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A CRITERION FOR THE LIMIT OF A RATIO OF FUNCTIONS

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Abstract

In this paper we establish a criterion for the existence of the limit of a ratio of real functions. In particular, this criterion offers a converse of l'Hôpital's rule.

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1 Introduction

L'Hospital's theorem is one of the most popular mathematical tools in one-variable calculus. In this paper we propose a converse theorem. Namely, we investigate the implication:

$$\lim_{x\uparrow b}\frac{\int_a^x f(t)\,dt}{\int_a^x g(t)\,dt}=L\ \Rightarrow\ \lim_{x\uparrow b}\frac{f(x)}{g(x)}=L, \tag{1}$$

where f and g are two locally Riemann integrable positive functions on $[a,b)$. Our main result (Theorem 1) states sufficient conditions for the above implication. These conditions are motivated by suitable counterexamples. A particular case of (1) was formulated by Călin Popescu (see [1], pages 320-321).

2 Main results

Our aim is to formulate sufficient conditions for the implication (1). The theorem below shows that (1) holds under certain conditions of monotony. In this paper, an increasing function means a non-decreasing function.

Theorem 1. *Let $f,g:[a,b)\to(0,\infty)$ be two Riemann locally integrable functions, where $-\infty<a<b\le\infty$. Assume that*

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1. *f is increasing on $[a,b)$;*

2. $\lim\limits_{x\uparrow b}\int\limits_a^x g(t)\,dt=\infty$;

3. $\dfrac{\int\limits_a^x g(t)\,dt}{g(x)}$ *is an increasing function in $x\in[a,b)$;*

4. *there exists the limit* $\lim\limits_{x\uparrow b}\dfrac{\int_a^x f(t)\,dt}{\int_a^x g(t)\,dt}=L.$

If L is finite, i.e. $L\in[0,\infty)$, then $\lim\limits_{x\uparrow b}\dfrac{f(x)}{g(x)}=L.$

Proof. Let us denote $F(x)=\displaystyle\int_a^x f(t)\,dt$ and $G(x)=\displaystyle\int_a^x g(t)\,dt$, for $x\in[a,b)$.

Case I. Firstly, suppose $L\in(0,\infty)$. Since f/L is increasing and $\frac{\int_a^x f(t)/L\,dt}{\int_a^x g(t)\,dt}\to 1$, we observe that it suffices to prove the theorem for $L=1$. Thus, we assume that

$$\lim_{x\uparrow b}\frac{F(x)}{G(x)}=1. \tag{2}$$

We will prove by *reductio ad absurdum* that

$$\lim_{x\uparrow b}\frac{f(x)}{g(x)}=1. \tag{3}$$

Assume $r(x):=\dfrac{f(x)}{g(x)}\not\to 1$ for $x\uparrow b$. Then, there are $L'\in[0,\infty]\setminus\{1\}$ and a strictly increasing sequence $(x_n)_{n\geq 1}$, with the terms in the interval $[a,b)$, such that $x_n\uparrow b$ and $\lim\limits_{n\to\infty} r(x_n)=L'$.

Assume $L'>1$. Let us consider two fixed numbers $c\in(1,L')$ and $\lambda\in(1,c)$. Since $r(x_n)\to L'$, there is a positive integer n_0 such that $r(x_n)>c$, $\forall\ n\geq n_0$. Seeing that G is strictly increasing and continuous, with $G(x)\to\infty$ for $x\uparrow b$, we can associate to each term x_n $(n\geq n_0)$ a unique number $y_n\in(x_n,b)$ with the property $G(y_n)=\lambda G(x_n)$. From the assumption 1, we find

$$F(y_n)=F(x_n)+\int_{x_n}^{y_n} f(t)\,dt\geq F(x_n)+f(x_n)(y_n-x_n)$$

$$>F(x_n)+cg(x_n)(y_n-x_n),$$

for all $n\geq n_0$. From the condition 3 of the hypothesis, we obtain

$$g(x_n)\geq\frac{g(t)\,G(x_n)}{G(t)}\geq\frac{g(t)\,G(x_n)}{G(y_n)}=\frac{g(t)}{\lambda},$$

for all $t \in [x_n, y_n]$. Hence

$$g(x_n)(y_n - x_n) \geq \frac{1}{\lambda} \int_{x_n}^{y_n} g(t)dt = \frac{1}{\lambda} [G(y_n) - G(x_n)].$$

Therefore, for $n \geq n_0$, we have

$$\frac{F(y_n)}{G(y_n)} > \frac{F(x_n) + c[G(y_n) - G(x_n)]/\lambda}{G(y_n)} = \frac{F(x_n)}{\lambda G(x_n)} + \frac{c(\lambda - 1)}{\lambda^2}.$$

Based on (2), the above inequality becomes for $n \to \infty$

$$1 \geq \frac{1}{\lambda} + \frac{c(\lambda - 1)}{\lambda^2},$$

or $(\lambda - 1)(\lambda - c) \geq 0$, in contradiction with our assumption $1 < \lambda < c$.
Let us suppose now that $L' \in [0, 1)$. Consider the fixed numbers $c \in (L', 1)$, $\lambda \in (c, 1)$ and $a_1 \in (a, b)$. Since G is continuous, $G(a_1) > 0$ and $\lim\limits_{x \uparrow b} G(x) = \infty$, we can find $a_2 \in (a_1, b)$ such that $G(a_2) = G(a_1)/\lambda$. There is a positive integer n_0 such that $x_n > a_2$ and $r(x_n) < c$, $\forall\ n \geq n_0$. We can associate to each term x_n $(n \geq n_0)$ a unique number $y_n \in (a_1, x_n)$ with the property $G(y_n) = \lambda G(x_n)$. Then we have

$$F(x_n) = F(y_n) + \int_{y_n}^{x_n} f(t)dt \leq F(y_n) + f(x_n)(x_n - y_n)$$

$$< F(y_n) + cg(x_n)(x_n - y_n), \ \forall\ n \geq n_0.$$

From the hypothesis, we get

$$g(x_n) \leq \frac{g(t)\, G(x_n)}{G(t)} \leq \frac{g(t)\, G(x_n)}{G(y_n)} = \frac{g(t)}{\lambda}, \ \forall\ t \in [y_n, x_n].$$

By integrating on $[y_n, x_n]$, we find $g(x_n)(x_n - y_n) \leq \dfrac{1}{\lambda}[G(x_n) - G(y_n)]$. Thus,

$$\frac{F(x_n)}{G(x_n)} < \frac{F(y_n) + c[G(x_n) - G(y_n)]/\lambda}{G(x_n)} = \lambda \frac{F(y_n)}{G(y_n)} + \frac{c(1 - \lambda)}{\lambda}, \ \forall\ n \geq n_0.$$

Since $x_n \uparrow b$, we also have $y_n \uparrow b$. Then, passing to the limit for $n \to \infty$, we get $1 \leq \lambda + c(1 - \lambda)/\lambda$, that is $(\lambda - 1)(\lambda - c) \geq 0$, in contradiction with the condition $c < \lambda < 1$.

Thus, we conclude that the relation (3) holds.

Case II. Assume $L = 0$. If we suppose that $r(x) := \dfrac{f(x)}{g(x)} \not\to 0$, for $x \uparrow b$, then there are $L' \in (0, \infty]$ and a strictly increasing sequence $(x_n)_{n \geq 1}$, with the terms in the interval $[a, b)$, such that $x_n \uparrow b$ and $\lim\limits_{n \to \infty} r(x_n) = L'$. Let us consider $c \in (0, L')$ and $\lambda > 1$. There is $n_0 \in \mathbb{N}$ such that $r(x_n) > c$, $\forall\ n \geq n_0$. We associate to each term x_n the term $y_n \in (x_n, b)$ such that $G(y_n) = \lambda G(x_n)$. Thus, following the arguments used in the previous case, we obtain

$$\frac{F(y_n)}{G(y_n)} > \frac{F(x_n)}{\lambda G(x_n)} + \frac{c(\lambda-1)}{\lambda^2},$$

for all $n \geq n_0$. Hence $\lim\limits_{n\to\infty} \dfrac{F(y_n)}{G(y_n)} \geq \lim\limits_{n\to\infty} \dfrac{F(x_n)}{\lambda G(x_n)} + \dfrac{c(\lambda-1)}{\lambda^2}$, or $\dfrac{c(\lambda-1)}{\lambda^2} \leq 0$, in contradiction with the inequalities $c > 0$ and $\lambda > 1$. So, $\lim\limits_{x\uparrow b} \dfrac{f(x)}{g(x)} = 0$. □

3 Comments and examples

If in Theorem 1 the functions f and g are supposed to be continuous on $[a, b)$, then F and G are their primitive functions on $[a, b)$, with $F(a) = G(a) = 0$. In this case, Theorem 1 is a converse of the theorem of l'Hospital (for the case "$\frac{\cdot}{\infty}$").

Further, we will comment on the hypothesis of Theorem 1. An implication of type (1) does not hold without some monotony requirements. This statement is confirmed by the following elementary example (see also [1]).

Counterexample 1. *Consider $f, g : [a,\infty) \to (0,\infty)$, with $a > 0$, given by $f(x) = x(2+\sin x)$ and $g(x) = 2x$, for $x \geq 0$. We obtain*

$$\lim_{x\to\infty} \frac{F(x)}{G(x)} = \lim_{x\to\infty} \frac{x^2 - x\cos x + \sin x - a^2 + a\cos a - \sin a}{x^2 - a^2} = 1.$$

But the function $\dfrac{f(x)}{g(x)} = 1 + \dfrac{\sin x}{2}$ has no limit at infinity.

Note that, in the above example, only the condition 1 of the hypothesis of Theorem 1 is not fulfilled. Observe that this condition ensures the convexity of F on $[a, b)$. A natural question is the following: for the conclusion, is it sufficient to assume the monotony of the two positive functions f and g? More precisely: can we replace in the hypothesis of Theorem 1 the increasing monotony of G/g by the increasing monotony of g? The following counterexample shows that this replacement does not provide the desired conclusion.

Counterexample 2. *Let us consider the increasing functions $f, g : [0,\infty) \to (0,\infty)$, defined by $f(x) = 2^{[x]}$, for $x \in [0,\infty)$, and $g(x) = 2^{[x]}$, for $x \in [0,\infty) \setminus \mathbb{N}$, with $g(n) = 2^{n-1}$, for $n \in \mathbb{N}$. We have $F(x) = G(x) = 2^{[x]}(1+\{x\})-1$, for $x \geq 0$. Thus $\lim\limits_{x\to\infty} G(x) = \infty$ and $\lim\limits_{x\to\infty} \dfrac{F(x)}{G(x)} = 1$, but $\dfrac{f(n)}{g(n)} = 2$, for all $n \in \mathbb{N}$.*

Observe that, in this counterexample, the function $\dfrac{G}{g}$ is not increasing on $[0,\infty)$. Indeed, we have

$$\frac{G(n)}{g(n)} = 2 - \frac{1}{2^{n-1}} > \frac{3}{2} - \frac{1}{2^n} = \frac{G\left(n+\frac{1}{2}\right)}{g\left(n+\frac{1}{2}\right)}, \text{ for } n \in \mathbb{N},\ n > 1.$$

Remark that, if the condition 3 of the hypothesis of Theorem 1 is fulfilled by an integrable function $g : [a, b) \to (0, \infty)$, then

$$\limsup_{t \downarrow x} g(t) \leq \lim_{t \downarrow x} \frac{G(t)g(x)}{G(x)} = g(x)$$

and

$$\liminf_{t \uparrow x} g(t) \geq \lim_{t \uparrow x} \frac{G(t)g(x)}{G(x)} = g(x),$$

for all $x \in (a, b)$. Therefore, if g is increasing on (a, b) then g is continuous on (a, b). Of course, a function g satisfying the condition 3 is not necessarily increasing. For example, $g(x) = \dfrac{1}{x}$ is decreasing on $[1, \infty)$ and $\dfrac{G(x)}{g(x)} = x \ln x$ is increasing on $[1, \infty)$.

Finally, let us show that Theorem 1 does not hold for $L = \infty$.

Counterexample 3. *Consider $g : [0, \infty) \to (0, \infty)$, $g(x) = e^x$. We define the strictly increasing and divergent sequence $(x_n)_{n \geq 0}$, by $x_0 = 0$ and $x_{n+1} = e^{x_n}$, for $n \in \mathbb{N}$. Let $f : [0, \infty) \to (0, \infty)$ be the increasing function defined by $f(x) = x_{n+2} = e^{x_{n+1}}$, for $x \in (x_n, x_{n+1}]$, $n \in \mathbb{N}$ (with $f(0) = 1$). The assumptions 1-4 of Theorem 1 are met, with $L = \infty$. But we have $\dfrac{f(x_n)}{g(x_n)} = 1$, for all $n \in \mathbb{N} \setminus \{0\}$.*

To prove that $\lim_{x \to \infty} \dfrac{F(x)}{G(x)} = \infty$, we can use the following method. From $\lim_{n \to \infty} x_n = \infty$, $\lim_{n \to \infty} (x_{n+1} - x_n) = \lim_{n \to \infty} (e^{x_n} - x_n) = \infty$ and $\lim_{t \to \infty} \dfrac{t}{1 - e^{-t}} = \infty$, we obtain

$$\lim_{n \to \infty} \frac{F(x_{n+1}) - F(x_n)}{G(x_{n+1}) - G(x_n)} = \lim_{n \to \infty} \frac{e^{x_{n+1}} (x_{n+1} - x_n)}{e^{x_{n+1}} - e^{x_n}} = \lim_{n \to \infty} \frac{x_{n+1} - x_n}{1 - e^{-(x_{n+1} - x_n)}} = \infty.$$

Hence, by using Stolz-Cesàro's Theorem, we find $\lim_{n \to \infty} \dfrac{F(x_n)}{G(x_n)} = \infty$. Then, for $x > 0$, there is $n \in \mathbb{N}$ such that $x \in (x_n, x_{n+1}]$. We have

$$\frac{F(x) - F(x_n)}{G(x) - G(x_n)} = \frac{e^{x_{n+1}} (x - x_n)}{e^x - e^{x_n}} = e^{x_{n+1} - x_n} \cdot \frac{x - x_n}{e^{x - x_n} - 1}.$$

Since the function $\varphi(t) = \dfrac{t}{e^t - 1}$ is decreasing on $(0, \infty)$, we find

$$\frac{F(x) - F(x_n)}{G(x) - G(x_n)} \geq e^{x_{n+1} - x_n} \cdot \frac{x_{n+1} - x_n}{e^{x_{n+1} - x_n} - 1} = \frac{F(x_{n+1}) - F(x_n)}{G(x_{n+1}) - G(x_n)}, \ \forall\, x \in (x_n, x_{n+1}].$$

So, we easily deduce that $\lim_{x \to \infty} \dfrac{F(x)}{G(x)} = \infty$.

References

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