpha, \phi), \alpha \in [0, 1]$ and let F, given by (5), be the k-th root of the function f. Then*

$$|b_{k+1}| \leq \frac{B_1}{2k}$$

$$|b_{2k+1}| \leq \frac{1}{3k(1+\alpha)} \left[B_1 + \max\left\{ B_1; \frac{|k(5\alpha - 3) + 3(1+\alpha)|}{8k} B_1^2 + |B_2| \right\} \right]$$

and, for any complex number μ

$$\begin{aligned} &|b_{2k+1} - \mu b_{k+1}^2| \\ \leq \; &\frac{1}{3k(1+\alpha)} \left[B_1 + \max\left\{ B_1; \frac{|k(5\alpha - 3) + 3(1+\alpha)(1 - 2\mu)|}{8k} B_1^2 + |B_2| \right\} \right]. \end{aligned}$$

Proof. Let $f \in \mathcal{G}_q(\alpha, \phi)$. Then, in view of Definition 1, there exist two analytic functions φ and w in $\mathbb{D}$ with $|\varphi(z)| \leq 1$ and $w \in \mathcal{B}$ such that

$$(1 - \alpha) f'(z) + \alpha \left(1 + \frac{z f''(z)}{f'(z)} \right) - 1 = \varphi(z)[\phi(w(z)) - 1]. \tag{8}$$

We have

$$(1 - \alpha) f'(z) + \alpha \left(1 + \frac{z f''(z)}{f'(z)} \right) = 1 + 2a_2 z + [3a_3(1+\alpha) - 4\alpha a_2^2] z^2 + \dots$$

and

$$\varphi(z)[\phi(w(z))-1]=B_1C_0w_1z+(B_1C_1w_1+B_1C_0w_2+B_2C_0w_1^2)z^2+\dots.$$

Equating the coefficients of z and z^2 on both sides of (8) we find

$$a_2=\frac{B_1C_0w_1}{2} \tag{9}$$

and

$$a_3=\frac{1}{3(1+\alpha)}[B_1C_1w_1+B_1C_0w_2+C_0(\alpha B_1^2C_0+B_2)w_1^2]. \tag{10}$$

For f given by (1), a computation shows that

$$F(z)=z\left[\frac{f(z^k)}{z^k}\right]^{1/k}=z+\frac{1}{k}a_2z^{k+1}+\left(\frac{1}{k}a_3-\frac{1}{2}\frac{k-1}{k^2}a_2^2\right)z^{2k+1}+\dots. \tag{11}$$

The equations (5) and (11) lead to

$$b_{k+1}=\frac{1}{k}a_2 \ \text{ and } \ b_{2k+1}=\frac{1}{k}a_3-\frac{1}{2}\frac{k-1}{k^2}a_2^2. \tag{12}$$

Substituting (9) and (10) in (12), we get

$$b_{k+1}=\frac{B_1C_0w_1}{2k}$$

and

$$b_{2k+1} = \frac{1}{3k(1+\alpha)}\left[B_1C_1w_1+B_1C_0w_2+C_0\left(\frac{k(5\alpha-3)+3(1+\alpha)}{8k}B_1^2C_0+B_2\right)w_1^2\right].$$

Therefore, for any complex number μ, we have

$$b_{2k+1}-\mu b_{k+1}^2 = \frac{B_1}{3k(1+\alpha)}\left\{C_1w_1+C_0\left[w_2-\left(-\frac{k(5\alpha-3)+3(1+\alpha)(1-2\mu)}{8k}B_1C_0-\frac{B_2}{B_1}\right)w_1^2\right]\right\}.$$

Making use of inequality (7), from Lemma 2, and the well-known inequality $|w_1|\le 1$, we obtain

$$|b_{k+1}|\le\frac{B_1}{2k}$$

and

$$|b_{2k+1}-\mu b_{k+1}^2|\le \frac{B_1}{3k(1+\alpha)}\left[1+\left|w_2-\left(-\frac{k(5\alpha-3)+3(1+\alpha)(1-2\mu)}{8k}B_1C_0-\frac{B_2}{B_1}\right)w_1^2\right|\right].$$

Applying Lemma 1 to

$$\left|w_2 - \left(-\frac{k(5\alpha-3)+3(1+\alpha)(1-2\mu)}{8k}B_1C_0 - \frac{B_2}{B_1}\right)w_1^2\right|$$

we find

$$\begin{aligned}&|b_{2k+1}-\mu b_{k+1}^2|\\ \leq\ & \frac{B_1}{3k(1+\alpha)}\left[1+\max\left\{1;\left|\frac{k(5\alpha-3)+3(1+\alpha)(1-2\mu)}{8k}B_1C_0+\frac{B_2}{B_1}\right|\right\}\right].\end{aligned}$$

Since

$$\begin{aligned}&\left|\frac{k(5\alpha-3)+3(1+\alpha)(1-2\mu)}{8k}B_1C_0+\frac{B_2}{B_1}\right|\\ &\qquad\leq \frac{|k(5\alpha-3)+3(1+\alpha)(1-2\mu)|}{8k}B_1|C_0|+\left|\frac{B_2}{B_1}\right|\end{aligned}$$

it follows that

$$\begin{aligned}&|b_{2k+1}-\mu b_{k+1}^2|\\ \leq\ & \frac{1}{3k(1+\alpha)}\left[B_1+\max\left\{B_1;\frac{|k(5\alpha-3)+3(1+\alpha)(1-2\mu)|}{8k}B_1^2+|B_2|\right\}\right].\end{aligned}$$

For $\mu = 0$, the above inequality gives

$$|b_{2k+1}| \leq \frac{1}{3k(1+\alpha)}\left[B_1+\max\left\{B_1;\frac{|k(5\alpha-3)+3(1+\alpha)|}{8k}B_1^2+|B_2|\right\}\right]$$

which completes the proof of our theorem. □

Setting $k = 1$ in Theorem 1 we obtain the following result.

Corollary 1. *Let f, given by (1), be in the class $\mathcal{G}_q(\alpha,\phi)$. Then,*

$$|a_2| \leq \frac{B_1}{2},\quad |a_3| \leq \frac{1}{3(1+\alpha)}[B_1+\max\{B_1;\alpha B_1^2+|B_2|\}]$$

and, for any complex number μ

$$|a_3-\mu a_2^2| \leq \frac{1}{3(1+\alpha)}\left[B_1+\max\left\{B_1;\left|\alpha-\frac{3(1+\alpha)}{4}\mu\right|B_1^2+|B_2|\right\}\right].$$

Remark 1. *The case $\alpha = 1$ in Corollary 1 reduces to the result obtained by Mohad and Darus for the class $C_q(\phi)$ in [14, Theorem 2.4]. Setting $\alpha = 0$ in Corollary 1, we find the estimates for the class $R_q(\phi)$ obtained by the same authors in [14, Theorem 2.6].*

By taking $w(z) = z$ in the proof of Theorem 1, we get the next result.

Theorem 2. *Let $f \in \mathcal{S}, \alpha \in [0,1]$ and let F, given by (5), be the k-th root of f. If*

$$(1-\alpha)f'(z)+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)-1 \ll \phi(z)-1,\ z\in\mathbb{D}$$

then,

$$|b_{k+1}|\le\frac{B_1}{2k},\quad |b_{2k+1}|\le\frac{1}{3k(1+\alpha)}\left[B_1+\frac{|k(5\alpha-3)+3(1+\alpha)|}{8k}B_1^2+|B_2|\right]$$

and, for any complex number μ

$$|b_{2k+1}-\mu b_{k+1}^2|\le\frac{1}{3k(1+\alpha)}\left[B_1+\frac{|k(5\alpha-3)+3(1+\alpha)(1-2\mu)|}{8k}B_1^2+|B_2|\right].$$

Setting $\varphi(z)=1$ in the proof of Theorem 1, we obtain the following theorem.

Theorem 3. *Let $f \in \mathcal{S}, \alpha \in [0,1]$ and let F, given by (5), be the k-th root of f. If*

$$(1-\alpha)f'(z)+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\prec\phi(z),\ z\in\mathbb{D}$$

then,

$$|b_{k+1}|\le\frac{B_1}{2k},\quad |b_{2k+1}|\le\frac{B_1}{3k(1+\alpha)}\max\left\{1;\left|\frac{k(5\alpha-3)+3(1+\alpha)}{8k}B_1+\frac{B_2}{B_1}\right|\right\}$$

and, for any complex number μ

$$|b_{2k+1}-\mu b_{k+1}^2|\le\frac{B_1}{3k(1+\alpha)}\max\left\{1;\left|\frac{k(5\alpha-3)+3(1+\alpha)(1-2\mu)}{8k}B_1+\frac{B_2}{B_1}\right|\right\}.$$

For $\phi(z)=\dfrac{1+z}{1-z}$ and $k=1$ in Theorem 3 we find the following result due to Al-Amiri and Reade [3, Theorem 3].

Corollary 2. *Let $f\in\mathcal{A}$ be given by (1) and let $\alpha\in[0,1]$. If*

$$\Re\left[(1-\alpha)f'(z)+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right]>0,\ z\in\mathbb{D}$$

then

$$|a_2|\le 1,\quad |a_3|\le\frac{2(1+2\alpha)}{3(1+\alpha)}$$

and, for any complex number μ

$$|a_3-\mu a_2^2|\le\frac{2}{3(1+\alpha)}\max\left\{1;\left|1+2\alpha-\frac{3(1+\alpha)}{2}\mu\right|\right\}.$$

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