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SOME REMARKS FOR CERTAIN SUBCLASSES OF MEROMORPHIC $1-valent$ FUNCTIONS

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Abstract

In this paper, a boundary version of the Schwarz lemma for classes $\mathcal{N}(\lambda)$ is investigated. For the function $f(z)=\frac{1}{z}+c_1z+c_2z^2+...$ defined in the punctured unit disc $D=\{z\in\mathbb{C}:0<|z|<1\}$ such that $f(z)\in\mathcal{N}(\lambda)$, we estimate a modulus of the angular derivative of $f''(z)$ function at the boundary point b with $f'(b)=-\frac{1+\lambda}{2b^2}$. The sharpness of these inequalities is also proved.

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1 Introduction

The most classical version of the Schwarz lemma involves the behavior at the origin of a bounded, holomorphic function on the unit disc $E=\{z:|z|<1\}$. Also, the Schwarz lemma is one of the most important results in the complex analysis and it is widely applied in many branches of mathematical research. In its most basic form, the familiar Schwarz lemma says this ([8], p.329):

Let E be the unit disc in the complex plane $\mathbb{C}$. Let $f:E\to E$ be a holomorphic function with $f(0)=0$. Under these circumstances $|f(z)|\leq|z|$ for all $z\in E$, and $|f'(0)|\leq 1$. In addition, if the equality $|f(z)|=|z|$ holds for any $z\neq 0$, or $|f'(0)|=1$ then f is a rotation, that is, $f(z)=ze^{i\theta}$, θ real.

In order to show our main results, we need the following lemma due to Jack's Lemma [9].

Lemma 1 (Jack's Lemma). *Let $f(z)$ be a non-constant and holomorphic function in the unit disc E with $f(0)=0$. If $|f(z)|$ attains its maximum value on the circle $|z|=r$ at the point z_0, then*

$$\frac{z_0f'(z_0)}{f(z_0)}=k,$$

where $k\geq 1$ is a real number.

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Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = \frac{1}{z} + c_1 z + c_2 z^2 + c_3 z^3 + ...$$

which are holomorphic and $1 - valent$ in the punctured unit disc $D = \{z \in \mathbb{C} : 0 < |z| < 1\}$. Also, let $\mathcal{N}(\lambda)$ be the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy

$$-\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \frac{3+\lambda}{2(1+\lambda)} \quad (z \in E),$$

where $\lambda > 1$.

A function $f(z) \in \mathcal{A}$ is said to be meromorphically $1 - valent$ convex of order γ if and only if

$$\Re\left(-1 - \frac{zf''(z)}{f'(z)}\right) > \gamma$$

for some $\gamma\,(0 < \gamma < 1)$. Therefore, we denote by $S(\gamma)$ the class of all meromorphically $1 - valent$ convex of order γ. Thus, the class of $\mathcal{N}(\lambda)$ and $S(\gamma)$ we have given above coincide with $\gamma = \frac{3+\lambda}{2(1+\lambda)}$.

Let $f(z) \in \mathcal{N}(\lambda)$ and consider the function

$$\psi(z) = -\frac{1 + z^2 f'(z)}{\lambda + z^2 f'(z)}. \tag{1.1}$$

Clearly, $\psi(z)$ is holomorphic function in E and $\psi(0) = 0$.

We want to prove that $|\psi(z)| < 1$ in E. From (1.1), we have

$$-\left(1 + \frac{zf''(z)}{f'(z)}\right) = 1 - \frac{\lambda z\psi'(z)}{1 + \lambda\psi(z)} + \frac{z\psi'(z)}{1 + \psi(z)}. \tag{1.2}$$

Assume that there exists a point $z_0 \in E$ such that

$$\max_{|z| \le |z_0|} |\psi(z)| = |\psi(z_0)| = 1.$$

From Jack's lemma, we obtain

$$\psi(z_0) = e^{i\theta} \quad \text{and} \quad \frac{z_0\psi'(z_0)}{\psi(z_0)} = k.$$

Therefore, by using (1.2) and Jack's lemma, we have

$$\begin{aligned} -\Re\left(1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right) &= 1 - \Re\left(\frac{\lambda z_0\psi'(z_0)}{1 + \lambda\psi(z_0)}\right) + \Re\left(\frac{z_0\psi'(z_0)}{1 + \psi(z_0)}\right) \\ &= 1 - \Re\left(\frac{k\lambda\psi(z_0)}{1 + \lambda\psi(z_0)}\right) + \Re\left(\frac{k\psi(z_0)}{1 + \psi(z_0)}\right). \end{aligned}$$

Since

$$\begin{aligned}
\Re\left(\frac{k\lambda\psi(z_0)}{1+\lambda\psi(z_0)}\right) &= \Re\left(\frac{k\lambda e^{i\theta}}{1+\lambda e^{i\theta}}\right) = \lambda k\Re\left(\frac{e^{i\theta}}{1+\lambda e^{i\theta}}\right) = \lambda k\Re\left(\frac{1}{\lambda+e^{-i\theta}}\right) \\
&= \lambda k\Re\left(\frac{1}{\lambda+\cos\theta-i\sin\theta}\right) = \lambda k\Re\left(\frac{\lambda+\cos\theta+i\sin\theta}{(\lambda+\cos\theta)^2+\sin^2\theta}\right) \\
&= \lambda k\Re\left(\frac{\lambda+\cos\theta}{(\lambda+\cos\theta)^2+\sin^2\theta}+i\frac{\sin\theta}{(\lambda+\cos\theta)^2+\sin^2\theta}\right) \\
&= \lambda k\frac{\lambda+\cos\theta}{(\lambda+\cos\theta)^2+\sin^2\theta} = \lambda k\frac{\lambda+\cos\theta}{\lambda^2+2\lambda\cos\theta+\cos^2\theta+\sin^2\theta} \\
&= \lambda k\frac{\lambda+\cos\theta}{1+\lambda^2+2\lambda\cos\theta}
\end{aligned}$$

and

$$\begin{aligned}
\Re\left(\frac{z_0\psi'(z_0)}{1+\psi(z_0)}\right) &= \Re\left(\frac{ke^{i\theta}}{1+e^{i\theta}}\right) = k\Re\left(\frac{1}{1+e^{-i\theta}}\right) \\
&= k\Re\left(\frac{1}{1+\cos\theta-i\sin\theta}\right) = k\Re\left(\frac{1+\cos\theta+i\sin\theta}{(1+\cos\theta)^2+\sin^2\theta}\right) \\
&= \frac{k}{2},
\end{aligned}$$

we obtain

$$-\Re\left(1+\frac{z_0f''(z_0)}{f'(z_0)}\right) = 1-\lambda k\frac{\lambda+\cos\theta}{1+\lambda^2+2\lambda\cos\theta}+\frac{k}{2} \le \frac{3+\lambda}{2\,(1+\lambda)}.$$

This contradicts the condition $f(z)\in\mathcal{N}(\lambda)$. This means that there is no point $z_0\in E$ such that $|\psi(z_0)|=1$ for all $z\in E$. Therefore, $|\psi(z)|<1$ for $|z|<1$. By the Schwarz lemma, we obtain

$$|c_1|\le\lambda-1.$$

The result is sharp and the extremal function is

$$f(z)=\frac{1}{z}+(1-\lambda)\arctan z.$$

That proves

Lemma 2. *If $f(z)\in\mathcal{N}(\lambda)$, then we have*

$$|c_1|\le\lambda-1. \tag{1.2}$$

The result is sharp and the extremal function is

$$f(z)=\frac{1}{z}+(1-\lambda)\arctan z.$$

It is an elementary consequence of Schwarz lemma that if f extends continuously to some boundary point b with $|b|=1$, and if $|f(b)|=1$ and $f'(b)$ exists, then $|f'(b)|\geq 1$, which is known as the Schwarz lemma on the boundary. The equality in $|f'(b)|\geq 1$ holds if and only if $f(z)=ze^{i\theta}$, θ real. This result of Schwarz lemma and its generalization are described as Schwarz lemma at the boundary in the literature.

Throughout the last decade, there have been tremendous studies on Schwarz lemma at the boundary (see,[1], [2], [5], [6], [13], [16], [17], [22] and references therein). Some of them are about the boundary of modulus of the functions derivation at the points (contact points) which satisfies $|f(b)|=1$ condition of the boundary of the unit circle.

Osserman [16] offered the following boundary refinement of the classical Schwarz lemma. It is very much in the spirit of the sort of result we wish to consider here.

Lemma 3. *Let $f(z)=c_pz^p+c_{p+1}z^{p+1}+...$ be holomorphic function in the unit disc E with $f(0)=0$ and $|f(z)|<1$ for $|z|<1$. Assume that there is a $b\in\partial E$*

so that f extends continuously to b, $|f(b)|=1$ and $f'(b)$ exists. Then

$$\left|f'(b)\right|\geq p+\frac{1-|c_p|}{1+|c_p|}. \tag{1.3}$$

and

$$\left|f'(b)\right|\geq p. \tag{1.4}$$

In addition, the equality in (1.4) holds if and only if $f(z)=z^pe^{i\theta}$, where θ is a real number. Also, the equality in (1.3) holds if and only if f is of the form $f(z)=-z^p\frac{a_0-z}{1-\overline{a_0}z}$, $\forall z\in E$, for some constant $a_0\in(-1,0]$.

The following lemma, known as the Julia-Wolff lemma, is needed in the sequel (see [18]).

Lemma 4 (Julia-Wolff lemma)**.** *Let f be a holomorphic function in E, $f(0)=0$ and $f(E)\subset E$. If, in addition, the function f has an angular limit $f(b)$ at $b\in\partial E$, $|f(b)|=1$, then the angular derivative $f'(b)$ exists and $1\leq|f'(b)|\leq\infty$.*

Corollary 1. *The holomorphic function f has a finite angular derivative $f'(b)$ if and only if f' has the finite angular limit $f'(b)$ at $b\in\partial E$.*

D. M. Burns and S. G. Krantz [3] and D. Chelst [4] studied the uniqueness part of the Schwarz lemma. The similar types of results which are related to the subject of the paper can be found in ([12], [14] and [15]). In addition, the results more general aspects are discussed by M. Mateljevic in [13] and they were was announced on ResearchGate.

The inequality (1.3) is a particular case of a result due to Vladimir N. Dubinin in [5], who strengthened the inequality $|f'(b)|\geq 1$ by involving zeros of the function f.

X. Tang, T. Liu and J. Lu [20] established a new type of the classical boundary Schwraz lemma for holomorphic self-mappings of the unit polydisk D^n in $\mathbb{C}^n$. They extended the classical Schwarz lemma at the boundary to high dimensions.

Also, M. Jeong [11] showed some inequalities at a boundary point for a different form of holomorphic functions and found the condition for equality and in [10] a holomorphic self map defined on the closed unit disc with fixed points only on the boundary of the unit disc. Furthermore, X. Tang, T. Liu and W. Zhang [21] established a new type of the classical Schwarz lemma at the boundary for holomorphic self-mappings of the unit ball in $\mathbb{C}^n$, and then gave the boundary version of the rigidity theorem. S.L. Wail and W.M. Shah [22] established some results by using a boundary refinement of the classical Schwarz lemma. For historical background about the Schwarz lemma and its applications on the boundary of the unit disc, we refer to (see [2], [7]).

2 Main Results

In this section, for a meromorphic function $f(z)$ belonging to the class of $\mathcal{N}(\lambda)$, it has been estimated from below the modulus of the angular derivative of the function $f''(z)$ on the boundary point of the unit disc. It has been proved that these results are sharp. Also, we derive an improvement of the above inequalities (1.3) and (1.4) as the special cases of our main result.

Theorem 1. *Let $f(z) \in \mathcal{N}(\lambda)$. Suppose that, for some $b \in \partial E$, f' has an angular limit $f'(b)$ at b, $f'(b) = -\frac{1+\lambda}{2b^2}$. Then we have the inequality*

$$\left|f''(b)\right| \geq -\frac{3+\lambda}{2}. \tag{2.1}$$

The inequality (2.1) *is sharp with extremal function*

$$f(z) = \frac{1}{z} + (1-\lambda)\arctan z.$$

Proof. Let us consider the following function

$$\psi(z) = -\frac{1+z^2 f'(z)}{\lambda + z^2 f'(z)}.$$

Then $\psi(z)$ is holomorphic function in the unit disc E and $\psi(0) = 0$. By the Jack's lemma and since $f(z) \in \mathcal{N}(\lambda)$, we take $|\psi(z)| < 1$ for $|z| < 1$. Also, we have $|\psi(b)| = 1$ for $b \in \partial E$. It is clear that

$$\psi'(z) = -\frac{2zf'(z)(\lambda-1) + z^2 f''(z)(\lambda-1)}{\left(\lambda + z^2 f'(z)\right)^2}$$

and

$$\left|\psi'(b)\right| = \left|\frac{2bf'(b)(\lambda-1) + b^2 f''(b)(\lambda-1)}{\left(\lambda + b^2 f'(b)\right)^2}\right|$$

Therefore, we take from (1.4) for $p=2$, we obtain

$$2 \leq \left|\psi'(b)\right| = \left|\frac{2b\left(-\frac{1+\lambda}{2b^2}\right)(\lambda-1) + b^2 f''(b)(\lambda-1)}{\left(\lambda + b^2\left(-\frac{1+\lambda}{2b^2}\right)\right)^2}\right|,$$

$$2 \leq (\lambda-1)\frac{\left|-\frac{1+\lambda}{b} + b^2 f''(b)\right|}{\left(\frac{\lambda-1}{2}\right)^2} \leq \frac{4}{\lambda-1}\left(1+\lambda+\left|f''(b)\right|\right)$$

and

$$\left|f''(b)\right| \geq -\frac{\lambda+3}{2}.$$

Now, we shall show that the inequality (2.1) is sharp. Let

$$f(z) = \frac{1}{z} + (1-\lambda)\arctan z.$$

Then

$$f'(z) = -\frac{1}{z^2} + \frac{1-\lambda}{1+z^2} = -\frac{1+\lambda z^2}{z^2(1+z^2)},$$

$$f''(z) = -\frac{-2\lambda z^5 - 4z^3 - 2z}{(z^2+z^4)^2}$$

and

$$f''(-1) = -\frac{\lambda+3}{2}.$$

□

Theorem 2. *Let $f(z) \in \mathcal{N}(\lambda)$. Suppose that, for some $b \in \partial E$, f' has an angular limit $f'(b)$ at b, $f'(b) = -\frac{1+\lambda}{2b^2}$. Then we have the inequality*

$$\left|f''(b)\right| \geq -\frac{\lambda+3}{2} + \frac{\lambda-1}{4}\left(\frac{\lambda-1-2|c_1|}{\lambda-1+2|c_1|}\right). \tag{2.2}$$

The inequality (2.2) is sharp with extremal function

$$f(z) = \frac{1}{z} + (1-\lambda)\arctan z.$$

Proof. Let $\psi(z)$ be the same as in the proof of Theorem1. Therefore, we take from (1.3), we obtain

$$2 + \frac{1-|d_2|}{1+|d_2|} \leq \left|\psi'(b)\right| = (\lambda-1)\frac{\left|-\frac{1+\lambda}{b} + b^2 f''(b)\right|}{\left(\frac{\lambda-1}{2}\right)^2}.$$

Since

$$\begin{aligned}\psi(z) &= -\frac{1+z^2 f'(z)}{\lambda + z^2 f'(z)} = -\frac{1+z^2\left(-\frac{1}{z^2} + c_1 + 2c_2 z + 3c_3 z^2 + \ldots\right)}{\lambda + z^2\left(-\frac{1}{z^2} + c_1 + 2c_2 z + 3c_3 z^2 + \ldots\right)} \\ &= -\frac{c_1 z^2 + 2c_2 z^3 + 3c_3 z^3 + \ldots}{\lambda - 1 + c_1 z^2 + 2c_2 z^3 + 3c_3 z^3 + \ldots} = -\frac{c_1}{\lambda-1}z^2 - \frac{2c_2}{\lambda-1}z^3 - \ldots,\end{aligned}$$

$$\psi(z) = -\frac{c_1}{\lambda-1}z^2 - \frac{2c_2}{\lambda-1}z^3 - ...$$

and

$$|d_2| = \frac{|c_1|}{\lambda-1},$$

we take

$$\begin{aligned} 2+\frac{1-\frac{|c1|}{\lambda-1}}{1+\frac{|c1|}{\lambda-1}} &\leq (\lambda-1)\frac{\left|-\frac{1+\lambda}{b}+b^2f''(b)\right|}{\left(\frac{\lambda-1}{2}\right)^2} \\ 2+\frac{\lambda-1-|c_1|}{\lambda-1+|c_1|} &\leq \frac{4}{\lambda-1}\left(1+\lambda+\left|f''(b)\right|\right) \end{aligned}$$

and

$$\left|f''(b)\right| \geq -\frac{\lambda+3}{2}+\frac{\lambda-1}{4}\left(\frac{\lambda-1-|c_1|}{\lambda-1+|c_1|}\right).$$

Now, we shall show that the inequality (2.2) is sharp. Let

$$f(z) = \frac{1}{z} + (1-\lambda)\arctan z.$$

Then

$$f'(z) = -\frac{1}{z^2}+\frac{1-\lambda}{1+z^2} = -\frac{1+\lambda z^2}{z^2\left(1+z^2\right)},$$

$$f''(z) = -\frac{-2\lambda z^5-4z^3-2z}{\left(z^2+z^4\right)^2}$$

and

$$f''(-1) = -\frac{\lambda+3}{2}.$$

Since $|c_1| = \lambda-1$, (2.2) is satisfied with equality. That is;

$$\begin{aligned} -\frac{\lambda+3}{2}+\frac{\lambda-1}{4}\left(\frac{\lambda-1-|c_1|}{\lambda-1+|c_1|}\right) &= -\frac{\lambda+3}{2}+\frac{\lambda-1}{4}\left(\frac{\lambda-1-(\lambda-1)}{\lambda-1+\lambda-1}\right) \\ &= -\frac{\lambda+3}{2}. \end{aligned}$$

□

The inequality (2.2) can be strengthened as below by taking into account c_2 which is second coefficient in the expansion of the function $f(z)$.

Theorem 3. *Let $f(z) \in \mathcal{N}(\lambda)$. Suppose that, for some $b \in \partial E$, f' has an angular limit $f'(b)$ at b, $f'(b) = -\frac{1+\lambda}{2b^2}$. Then we have the inequality*

$$\left|f''(b)\right| \geq -\frac{\lambda+3}{2}+\frac{\lambda-1}{2}\left(\frac{\left(\lambda-1-|c_1|\right)^2}{(\lambda-1)^2-|c_1|^2+2(\lambda-1)|c_2|}\right). \tag{2.3}$$

The equality in (2.3) *occurs for the function*

$$f(z) = \frac{1}{z} + (1-\lambda)\arctan z.$$

Proof. Let $\psi(z)$ be the same as in the proof of Theorem1. Let us consider the function

$$h(z)=\frac{\psi(z)}{B(z)},$$

where $B(z)=z^2$. The function $h(z)$ is holomorphic in E. According to the maximum principle, we have $|h(z)|<1$ for each $z\in E$. In particular, we have

$$|h(0)|=\frac{|c_1|}{\lambda-1}\leq 1 \tag{2.4}$$

and

$$\left|h'(0)\right|=\frac{2\left|c_2\right|}{\lambda-1}.$$

Since the expression $\frac{b\psi'(b)}{\psi(b)}$ is a real number greater or equal to 1 (see, [2]) and $f'(b)=-\frac{1+\lambda}{2b^2}$ yields $|\psi(b)|=1$, we take

$$\frac{b\psi'(b)}{\psi(b)}=\left|\frac{b\psi'(b)}{\psi(b)}\right|=\left|\psi'(b)\right|.$$

Also, by the maximum principle for each $z\in E$, we have $|\psi(z)|\leq|B(z)|$. So, we get

$$\frac{1-|\psi(z)|}{1-|z|}\geq\frac{1-|B(z)|}{1-|z|}.$$

Moving to the angular limit in the last inequality yields

$$\left|\psi'(b)\right|\geq\left|B'(b)\right|.$$

Therefore, we obtain

$$\frac{b\psi'(b)}{\psi(b)}=\left|\psi'(b)\right|\geq\left|B'(b)\right|=\frac{bB'(b)}{B(b)}.$$

Consider the function

$$\Phi(z)=\frac{h(z)-h(0)}{1-\overline{h(0)}h(z)}.$$

That function is holomorphic in E, $|\Phi(z)|\leq 1$ for $|z|<1$, $\Phi(0)=0$, and $|\Phi(b)|=1$ for $b\in\partial E$. From (1.3) for $p=1$, we obtain

$$\begin{aligned}\frac{2}{1+|\Phi'(0)|} &\leq \left|\Phi'(b)\right|=\frac{1-|h(0)|^2}{\left|1-\overline{h(0)}h(b)\right|^2}\left|h'(b)\right| \\ &\leq \frac{1+|h(0)|}{1-|h(0)|}\left\{\left|\psi'(b)\right|-\left|B'(b)\right|\right\}.\end{aligned}$$

Since

$$\Phi'(z)=\frac{1-|h(0)|^2}{\left(1-\overline{h(0)}h(z)\right)^2}h'(z)$$

and

$$\left|\Phi'(0)\right| = \frac{|h'(0)|}{1-|h(0)|^2} = \frac{\frac{2|c_2|}{\lambda-1}}{1-\left(\frac{|c_1|}{\lambda-1}\right)^2} = \frac{2(\lambda-1)|c_2|}{(\lambda-1)^2-|c_1|^2},$$

we take

$$\begin{aligned}
\frac{2}{1+\frac{2(\lambda-1)|c_2|}{(\lambda-1)^2-|c_1|^2}} &\leq \frac{1+\frac{|c_1|}{\lambda-1}}{1-\frac{|c_1|}{\lambda-1}}\left\{\frac{4}{\lambda-1}\left(1+\lambda+\left|f''(b)\right|\right)-2\right\},\\
\frac{2\left((\lambda-1)^2-|c_1|^2\right)}{(\lambda-1)^2-|c_1|^2+2(\lambda-1)|c_2|} &\leq \frac{\lambda-1+|c_1|}{\lambda-1-|c_1|}\left\{\frac{4}{\lambda-1}\left(1+\lambda+\left|f''(b)\right|\right)-2\right\}\\
\frac{2(\lambda-1-|c_1|)^2}{(\lambda-1)^2-|c_1|^2+2(\lambda-1)|c_2|} &\leq \frac{4}{\lambda-1}\left(1+\lambda+\left|f''(b)\right|\right)-2\\
\frac{(\lambda-1-|c_1|)^2}{(\lambda-1)^2-|c_1|^2+2(\lambda-1)|c_2|} &\leq \frac{2}{\lambda-1}\left(1+\lambda+\left|f''(b)\right|\right)-1
\end{aligned}$$

and

$$\left|f''(b)\right| \geq -\frac{\lambda+3}{2}+\frac{\lambda-1}{2}\left(\frac{(\lambda-1-|c_1|)^2}{(\lambda-1)^2-|c_1|^2+2(\lambda-1)|c_2|}\right).$$

Now, we shall show that the inequality (2.3) is sharp. Let

$$f(z) = \frac{1}{z}+(1-\lambda)\arctan z$$

Then

$$\left|f''(1)\right| = -\frac{\lambda+3}{2}.$$

Since $|c_1| = \lambda-1$, (2.3) is satisfied with equality. □

If $z^2 f''(z)+1$ has no zeros different from $z=0$ in Theorem 3, the inequality (2.3) can be further strengthened. This is given by the following Theorem.

Theorem 4. *Let $f(z) \in \mathcal{N}(\lambda)$, $z^2 f''(z)+1$ has no zeros in E except $z=0$ and $c_1 < 0$. Suppose that, for some $b \in \partial E$, f' has an angular limit $f'(b)$ at b, $f'(b) = -\frac{1+\lambda}{2b^2}$. Then we have the inequality*

$$\left|f''(b)\right| \geq -\frac{\lambda+3}{2}-\frac{\lambda-1}{4}\left(\frac{|c_1|\ln^2\frac{|c_1|}{\lambda-1}}{|c_1|\ln\frac{|c_1|}{\lambda-1}-|c_2|}\right). \tag{2.5}$$

The equality in (2.5) *occurs for the function*

$$f(z) = \frac{1}{z}+(1-\lambda)\arctan z.$$

Proof. Let $c_1 < 0$ and let us consider the function $h(z)$ as in Theorem 3. Taking into account equality (2.4), we denote by $\ln h(z)$ the holomorphic branch of the logarithm normed by condition

$$\ln h(0) = \ln\left(-\frac{c_1}{\lambda - 1}\right) = \ln\left|\frac{c_1}{\lambda - 1}\right| + i\arg\left(-\frac{c_1}{\lambda - 1}\right) < 0, \quad c_1 < 0$$

and

$$\ln\left|\frac{c_1}{\lambda - 1}\right| < 0.$$

Take the following auxiliary function

$$\phi(z) = \frac{\ln h(z) - \ln h(0)}{\ln h(z) + \ln h(0)}.$$

It is obvious that $\phi(z)$ is a holomorphic function in E, $\phi(0) = 0$, $|\phi(z)| < 1$ for $|z| < 1$, and also $|\phi(b)| = 1$ for $b \in \partial E$. That is; since

$$\begin{aligned} |\ln h(z) - \ln h(0)|^2 &= (\ln h(z) - \ln h(0))\left(\overline{\ln h(z)} - \ln h(0)\right) \\ &= |\ln h(z)|^2 - \ln h(z)\ln h(0) - \ln h(0)\overline{\ln h(z)} + \ln^2 h(0) \end{aligned}$$

and

$$\begin{aligned} |\ln h(z) + \ln h(0)|^2 &= (\ln h(z) + \ln h(0))\left(\overline{\ln h(z)} + \ln h(0)\right) \\ &= |\ln h(z)|^2 + \ln h(z)\ln h(0) + \ln h(0)\overline{\ln h(z)} + \ln^2 h(0), \end{aligned}$$

we obtain

$$\begin{aligned} |\ln h(z) - \ln h(0)|^2 - |\ln h(z) + \ln h(0)|^2 &= -2\ln h(z)\ln h(0) - 2\ln h(0)\overline{\ln h(z)} \\ &= -2\ln h(0)\left(\ln h(z) + \overline{\ln h(z)}\right) \\ &= -4\ln h(0)\ln|h(z)| \\ &< 0, \end{aligned}$$

$$|\ln h(z) - \ln h(0)|^2 < |\ln h(z) + \ln h(0)|^2$$

and

$$|\phi(z)| = \left|\frac{\ln h(z) - \ln h(0)}{\ln h(z) + \ln h(0)}\right| < 1$$

Also, since

$$\ln h(0) = \ln|h(0)| + i\arg h(0) = \ln|h(0)|$$

and

$$\ln h(b) = \ln|h(b)| + i\arg h(b) = i\arg h(b),$$

we obtain

$$|\phi(b)| = \left|\frac{\ln h(b) - \ln h(0)}{\ln h(b) + \ln h(0)}\right| = \left|\frac{i\arg h(b) - \ln|h(0)|}{i\arg h(b) + \ln|h(0)|}\right| = 1$$

So, we can apply (1.3) to the function $\phi(z)$ for $p=1$. Since

$$\phi'(z) = 2\ln h(0)\frac{h'(z)}{h(z)\left(\ln h(z)+\ln h(0)\right)^2},$$

and

$$\phi'(b) = 2\ln h(0)\frac{h'(b)}{h(b)\left(\ln h(b)+\ln h(0)\right)^2},$$

we obtain

$$\begin{aligned}
\frac{2}{1+|\phi'(0)|} &\leq \left|\phi'(b)\right| = \frac{2\left|\ln h(0)\right|}{\left|\ln h(b)+\ln h(0)\right|^2}\left|\frac{h'(b)}{h(b)}\right|, \\
&= \frac{-2\ln h(0)}{\ln^2 h(0)+\arg^2 h(b)}\left|\frac{\psi'(b)}{B(b)}-\frac{\psi(b)B'(b)}{B(b)^2}\right| \\
&= \frac{-2\ln h(0)}{\ln^2 h(0)+\arg^2 h(b)}\left|\frac{\psi(b)}{b^2}\right|\left|\frac{b\psi'(b)}{\psi(b)}-\frac{bB'(b)}{B(b)}\right| \\
&= \frac{-2\ln h(0)}{\ln^2 h(0)+\arg^2 h(b)}\left\{\left|\psi'(b)\right|-\left|B'(b)\right|\right\} \\
&\leq \frac{-2\ln h(0)}{\ln^2 h(0)}\left\{\frac{4}{\lambda-1}\left(1+\lambda+\left|f''(b)\right|\right)-2\right\} \\
&= \frac{-2}{\ln\frac{|c_1|}{\lambda-1}}\left\{\frac{4}{\lambda-1}\left(1+\lambda+\left|f''(b)\right|\right)-2\right\}.
\end{aligned}$$

Since

$$\phi'(0) = \frac{h'(0)}{2h(0)\ln h(0)}$$

and thus,

$$\left|\phi'(0)\right| = \frac{\frac{2|c_2|}{\lambda-1}}{-2\frac{|c_1|}{\lambda-1}\ln\frac{|c_1|}{\lambda-1}} = \frac{|c_2|}{-|c_1|\ln\frac{|c_1|}{\lambda-1}},$$

we have

$$\frac{2}{1-\frac{|c_2|}{|c_1|\ln\frac{|c_1|}{\lambda-1}}} \leq \frac{-2}{\ln\frac{|c_1|}{\lambda-1}}\left\{\frac{4}{\lambda-1}\left(1+\lambda+\left|f''(b)\right|\right)-2\right\},$$

$$\frac{|c_1|\ln\frac{|c_1|}{\lambda-1}}{|c_1|\ln\frac{|c_1|}{\lambda-1}-|c_2|} \leq \frac{-1}{\ln\frac{|c_1|}{\lambda-1}}\left\{\frac{4}{\lambda-1}\left(1+\lambda+\left|f''(b)\right|\right)-2\right\},$$

$$2-\frac{|c_1|\ln^2\frac{|c_1|}{\lambda-1}}{|c_1|\ln\frac{|c_1|}{\lambda-1}-|c_2|} \leq \frac{4}{\lambda-1}\left(1+\lambda+\left|f''(b)\right|\right)$$

and

$$\left|f''(b)\right| \geq -\frac{\lambda+3}{2}-\frac{\lambda-1}{4}\left(\frac{|c_1|\ln^2\frac{|c_1|}{\lambda-1}}{|c_1|\ln\frac{|c_1|}{\lambda-1}-|c_2|}\right).$$

Since $|c_1|=\lambda-1$, (2.5) is satisfied with equality. □

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