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A CHOICE OF THE KNOTS IN SCHOENBERG SPLINE OPERATOR WITH AN IMPROVEMENT OF THE ORDER OF APPROXIMATION

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Abstract

In our paper, we show that a small modification of the equidistant knots, in the quadratic Schoenberg operator, namely considering the knots $x_0 = 0$, $x_j = \frac{1}{2n} + (j-1)\frac{n-1}{n(n-2)}$, $(1 \le j \le n-1)$, $x_n = 1$, ensures a better order of approximation in Voronovskaya theorem.

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1 Introduction

Schoenberg [7] has built an operator that attaches to a function a k-th degree spline with n intermediate knots.

A k spline on [0,1] attached to a finite sequence of knots is a function which is polynomial of degree k on each interval between two consecutive knots, and which has a continue derivative of order k-1 on the whole interval [0,1].

An algebraic base of k-th degree spline functions is represented by the B-splines introduced by Schoenberg. To build these B-splines k extra-nodes equal to the left end of the interval and k-extra-knots equal to the right end of the interval are necessary.

More specifically, we consider a function $f : [0,1] \to \mathbb{R}$ and the knots which form the division $\Delta_{n,k}$, with $k > 0, n > 0$ natural numbers:

$$0 = x_{-k} = x_{-k+1} = ... = x_0 < x_1 < x_2 < ... < x_n = x_{n+1} = ... = x_{n+k} = 1 \quad (1)$$

B-splines are defined [1] by:

$$N_{j,k}(x) := (x_{j+k+1} - x_j)[x_j, x_{j+1}, ..., x_{j+k+1}](\bullet - x)^k_+, \quad -k \le j \le n-1 \quad (2)$$

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where

$$(t-x)_+^k = \begin{cases} (t-x)^k, & t > x \\ 0, & t \le x \end{cases}$$

and

$$[x_j, x_{j+1}, ..., x_{j+k+1}]f$$

means the divided difference of function f on nodes $x_j, x_{j+1}, ..., x_{j+k+1}$[1].

$N_{j,k}(x)$ is null for $x < x_j$ and $x > x_{j+k+1}$;

$N_{j,k}(x)$ is positive for $x_j \le x \le x_{j+k+1}$.

We further take the nodes (Greville abscissas)[1] :

$$\xi_{i,k} := \frac{x_{j+1} + ... + x_{j+k}}{k}, \ -k \le j \le n-1 \tag{3}$$

With the help of these data, Schoenberg operator [7],[5],[4] is defined:

$$S_{\Delta_{n,k}} f(x) := \sum_{j=-k}^{n-1} f(\xi_{j,k}) N_{j,k}(x), 0 \le x \le 1 \tag{4}$$

In the case when $\Delta_{n,k}$ is formed with equidistant knots, we denote $S_{n,k}f(x)$ instead of $S_{\Delta_{n,k}} f(x)$.

In the case $n = 1$ and $k \ge 2$, Schoenberg operator coincides with k-th Bernstein operator.

Because operators are linear and positive, for the study of convergence to f, it is useful to know the second moment.

The following formula established by DeVore [2], gives an estimate of the second moment in the case of arbitrary knots.

Theorem A For $n \ge 1$, $k > 1$ and arbitrary knots we have:

$$0 \le (S_{\Delta_{n,k}}(e_1 - x)^2)(x) = \sum_{j=-k}^{n-1} \frac{1}{k^2(k-1)} \sum_{1 \le r < s \le k} (x_{j+r} - x_{j+s})^2 N_{j,k}(x). \tag{5}$$

In the literature,[1], in case when $\Delta_{n,k}$ is formed by *equidistant knots* and k=2, the following result:

$$\frac{(S_{n,2}(e_1 - x)^2)(x)}{x(1-x)} = \frac{\frac{1}{2n} - \frac{x}{4}}{1-x} \le \frac{1}{2n} \tag{6}$$

hold.

The purpose of the paper is to obtain a better constant than $\frac{1}{2}$ in (6) for the second moment Schoenberg operators, working with a different choice of knots. This leads to a better order of approximation using Voronovskaya theorem.

2 A different way to choose the knots. The second moment of the quadratic Schoenberg operator

In the sequel we consider $k = 2$. Take a number $\lambda \in (0,1)$ and $n \geq 1$. We consider the knots and extra-knots of interval $[0,1]$:

$$0 = x_{-2} = x_{-1} = x_0 < x_1 < x_2 < ... < x_n = x_{n+1} = x_{n+2} = 1, \qquad (7)$$

where

$$\begin{aligned} x_0 &= 0 \\ x_1 &= \frac{\lambda}{n}, \\ x_j &= \frac{\lambda}{n} + (j-1)\frac{n-2\lambda}{n(n-2)},\ for\ \ 2 \leq j \leq n-2, \\ x_{n-1} &= 1 - \frac{\lambda}{n}, \\ x_n &= 1. \end{aligned} \qquad (8)$$

For simplicity we denote simply $N_j(x) := N_{j,2}(x)$, i.e.
$N_j(x) = [x_j, x_{j+1}, x_{j+2}, x_{j+4}](\bullet - x)_+^2$, for $x \in [0,1]$.

Lemma 1. *Let $n \geq 7$, $\lambda \in (0,1)$ and the knots and extra-nodes given in (7), (11).*

$$x_{-2} = x_{-1} = 0,\ x_{n+1} = x_{n+2} = 1.$$

The following relationships take place:

Case 1. $x \in [x_0, x_1]$:

$$\begin{aligned} N_{-1}(x) &= \frac{2n}{\lambda}x - \frac{n^2}{\lambda^2}\frac{2n\lambda + n - 6\lambda}{n\lambda + n - 4\lambda}x^2, \\ N_0(x) &= \frac{n^2(n-2)}{\lambda(n\lambda + n - 4\lambda)}x^2, \\ N_j(x) &= 0\ \ for\ \ 1 \leq j \leq n-1. \end{aligned}$$

Case 2. $x \in [x_1, x_2]$:

$$\begin{aligned} N_{-1}(x) &= \frac{n^2(n-2)^2}{(n\lambda + n - 4\lambda)(n-2\lambda)}x^2 - \frac{2n(n-2)}{n-2\lambda}x + \frac{n\lambda + n - 4\lambda}{n-2\lambda}, \\ N_0(x) &= -\frac{(n\lambda + 3n - 8\lambda)n^2(n-2)^2}{2(n\lambda + n - 4\lambda)(n-2\lambda)^2}x^2 + \frac{(n\lambda + 2n - 6\lambda)n(n-2)}{(n-2\lambda)^2}x \\ &\quad -\frac{\lambda(n\lambda + 2n - 6\lambda)(n-2)}{2(n-2\lambda)^2}, \\ N_1(x) &= \frac{n^2(n-2)^2}{2(n-2\lambda)^2}x^2 - \frac{n\lambda(n-2)^2}{(n-2\lambda)^2}x + \frac{\lambda^2(n-2)^2}{2(n-2\lambda)^2}, \\ N_j(x) &= 0\ \ for\ \ 2 \leq j \leq n-1\ \ or\ \ j = -2. \end{aligned}$$

Case 3. $x \in [x_2, x_3]$

$$\begin{aligned}
N_0(x) &= x^2\frac{n^2(n-2)^2}{2(n-2\lambda)^2} - x\frac{n(n-2)(n\lambda+2n-6\lambda)}{(n-2\lambda)^2} + \frac{(n\lambda+2n-6\lambda)^2}{2(n-2\lambda)^2},\\
N_1(x) &= -x^2\frac{n^2(n-2)^2}{(n-2\lambda)^2} + x\frac{n(n-2)(2n\lambda+3n-10\lambda)}{(n-2\lambda)^2}\\
&\quad -\frac{2\lambda^2(n-2)^2+6\lambda(n-2)(n-2\lambda)+3(n-2\lambda)^2}{2(n-2\lambda)^2},\\
N_2(x) &= x^2\frac{n^2(n-2)^2}{2(n-2\lambda)^2} - x\frac{n(n-2)(n+n\lambda-4\lambda)}{(n-2\lambda)^2} + \frac{(n+n\lambda-4\lambda)^2}{2(n-2\lambda)^2},\\
N_j(x) &= 0 \;\; for \;\; 3 \le j \le n-1 \;\; or \;\; j \in \{-2,-1\}.
\end{aligned}$$

Case 4. $x \in [x_q, x_{q+1}]$ *with* $3 \le q \le n-4$

$$N_j(x) = \frac{n^2(n-2)^2}{2(n-2\lambda)^2}\sum_{i=0}^{3} d_i\Big((i+j-1)\frac{n-2\lambda}{n(n-2)} + \frac{\lambda}{n} - x\Big)_+^2,$$

for $q-2 \le j \le q$, *where*

$$d_0 = -1,\; d_1 = 3,\; d_2 = -3,\; d_3 = 1. \tag{9}$$

$N_j(x) = 0$ *for* $2 \le j \le q-3$ *or* $q+1 \le j \le n-1$.

Case 5. $x \in [x_{n-3}, x_{n-2}]$

$$\begin{aligned}
N_{n-5}(x) &= (1-x)^2\frac{n^2(n-2)^2}{2(n-2\lambda)^2} - (1-x)\frac{n(n-2)(n+n\lambda-4\lambda)}{(n-2\lambda)^2}\\
&\quad +\frac{(n+n\lambda-4\lambda)^2}{2(n-2\lambda)^2},\\
N_{n-4}(x) &= -(1-x)^2\frac{n^2(n-2)^2}{(n-2\lambda)^2} + (1-x)\frac{n(n-2)(2n\lambda+3n-10\lambda)}{(n-2\lambda)^2}\\
&\quad -\frac{2\lambda^2(n-2)^2+6\lambda(n-2)(n-2\lambda)+3(n-2\lambda)^2}{2(n-2\lambda)^2},\\
N_{n-3}(x) &= \frac{x^2n^2(n-2)^2}{2(n-2\lambda)^2} - x\frac{n(n-2)(n^2-4n-n\lambda+6\lambda)}{(n-2\lambda)^2}\\
&\quad +\frac{(n^2-4n-n\lambda+6\lambda)^2}{2(n-2\lambda)^2},\\
N_j(x) &= 0 \;\; for \;\; -2 \le j \le n-6, \; or \; j \ge n-2.
\end{aligned}$$

Case 6. $x \in [x_{n-2}, x_{n-1}]$

$$\begin{aligned}
N_{n-4}(x) &= \frac{n^2(n-2)^2}{2(n-2\lambda)^2}(1-x)^2 - \frac{n\lambda(n-2)^2}{(n-2\lambda)^2}(1-x) + \frac{\lambda^2(n-2)^2}{2(n-2\lambda)^2},\\
N_{n-3}(x) &= -\frac{(n\lambda+3n-8\lambda)n^2(n-2)^2}{2(n\lambda+n-4\lambda)(n-2\lambda)^2}(1-x)^2\\
&\quad + \frac{(n\lambda+2n-6\lambda)n(n-2)}{(n-2\lambda)^2}(1-x) - \frac{\lambda(n\lambda+2n-6\lambda)(n-2)}{2(n-2\lambda)^2},\\
N_{n-2}(x) &= \frac{n^2(n-2)^2}{(n\lambda+n-4\lambda)(n-2\lambda)}(1-x)^2 - \frac{2n(n-2)}{n-2\lambda}(1-x)\\
&\quad + \frac{n\lambda+n-4\lambda}{n-2\lambda},\\
N_j(x) &= 0 \;\; for \;\; -2 \le j \le n-5 \; or \; j = n-1.
\end{aligned}$$

Case 7. $x \in [x_{n-1}, x_n]$

$$\begin{aligned}
N_{n-3}(x) &= \frac{(1-x)^2 n^2(n-2)}{\lambda(n\lambda+n-4\lambda)},\\
N_{n-2}(x) &= \frac{2n}{\lambda}(1-x) - \frac{n^2}{\lambda^2}\frac{2n\lambda+n-6\lambda}{n\lambda+n-4\lambda}(1-x)^2,\\
N_j(x) &= 0 \;\; for \;\; -2 \le j \le n-4.
\end{aligned}$$

Proof. Since $n \ge 7$, all the cases are possible.

The following formulas are well known for the divided difference, (see for instance [8], [6]) of function f on nodes $y_1, y_2, ..., y_m$:

$$[y_1, y_2, ..., y_m]f = \sum_{k=1}^{m} \Big(\prod_{1 \le i \le m, i \ne k} (y_k - y_i)^{-1} \Big) f(y_k) \qquad (10)$$

$$[y_1, y_2, ..., y_m]p = 0, \text{ if } p \text{ is a polinomial of degree at most } m-2,\ m \ge 2. \qquad (11)$$

$$[y_1, y_2, ..., y_m](\bullet - x)_+^k = 0, \text{ for } x > \max\{y_1, y_2, \dots, y_m\}. \qquad (12)$$

Case 1. $x \in [x_0, x_1]$, which means $x \in \left[0, \frac{\lambda}{n}\right]$

Consider first, that $x \in (x_0, x_1]$. We have

$$\begin{aligned}
&[0,0,x_1,x_2](\bullet - x)_+^2 = \lim_{\eta \searrow 0}[0,\eta,x_1,x_2](\bullet - x)_+^2\\
&= \lim_{\eta \searrow 0}\Big[-\frac{(0-x)_+^2}{\eta x_1 x_2} + \frac{(\eta - x)_+^2}{\eta(x_1-\eta)(x_2-\eta)}\\
&\qquad - \frac{(x_1-x)_+^2}{x_1(x_1-\eta)(x_2-x_1)} + \frac{(x_2-x)_+^2}{x_2(x_2-\eta)(x_2-x_1)}\Big]\\
&= \lim_{\eta \searrow 0}\Big[-\frac{(x_1-x)_+^2}{x_1(x_1-\eta)(x_2-x_1)} + \frac{(x_2-x)_+^2}{x_2(x_2-\eta)(x_2-x_1)}\Big]\\
&= -\frac{(x_1-x)^2}{x_1^2(x_2-x_1)} + \frac{(x_2-x)^2}{x_2^2(x_2-x_1)} = \frac{2x_1x_2x - (x_1+x_2)x^2}{x_1^2 x_2^2}.
\end{aligned}$$

Hence

$$\begin{aligned} N_{-1}(x) &= (x_2-x_{-1})[0,0,x_1,x_2](\bullet - x)^2_+ = \frac{2x_1x_2x-(x_1+x_2)x^2}{x_1^2x_2} \\ &= \frac{2n}{\lambda}x - \frac{n^2}{\lambda^2}\frac{2n\lambda+n-6\lambda}{n\lambda+n-4\lambda}x^2. \end{aligned}$$

Using the continuity, this relation can be extended from interval $(x_0, x_1]$ to the interval $[x_0, x_1]$. Also we have:

$$\begin{aligned} N_0(x) &= (x_3-x_0)[0,x_1,x_2,x_3](\bullet - x)^2_+ \\ &= (x_3-x_0)\Big[\frac{(x_1-x)^2_+}{(x_1-x_0)(x_2-x_1)(x_3-x_1)} \\ &\quad -\frac{(x_2-x)^2_+}{(x_2-x_0)(x_2-x_1)(x_3-x_2)} + \frac{(x_3-x)^2_+}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}\Big] \\ &= \frac{x^2}{x_1x_2} = \frac{x^2n^2(n-2)}{\lambda(n\lambda+n-4\lambda)}. \\ N_j(x) &= 0, \; for \;\; 1 \le j \le n-1 \;\; from \;\; relation \;\; (11). \end{aligned}$$

Case 2. $x \in [x_1, x_2]$, which means $x \in \left[\frac{\lambda}{n}, \frac{\lambda}{n} + \frac{n-2\lambda}{n(n-2)}\right]$

$$\begin{aligned} N_{-1}(x) &= (x_2-x_{-1})[0,0,x_1,x_2](\bullet - x)^2_+ = \frac{(x_2-x)^2_+}{x_2(x_2-x_1)} \\ &= \frac{1}{x_2(x_2-x_1)}x^2 - \frac{2}{x_2-x_1}x + \frac{x_2}{x_2-x_1} \\ &= \frac{n^2(n-2)^2}{(n\lambda+n-4\lambda)(n-2\lambda)}x^2 - \frac{2n(n-2)}{n-2\lambda}x + \frac{n\lambda+n-4\lambda}{n-2\lambda}. \\ N_0(x) &= (x_3-x_0)[0,x_1,x_2,x_3](\bullet - x)^2_+ \\ &= x_3\Big[\frac{(x_2-x)^2_+}{x_2(x_2-x_1)(x_2-x_3)} + \frac{(x_3-x)^2_+}{x_3(x_3-x_1)(x_3-x_2)}\Big] \\ &= \frac{x_2+x_3-x_1}{x_2(x_2-x_1)(x_1-x_3)}x^2 + \frac{2x_3}{(x_2-x_1)(x_3-x_1)}x + \frac{x_1x_3}{(x_2-x_1)(x_1-x_3)} \\ &= -\frac{(n\lambda+3n-8\lambda)n^2(n-2)^2}{2(n\lambda+n-4\lambda)(n-2\lambda)^2}x^2 + \frac{(n\lambda+2n-6\lambda)n(n-2)}{(n-2\lambda)^2}x \\ &\quad -\frac{\lambda(n\lambda+2n-6\lambda)(n-2)}{2(n-2\lambda)^2}. \\ N_1(x) &= (x_4-x_1)[x_1,x_2,x_3,x_4](\bullet - x)^2_+ \\ &= \Big[\frac{(x_2-x)^2_+}{(x_2-x_1)(x_2-x_3)(x_2-x_4)} + \frac{(x_3-x)^2_+}{(x_3-x_1)(x_3-x_2)(x_3-x_4)} \\ &\quad +\frac{(x_4-x)^2_+}{(x_4-x_1)(x_4-x_2)(x_4-x_3)}\Big](x_4-x_1) \end{aligned}$$

$$\begin{aligned}
&= \frac{(x_1-x)^2}{(x_2-x_1)(x_3-x_1)} \\
&= \frac{n^2(n-2)^2}{2(n-2\lambda)^2}x^2 - \frac{n\lambda(n-2)^2}{(n-2\lambda)^2}x + \frac{\lambda^2(n-2)^2}{2(n-2\lambda)^2}. \\
N_j(x) &= 0 \; for \;\; 2 \le j \le n-1 \;\; from \;\; relation \;\; (11).
\end{aligned}$$

Case 3. $x \in [x_2, x_3]$, which means $x \in \left[\frac{\lambda}{n} + \frac{n-2\lambda}{n(n-2)}, \frac{\lambda}{n} + 2\frac{n-2\lambda}{n(n-2)}\right]$

$$\begin{aligned}
N_0(x) &= (x_3-x_0)[x_0,x_1,x_2,x_3](\bullet - x)_+^2 \\
&= \frac{(x_3-x)_+^2-(x_2-x)_+^2}{(x_3-x_1)(x_3-x_2)} - \frac{(x_2-x)_+^2-(x_1-x)_+^2}{(x_3-x_1)(x_2-x_1)} \\
&\quad -\frac{(x_2-x)_+^2-(x_1-x)_+^2}{(x_2-x_0)(x_2-x_1)} + \frac{(x_1-x)_+^2-(x_0-x)_+^2}{(x_2-x_0)(x_1-x_0)} \\
&= \frac{(x_3-x)^2}{(x_3-x_1)(x_3-x_2)} \\
&= x^2\frac{n^2(n-2)^2}{2(n-2\lambda)^2} - x\frac{n(n-2)(n\lambda+2n-6\lambda)}{(n-2\lambda)^2} + \frac{(n\lambda+2n-6\lambda)^2}{2(n-2\lambda)^2}. \\
N_1(x) &= (x_4-x_1)[x_1,x_2,x_3,x_4](\bullet - x)^2 \\
&= (x_4-x_1)\Big[\frac{(x_3-x)^2}{(x_3-x_1)(x_3-x_2)(x_3-x_4)} \\
&\quad +\frac{(x_4-x)^2}{(x_4-x_1)(x_4-x_2)(x_4-x_3)}\Big] \\
&= -3\frac{n^2(n-2)^2}{2(n-2\lambda)^2}\Big(2\frac{n-2\lambda}{n(n-2)} + \frac{\lambda}{n} - x\Big)^2 \\
&\quad +\frac{n^2(n-2)^2}{2(n-2\lambda)^2}\Big(3\frac{n-2\lambda}{n(n-2)} + \frac{\lambda}{n} - x\Big)^2 \\
&\quad = -x^2\frac{n^2(n-2)^2}{(n-2\lambda)^2} + x\frac{n(n-2)(2n\lambda+3n-10\lambda)}{(n-2\lambda)^2} \\
&\quad -\frac{2\lambda^2(n-2)^2+6\lambda(n-2)(n-2\lambda)+3(n-2\lambda)^2}{2(n-2\lambda)^2}. \\
N_2(x) &= (x_5-x_2)[x_2,x_3,x_4,x_5](.-x)^2 \\
&= (x_5-x_2)\Big[\frac{(x_3-x)^2}{(x_3-x_2)(x_3-x_4)(x_3-x_5)} \\
&\quad +\frac{(x_4-x)^2}{(x_4-x_2)(x_4-x_3)(x_4-x_5)} + \frac{(x_5-x)^2}{(x_5-x_2)(x_5-x_3)(x_5-x_4)}\Big] \\
&= \frac{n^2(n-2)^2}{2(n-2\lambda)^2}x^2 \\
&\quad +\frac{3n^2(n-2)^2}{(n-2\lambda)^2}\Big[-2\frac{n-2\lambda}{n(n-2)} + 3\frac{n-2\lambda}{n(n-2)} - \frac{4}{3}\frac{n-2\lambda}{n(n-2)} - \frac{\lambda}{3n}\Big]x
\end{aligned}$$

$$+\frac{3n^2(n-2)^2}{2(n-2\lambda)^2}\Big[\Big(2\frac{n-2\lambda}{n(n-2)}+\frac{\lambda}{n}\Big)^2-\Big(3\frac{n-2\lambda}{n(n-2)}+\frac{\lambda}{n}\Big)^2$$
$$+\frac{1}{3}\Big(4\frac{n-2\lambda}{n(n-2)}+\frac{\lambda}{n}\Big)^2\Big]$$
$$= x^2\frac{n^2(n-2)^2}{2(n-2\lambda)^2}-x\frac{n(n-2)(n\lambda+n-4\lambda)}{(n-2\lambda)^2}+\frac{(n+n\lambda-4\lambda)^2}{2(n-2\lambda)^2}.$$

$N_j(x) \;=\; 0$ *for* $3\le j\le n-1$ *or* $j\in\{-2,-1\}$ *from relations* (11) *and* (12).

Case 4. $x\in[x_q,x_{q+1}]$ with $3\le q\le n-4$.
Let j such that $q-2\le j\le q$. Using the recurrence formula of divided differences, we obtain:

$$N_j(x) = (x_{j+3}-x_j)[x_j,x_{j+1},x_{j+2},x_{j+3}](\bullet-x)_+^2$$
$$= \frac{(x_{j+3}-x)_+^2-(x_{j+2}-x)_+^2}{(x_{j+3}-x_{j+1})(x_{j+3}-x_{j+2})}-\frac{(x_{j+2}-x)_+^2-(x_{j+1}-x)_+^2}{(x_{j+3}-x_{j+1})(x_{j+2}-x_{j+1})}$$
$$-\frac{(x_{j+2}-x)_+^2-(x_{j+1}-x)_+^2}{(x_{j+2}-x_j)(x_{j+2}-x_{j+1})}+\frac{(x_{j+1}-x)_+^2-(x_j-x)_+^2}{(x_{j+2}-x_j)(x_{j+1}-x_j)}$$
$$= \frac{n^2(n-2)^2}{2(n-2\lambda)^2}\Big((j+2)\frac{n-2\lambda}{n(n-2)}+\frac{\lambda}{n}-x\Big)_+^2-\frac{3n^2(n-2)^2}{2(n-2\lambda)^2}\Big((j$$
$$+1)\frac{n-2\lambda}{n(n-2)}+\frac{\lambda}{n}-x\Big)_+^2+\frac{3n^2(n-2)^2}{2(n-2\lambda)^2}\Big(j\frac{n-2\lambda}{n(n-2)}+\frac{\lambda}{n}-x\Big)_+^2$$
$$-\frac{n^2(n-2)^2}{2(n-2\lambda)^2}\Big((j-1)\frac{n-2\lambda}{n(n-2)}+\frac{\lambda}{n}-x\Big)_+^2.$$

Hence, we obtain the desired relation, for $q-2\le j\le q$.

$N_j(x)=0$ *for* $-2\le j\le q-3$ *or* $q+1\le j\le n-1$ from from relations (11) and (12).

Case 5. $x\in[x_{n-3},x_{n-2}]$
By symmetry, using Case 3, we obtain $N_{n-5}(x), N_{n-4}(x)$.

$$N_{n-3}(x) = (x_n-x_{n-3})[x_{n-3},x_{n-2},x_{n-1},x_n](\bullet-x)_+^2$$
$$= [x_{n-2},x_{n-1},x_n](\bullet-x)_+^2-[x_{n-3},x_{n-2},x_{n-1}](\bullet-x)_+^2$$
$$= \frac{(x_n-x)_+^2-(x_{n-1}-x)_+^2}{(x_n-x_{n-2})(x_n-x_{n-1})}-\frac{(x_{n-1}-x)_+^2-(x_{n-2}-x)_+^2}{(x_n-x_{n-2})(x_{n-1}-x_{n-2})}$$
$$-\frac{(x_{n-1}-x)_+^2-(x_{n-2}-x)_+^2}{(x_{n-1}-x_{n-3})(x_{n-1}-x_{n-2})}+\frac{(x_{n-2}-x)_+^2-(x_{n-3}-x)_+^2}{(x_{n-2}-x_{n-3})(x_{n-1}-x_{n-3})}$$
$$= \frac{x^2}{(x_{n-2}-x_{n-3})(x_{n-1}-x_{n-3})}-\frac{2xx_{n-3}}{(x_{n-2}-x_{n-3})(x_{n-1}-x_{n-3})}$$
$$+\frac{x_{n-3}^2}{(x_{n-2}-x_{n-3})(x_{n-1}-x_{n-3})}=\frac{x^2n^2(n-2)^2}{2(n-2\lambda)^2}$$
$$-x\frac{n(n-2)(n^2-4n-n\lambda+6\lambda)}{(n-2\lambda)^2}+\frac{(n^2-4n-n\lambda+6\lambda)^2}{2(n-2\lambda)^2}.$$

$N_j(x) \;=\; 0$ *for* $-2\le j\le n-6$ *from relations* (11) *and* (12).

Case 6. $x \in [x_{n-2}, x_{n-1}]$. Using the symmetry, Case 2, demonstrates this case.

Case 7. $x \in [x_{n-1}, x_n]$. Using the symmetry, Case 1, demonstrates this case. □

Theorem 1. *The second moment of second degree Schoenberg spline operator for knots* (8) *is:* $(S_{\Delta_{n,2}}(e_1 - x)^2)(x)$

$$= \begin{cases} x^2 \frac{n^2 - n\lambda(n+2) - 2\lambda^2(n-1)(n-4)}{4\lambda(n-2)(n\lambda+n-4\lambda)} + x\frac{\lambda}{2n}, & x \in \left[0, \frac{\lambda}{n}\right] \\ x^2 \frac{n(\lambda-1)}{4(n-2\lambda)} + x\frac{(1-\lambda)(n\lambda+n-4\lambda)}{2(n-2)(n-2\lambda)} & \\ +\frac{\lambda^3(n^2-6n+4)+n\lambda^2(n+2)-n^2\lambda}{4n^2(n-2)(n-2\lambda)}, & x \in \left[\frac{\lambda}{n}, \frac{\lambda}{n} + \frac{n-2\lambda}{n(n-2)}\right] \\ \frac{1}{4n^2}\frac{(n-2\lambda)^2}{(n-2)^2}, & x \in \left[\frac{\lambda}{n} + \frac{n-2\lambda}{n(n-2)}, 1 - \frac{\lambda}{n} - \frac{n-2\lambda}{n(n-2)}\right] \\ \frac{(1-x)^2 n(\lambda-1)}{4(n-2\lambda)} + \frac{(1-x)(1-\lambda)(n\lambda+n-4\lambda)}{2(n-2)(n-2\lambda)} & \\ +\frac{\lambda^3(n^2-6n+4)+n\lambda^2(n+2)-n^2\lambda}{4n^2(n-2)(n-2\lambda)}, & x \in \left[1 - \frac{\lambda}{n} - \frac{n-2\lambda}{n(n-2)}, 1 - \frac{\lambda}{n}\right] \\ \frac{(1-x)^2[n^2-n\lambda(n+2)-2\lambda^2(n-1)(n-4)]}{4\lambda(n-2)(n\lambda+n-4\lambda)} + \frac{(1-x)\lambda}{2n}, & x \in \left[1 - \frac{\lambda}{n}, 1\right] \end{cases}$$

Proof. Formula (5) becomes in our case

$$(S_{\Delta_{n,2}}(e_1 - x)^2)(x) = \frac{1}{4}\sum_{j=-2}^{n-1}(x_{j+1} - x_{j+2})^2 N_j(x). \tag{13}$$

From Lemma 1 we obtain the following cases:

Case 1. $x \in \left[0, \frac{\lambda}{n}\right]$. Since $x_2 = x_1 = 0$, using Case 1 of Lemma 1 we get:

$$\begin{aligned} (S_{\Delta_{n,2}}(e_1 - x)^2)(x) &= \frac{1}{4}(x_0 - x_1)^2 N_{-1}(x) + \frac{1}{4}(x_1 - x_2)^2 N_0(x) \\ &= \frac{\lambda^2}{4n^2}N_{-1}(x) + \frac{(n-2\lambda)^2}{4n^2(n-2)^2}N_0(x) \\ &= \frac{\lambda^2}{4n^2}\Big(\frac{2n}{\lambda}x - \frac{n^2}{\lambda^2}\frac{2n\lambda + n - 6\lambda}{n\lambda + n - 4\lambda}x^2\Big) \\ &\quad + \frac{(n-2\lambda)^2}{4n^2(n-2)^2}\frac{x^2 n^2(n-2)}{\lambda(n\lambda + n - 4\lambda)} \\ &= x^2\frac{n^2 - n\lambda(n+2) - 2\lambda^2(n-1)(n-4)}{4\lambda(n-2)(n\lambda + n - 4\lambda)} + x\frac{\lambda}{2n}. \end{aligned}$$

Case 2. For $x \in \left[\frac{\lambda}{n}, \frac{\lambda}{n} + \frac{n-2\lambda}{n(n-2)}\right]$, using Case 2 of Lemma 1 we get:

$$(S_{\Delta_{n,2}}(e_1 - x)^2)(x) = \frac{1}{4}\sum_{j=-2}^{n-1}(x_{j+1} - x_{j+2})^2 N_{j,2}(x)$$

$$= \frac{1}{4}(x_0 - x_1)^2 N_{-1}(x) + \frac{1}{4}(x_1 - x_2)^2 N_0(x) + \frac{1}{4}(x_2 - x_3)^2 N_{1,2}(x)$$

$$\begin{aligned}
&= \frac{\lambda^2}{4n^2}N_{-1}(x) + \frac{1}{4}\frac{(n-2\lambda)^2}{n^2(n-2)^2}N_0(x) + \frac{1}{4}\frac{(n-2\lambda)^2}{n^2(n-2)^2}N_1(x)\\
&= \frac{x^2}{4}\Big(\frac{\lambda^2(n-2)^2}{(n\lambda+n-4\lambda)(n-2\lambda)} - \frac{n\lambda+3n-8\lambda}{2(n\lambda+n-4\lambda)} + \frac{1}{2}\Big)\\
&\quad + \frac{x}{4}\Big(-\frac{2\lambda^2(n-2)}{n(n-2\lambda)} + \frac{n\lambda+2n-6\lambda}{n(n-2)} - \frac{\lambda}{n}\Big) + \frac{1}{4n^2}\Big(\frac{\lambda^2(n\lambda+n-4\lambda)}{n-2\lambda}\\
&\quad - \frac{\lambda(n\lambda+2n-6\lambda)}{2(n-2)} + \frac{\lambda^2}{2}\Big)\\
&= x^2\frac{n(\lambda-1)}{4(n-2\lambda)} + x\frac{(1-\lambda)(n\lambda+n-4\lambda)}{2(n-2)(n-2\lambda)}\\
&\quad + \frac{\lambda^3(n^2-6n+4)+n\lambda^2(n+2)-n^2\lambda}{4n^2(n-2)(n-2\lambda)}.
\end{aligned}$$

Case 3. For $x \in \left[\frac{\lambda}{n} + \frac{n-2\lambda}{n(n-2)}, \frac{\lambda}{n} + 2\frac{n-2\lambda}{n(n-2)}\right]$, using Case 3 of Lemma 1 we get:

$$\begin{aligned}
&(S_{\Delta_{n,2}}(e_1-x)^2)(x) = \frac{1}{4}\sum_{j=-2}^{n-1}(x_{j+1}-x_{j+2})^2 N_j(x)\\
&= \frac{1}{4}(x_1-x_2)^2N_0(x) + \frac{1}{4}(x_2-x_3)^2N_1(x) + \frac{1}{4}(x_3-x_4)^2N_2(x)\\
&= \frac{(n-2\lambda)^2}{4n^2(n-2)^2}[N_0(x)+N_1(x)+N_2(x)]\\
&= \frac{(n-2\lambda)^2}{4n^2(n-2)^2}\Big[x^2\frac{n^2(n-2)^2}{2(n-2\lambda)^2} - x\frac{n(n-2)(n\lambda+2n-6\lambda)}{(n-2\lambda)^2} + \frac{(n\lambda+2n-6\lambda)^2}{2(n-2\lambda)^2}\\
&\quad -x^2\frac{n^2(n-2)^2}{(n-2\lambda)^2} + x\frac{n(n-2)(2n\lambda+3n-10\lambda)}{(n-2\lambda)^2}\\
&\quad -\frac{2\lambda^2(n-2)^2+6\lambda(n-2)(n-2\lambda)+3(n-2\lambda)^2}{2(n-2\lambda)^2} + x^2\frac{n^2(n-2)^2}{2(n-2\lambda)^2}\\
&\quad -x\frac{n(n-2)(n+n\lambda-4\lambda)}{(n-2\lambda)^2} + \frac{(n+n\lambda-4\lambda)^2}{2(n-2\lambda)^2}\Big]\\
&= \frac{1}{4n^2}\frac{(n-2\lambda)^2}{(n-2)^2}.
\end{aligned}$$

Case 4. $x \in \left[\frac{\lambda}{n} + 2\frac{n-2\lambda}{n(n-2)}, 1 - \frac{\lambda}{n} - 2\frac{n-2\lambda}{n(n-2)}\right]$.
Then there is $3 \le q \le n-4$, such that $x \in [x_q, x_{q+1}]$. For $-2 \le j \le q-3$ and $q+1 \le j \le n-3$ we have $N_j(x) = 0$. Therefore, using Case 4 of Lemma 1 we get:

$$\begin{aligned}
(S_{\Delta_{n,2}}(e_1-x)^2)(x) &= \frac{1}{4}\sum_{j=-2}^{n-1}(x_{j+2}-x_{j+1})^2N_{j,2}(x)\\
&= \frac{n^2(n-2)^2}{4n^2(n-2\lambda)^2}\sum_{j=q-2}^{q}N_j(x)
\end{aligned}$$

$$\begin{aligned}
&= \frac{(n-2\lambda)^2}{4n^2(n-2)^2}\sum_{j=q-2}^{q}\frac{n^2(n-2)^2}{2(n-\lambda)^2}\sum_{i=0}^{3}d_i(x_{i+j}-x)_+^2\\
&= \frac{1}{8}\sum_{j=q-2}^{q}\sum_{i=0}^{3}d_i(x_{i+j}-x)_+^2\\
&= \frac{1}{8}((x_{q+1}-x)^2-2(x_{q+2}-x)^2+(x_{q+3}-x)^2)\\
&= \frac{1}{4}(x_{q+1}-x_q)^2\\
&= \frac{(n-2\lambda)^2}{4n^2(n-2)^2}.
\end{aligned}$$

Case 5. For $x \in \left[1-\frac{\lambda}{n}-2\frac{n-2\lambda}{n(n-2)}, 1-\frac{\lambda}{n}-\frac{n-2\lambda}{n(n-2)}\right]$, is obtained, by symmetry with Case 3:

$$(S_{\Delta_{n,2}}(e_1-x)^2)(x) = \frac{1}{4n^2}\frac{(n-2\lambda)^2}{(n-2)^2}.$$

Case 6. For $x \in \left[1-\frac{\lambda}{n}-\frac{n-2\lambda}{n(n-2)}, 1-\frac{\lambda}{n}\right]$ is obtained by symmetry with Case 2:

$$\begin{aligned}
(S_{\Delta_{n,2}}(e_1-x)^2)(x) &= \frac{(1-x)^2 n(\lambda-1)}{4(n-2\lambda)} + \frac{(1-x)(1-\lambda)(n\lambda+n-4\lambda)}{2(n-2)(n-2\lambda)}\\
&\quad + \frac{\lambda^3(n^2-6n+4)+n\lambda^2(n+2)-n^2\lambda}{4n^2(n-2)(n-2\lambda)}.
\end{aligned}$$

Case 7. For $x \in \left[1-\frac{\lambda}{n}, 1\right]$, is obtained, by symmetry with Case 1:

$$(S_{\Delta_{n,2}}(e_1-x)^2)(x) = (1-x)^2\frac{n^2-n\lambda(n+2)-2\lambda^2(n-1)(n-4)}{4\lambda(n-2)(n\lambda+n-4\lambda)} + (1-x)\frac{\lambda}{2n}.$$

□

For $\lambda = 1$ we obtain the particular case of *equidistant knots*:

Theorem 2. *The second moment of second degree Schoenberg spline operator for equidistant knots is:*

$$(S_{n,2}(e_1-x)^2)(x) = \begin{cases} -\frac{x^2}{4}+\frac{x}{2n}, & x \in \left[0, \frac{1}{n}\right] \\ \frac{1}{4n^2}, & x \in \left[\frac{1}{n}, \frac{n-1}{n}\right] \\ -\frac{(1-x)^2}{4}+\frac{1-x}{2n}, & x \in \left[\frac{n-1}{n}, 1\right] \end{cases}$$

In literature,[1], the case with equidistant knots is the case usually studied.

3 Applications to Voronovskaya's theorem

The following improved version of the Voronovskaja theorem was proved in [3]:

Theorem B *If $L_n : C[a,b] \to C[a,b]$ is a sequence of positive linear operator such that $L_n(e_j) = e_j$, $j \in \{0,1\}$, then, for $f \in C^2[a,b]$ and $x \in (0,1)$:*

$$L_n(f,x) - f(x) \quad = \quad \frac{1}{2} L_n((e_1 - x)^2, x) f''(x) + o\Big(L_n((e_1 - x)^2, x)\Big), \qquad (14)$$

uniformly with regard to $x \in [a,b]$.

Consequently, if $L_n : C[0,1] \to C[0,1]$ is a sequence of positive linear operator such that $L_n(e_j) = e_j$, $j \in \{0,1\}$, and there is a constant $K > 0$ such that

$$\frac{L_n((e_1 - x)^2, x)}{x(1-x)} \leq K \cdot \frac{1}{n},\ x \in (0,1)$$

then, for $f \in C^2[0,1]$:

$$\left| \frac{L_n(f,x) - f(x)}{x(1-x)} \right| \leq \frac{K}{2n} |f''(x)| + \mathrm{o}\Big(\frac{1}{n}\Big), \qquad (15)$$

uniformly with regard to $x \in (0,1)$.

First we apply this theorem in the case of equidistant knots and k=2.

Corollary 1.

$$\Big| \frac{S_{n,2}(f,x) - f(x)}{x(1-x)} \Big| \leq \frac{1}{4n} |f''(x)| + o\Big(\frac{1}{n}\Big),\ uniformly\ with\ regard\ to\ x \qquad (16)$$

for $f \in C^2[0,1], x \in (0,1)$; or

$$\Big|\Big| \frac{S_{n,2}(f) - f}{e_1 - e_2} \Big|\Big|_{(0,1)} \leq \frac{1}{4n} \|f''\| + o\Big(\frac{1}{n}\Big). \qquad (17)$$

Proof.

$$\sup_{x \in \left[0, \frac{1}{n}\right]} \frac{(S_{n,2}(e_1 - x)^2)(x)}{x(1-x)} = \sup_{x \in \left[0, \frac{1}{n}\right]} \frac{-\frac{x^2}{4} + \frac{x}{2n}}{x(1-x)} = \frac{-\frac{x}{4} + \frac{1}{2n}}{1-x} \Bigg|_{x=0} = \frac{1}{2n} + o\Big(\frac{1}{n}\Big)$$

$$\sup_{x \in \left[\frac{1}{n}, \frac{n-1}{n}\right]} \frac{(S_{n,2}(e_1 - x)^2)(x)}{x(1-x)} = \frac{\frac{1}{4n^2}}{x(1-x)} \Bigg|_{x=\frac{1}{n}} = \frac{1}{4(n-1)} = \frac{1}{4n} + o\Big(\frac{1}{n}\Big).$$

□

For our choice of knots (8) and for $\lambda = \frac{1}{2}$ we obtain an improvement given in the next corollary:

Corollary 2.

$$\Big|\frac{S^{\lambda=\frac{1}{2}}_{\Delta_{n,2}}(f,x)-f(x)}{x(1-x)}\Big| \le \frac{1}{8n}|f''(x)| + o\Big(\frac{1}{n}\Big),\ \textit{uniformly with regard to } x \qquad (18)$$

for $f \in C^2[0,1], x \in (0,1)$; *or*

$$\Big\|\frac{S^{\lambda=\frac{1}{2}}_{\Delta_{n,2}}(f)-f}{e_1-e_2}\Big\|_{(0,1)} \le \frac{1}{8n}\|f''\| + o\Big(\frac{1}{n}\Big). \qquad (19)$$

Proof.

$$\sup_{x\in\left[0,\frac{1}{2n}\right]} \frac{(S^{\lambda=\frac{1}{2}}_{\Delta_{n,2}}(e_1-x)^2)(x)}{x(1-x)} = \sup_{x\in\left[0,\frac{1}{2n}\right]} \frac{\frac{x^2}{2(n-2)}+\frac{x}{4n}}{x(1-x)} = \frac{\frac{x}{2(n-2)}+\frac{1}{4n}}{1-x}\Bigg|_{x=\frac{1}{2n}}$$
$$= \frac{1}{4n} + o\Big(\frac{1}{n}\Big)\ \textit{uniformly with regard to } x \in (0,1).$$

$$\sup_{x\in\left[\frac{1}{2n},\frac{3n-4}{2n(n-2)}\right]} \frac{(S^{\lambda=\frac{1}{2}}_{\Delta_{n,2}}(e_1-x)^2)(x)}{x(1-x)}$$
$$= \sup_{x\in\left[\frac{1}{2n},\frac{3n-4}{2n(n-2)}\right]} \frac{-\frac{nx^2}{8(n-1)}+\frac{x(3n-4)}{8(n-1)(n-2)}-\frac{n^2+2n-4}{32n^2(n-1)(n-2)}}{x(1-x)}$$
$$\le \frac{-\frac{nx^2}{8(n-1)}+\frac{x(3n-4)}{8(n-1)(n-2)}-\frac{n^2+2n-4}{32n^2(n-1)(n-2)}}{x(1-x)}\Bigg|_{x=\frac{1}{2n}} \cdot \frac{2n}{2n-1}$$
$$= \frac{1}{4n} + o\Big(\frac{1}{n}\Big)\ \textit{uniformly with regard to } x \in (0,1).$$

$$\sup_{x\in\left[\frac{3n-4}{2n(n-2)},\frac{5n-4}{2n(n-2)}\right]} \frac{(S^{\lambda=\frac{1}{2}}_{\Delta_{n,2}}(e_1-x)^2)(x)}{x(1-x)} = \sup_{x\in\left[\frac{3n-4}{2n(n-2)},\frac{5n-4}{2n(n-2)}\right]} \frac{\frac{(n-1)^2}{4n^2(n-2)^2}}{x(1-x)}$$
$$= \frac{\frac{(n-1)^2}{4n^2(n-2)^2}}{x(1-x)}\Bigg|_{x=\frac{3n-4}{2n(n-2)}} = \frac{1}{6n} + o\Big(\frac{1}{n}\Big)\ \textit{uniformly with regard to } x \in (0,1).$$

$$\sup_{x\in\left[\frac{5n-4}{2n(n-2)},1-\frac{\lambda}{n}-\frac{2(n-2\lambda)}{n(n-2)}\right]} \frac{(S^{\lambda=\frac{1}{2}}_{\Delta_{n,2}}(e_1-x)^2)(x)}{x(1-x)} = \sup_{x\in\left[\frac{5n-4}{2n(n-2)},1-\frac{\lambda}{n}-\frac{2(n-2\lambda)}{n(n-2)}\right]} \frac{\frac{(n-1)^2}{4n^2(n-2)^2}}{x(1-x)}$$
$$= \frac{\frac{(n-1)^2}{4n^2(n-2)^2}}{x(1-x)}\Bigg|_{x=\frac{5n-4}{2n(n-2)}} = \frac{1}{10n} + o\Big(\frac{1}{n}\Big)\ \textit{uniformly with regard to } x \in (0,1).$$

□

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